

QCD equation of state via the complex Langevin method

[arXiv:2203.13144](https://arxiv.org/abs/2203.13144)

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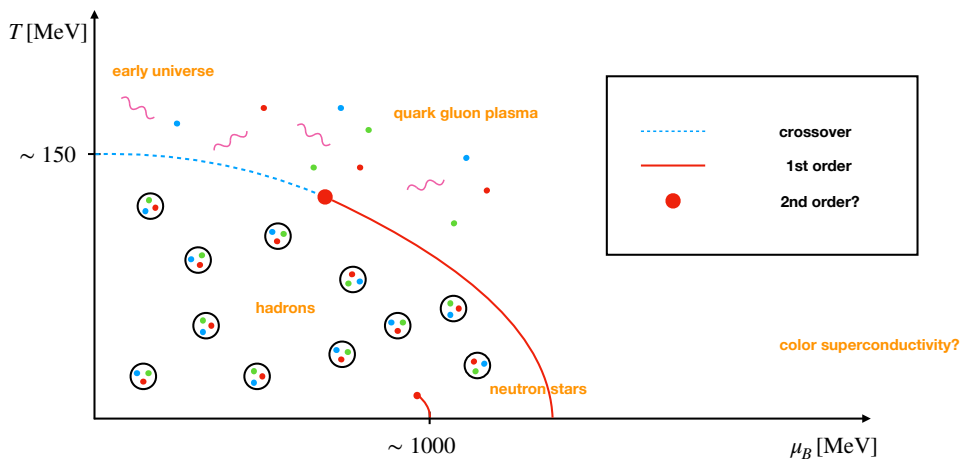


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1. The QCD phase diagram and the sign problem
2. The complex Langevin method
3. Numerical results
4. Conclusion and plans

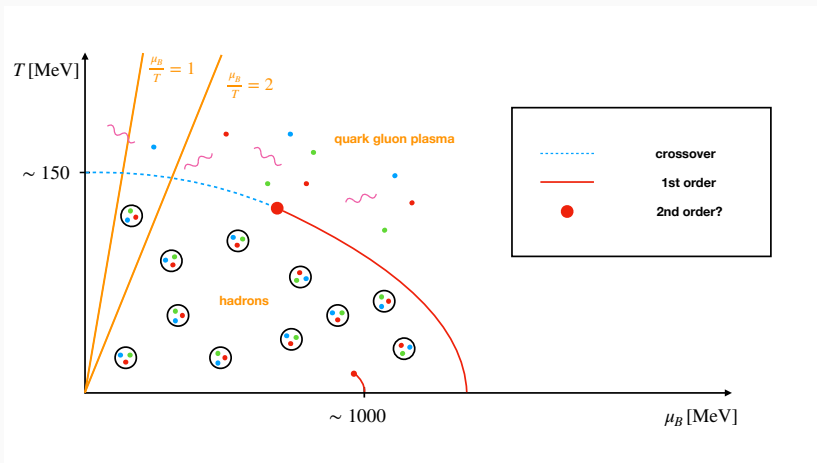
The QCD phase diagram and the sign problem

Sketching the QCD phase diagram

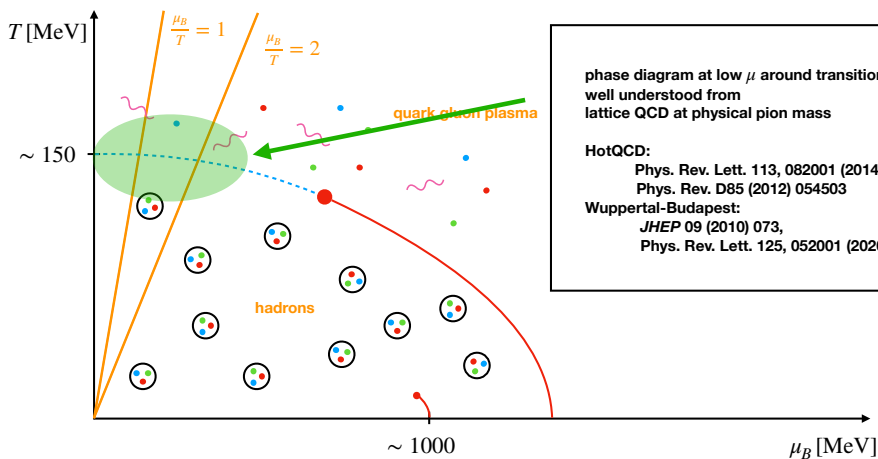


Mapping out the QCD phase diagram

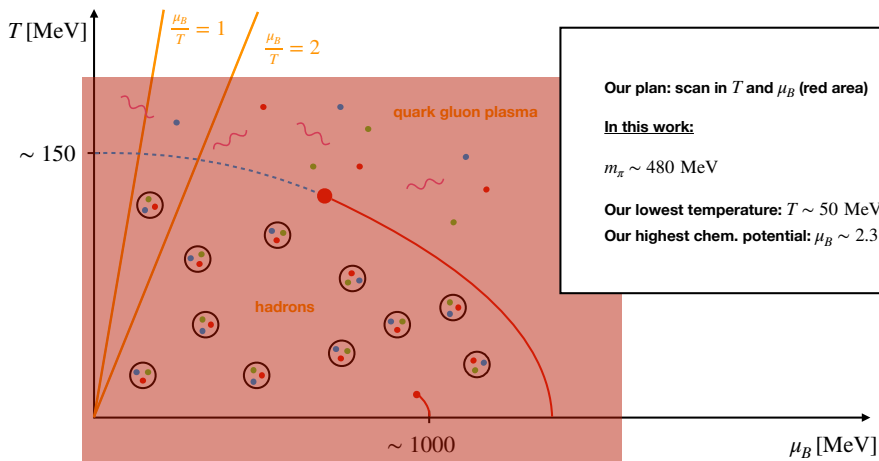
- experiment: heavy-ion collisions
LHC, RHIC, FAIR
- theory: **lattice QCD** or
functional methods
- finite μ_B restricts conventional
lattice Monte Carlo simulations
substantially → **SIGN
PROBLEM**



Status of the QCD phase diagram from lattice QCD



Our plan for lower T and higher μ_B



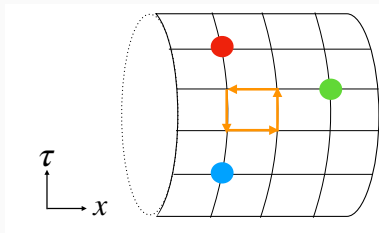
Lattice QCD at finite chemical potential

$$Z[T, \mu] = \int_{SU(3)^\Omega} \mathcal{D}U \exp(-S_{\text{eff}}[U; T, \mu]),$$

$$\mu = \mu_B/3, T = \frac{1}{aN_t}, \Omega = N_t N_s^3.$$

In this talk:

$$\Omega = 32 \times 24^3 = 1769472$$



Euclidean action

$$S_{\text{eff}}[U; T, \mu] = -N_f \log(\det M[U; T, \mu]) + \frac{\beta}{3} \sum_{x, \rho < \sigma} \text{Tr} \left[\mathbb{1} - \frac{1}{2} (U_{x, \rho\sigma} + U_{x, \rho\sigma}^{-1}) \right]$$

A closer look at the fermion matrix

$$M_{xy}[U; T, \mu] = (4+m)\delta_{xy} - \frac{1}{2} \sum_{\nu} \left[\Gamma_{\nu} e^{\mu\delta_{\nu,0}} U_{x,\nu} \delta_{x+\hat{\nu},y} + \Gamma_{-\nu} e^{-\mu\delta_{\nu,0}} U_{x-\hat{\nu},\nu}^{-1} \delta_{x-\hat{\nu},y} \right],$$

$$\Gamma_{\pm\nu} = 1 \mp \gamma_{\nu}$$

- μ renders $S_{\text{eff}}[U; T, \mu]$ complex
- no conventional Monte Carlo sampling applicable
- How bad is the sign problem?

For $U \in SU(3)$

$$M[U; T, \mu]^{\dagger} = \gamma_5 M(-\mu^*) \gamma_5$$

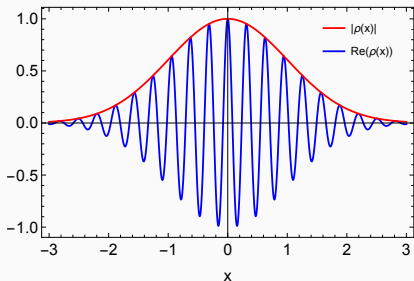
$$\det M[U; T, -\mu^*] = (\det M[U; T, \mu])^*$$

Severity of the sign problem

- In QCD cost of phase reweighting increases exponentially with the space-time volume.
- Average phase

$$\langle \sigma \rangle_{\text{PQ}} \propto \exp(-\Omega(F - F_{\text{PQ}})),$$

F (F_{PQ}) is free energy of the full (phase-quenched) theory.



Toy model for a free theory

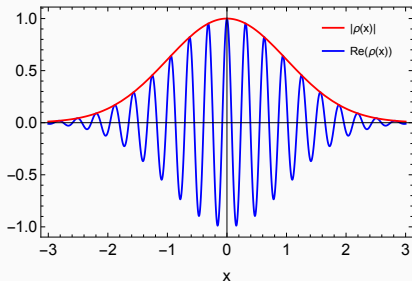
$$\rho(x, \lambda) = \exp\left(-\frac{x^2}{2} + i\lambda x\right), \quad \lambda \in \mathbb{R}$$

$$Z[\lambda] = \int_{\mathbb{R}} dx \rho(x, \lambda) = e^{-\lambda^2/2}$$

In the figure: $\lambda = 20$

Ideas to face the sign problem

In QCD cost of phase reweighting increases exponentially with the space-time volume.



reweighting, Taylor expansions, imaginary μ , dual formulations, density of states, **complex Langevin**, Lefschetz thimbles, flowed manifolds,... De Forcrand, PoS LAT2009, Alexandru et. al., Rev.Mod.Phys. 94 (2022), Attanasio, Jäger, FPGZ, Eur. Phys. J. A 56 (2020), Guenther PoS LAT2021

The complex Langevin method

The complex Langevin method in a nutshell

Stochastic Quantization

Parisi and Wu in Sci. Sin. 24 483 (1981)

- Evolve fields in fictitious time θ (Langevin or computer time)
- Langevin equation

$$\frac{\partial \phi(x, \theta)}{\partial \theta} = -\frac{\delta S}{\delta \phi(x, \theta)} + \eta(x, \theta)$$

- S is the Euclidean action (≥ 0) and η a Gaussian white noise field
- stationary solution of the associated Fokker-Planck equation is the equilibrium distribution e^{-S}
- observable expectation values

$$\langle \mathcal{O} \rangle = \lim_{\theta_{\max} \rightarrow \infty} \frac{1}{\theta_{\max} - \theta_{\text{therm}}} \int_{\theta_{\text{therm}}}^{\theta_{\max}} \mathcal{O}(\theta) d\theta.$$

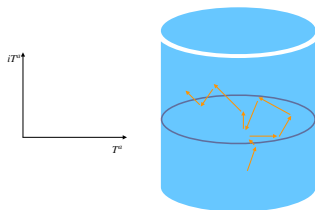
The complex Langevin method in a nutshell

- **Analytic continuation** in the field variables, Parisi, Phys. Lett. B, 131 (1983)
- extending **stochastic quantization** from $SU(3)$ to $SL(3, \mathbb{C})$.

Langevin equation

$$\frac{\partial \phi}{\partial \theta} = -\frac{\delta S}{\delta \phi} + \eta$$

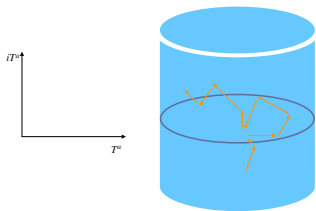
- S is a meromorphic action
- η Gaussian white noise
- θ fictitious time



The complex Langevin method in a nutshell

Langevin equation

$$\frac{\partial \phi}{\partial \theta} = -\frac{\delta S}{\delta \phi} + \eta$$



- mathematical foundations,
criteria of correctness and **boundary terms** Seiler et.al., Phys. Lett. B723 (2013),
Nishimura et.al., Phys. Rev. D 92 (2015),
Attanasio, Jäger, FPGZ, Eur. Phys. J. A 56 (2020),
Scherzer et. al., Phys. Rev. D 101 (2020)

Lattice QCD with complex Langevin

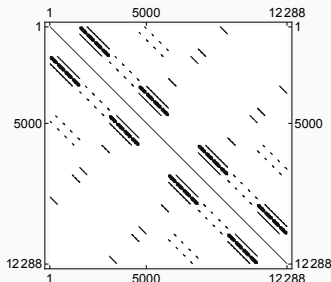
Discretized complex Langevin equation with finite step size ϵ

$$U_{x,\nu}(\theta + \epsilon) = \exp \left[i\epsilon\lambda^a \left(-D_{x,\nu}^a S + \eta_{x,\nu}^a \right) \right] U_{x,\nu}(\theta),$$

$$\langle \eta_{x,\nu}^a \eta_{y,\rho}^b \rangle = 2\delta_{x,y}\delta_{\nu,\rho}\delta_{a,b}, \quad a = 1, \dots, 8.$$

Fermionic drift term

$$-D_{x,\nu}^a S_F = N_f \text{Tr} \left[M^{-1} D_{x,\nu}^a M \right]$$



Stabilizing the complex Langevin evolution

Moving around in $SL(3, \mathbb{C})$ require to stabilize the CL process in numerical simulations \Rightarrow guarantee **correct convergence**.

Recipe:

- (1) Adaptive step size Aarts et. al., Eur. Phys. J. A 49 (2013)
- (2) **Gauge cooling**: exploit enlarged gauge freedom: move back to $SU(3)$ by minimizing the **unitarity norm** $\text{tr}[(UU^\dagger - 1)^2] \geq 0$
Seiler et. al., Phys. Lett. B 723 (2013)
- (3) **Dynamic stabilization**: keep CL trajectory close to $SU(3)$ by minimizing imaginary part of the gauge field, caveat: non-holomorphic modification of the drift term
but disappears as $a \rightarrow 0$
Attanasio and Jäger, Eur. Phys. J. C 79 (2019)

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Alternative approach: use kernels + CL,

see talk by Daniel Alvestad, 23/08/2022 at 11:00

Numerical results

Lattice setup

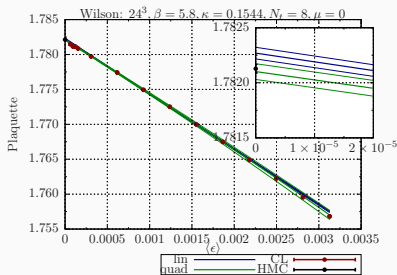
- $\beta = 5.8, \kappa = 0.144, V = 24^3, a \approx 0.06\text{fm}$
Del Debbio et. al., JHEP 02 (2006)
- $N_f = 2$ Wilson fermions ($c_{\text{SW}} = 0$)
- $m_\pi \approx 480\text{MeV}, m_N \approx 1.3\text{GeV}$
- temperature range: $N_t \in \{64, \dots, 4\} \leftrightarrow T \in \{50, \dots, 850\}\text{MeV}$
- We have data for a quark chemical potential range:
 $\mu \in \{0, \dots, 6500\}\text{MeV}$
- For EOS results focus on $\mu_B \in [0, 1.8m_N]$ and $T \in [50, 200]\text{MeV}$
- fermion matrix inversion: eoCG

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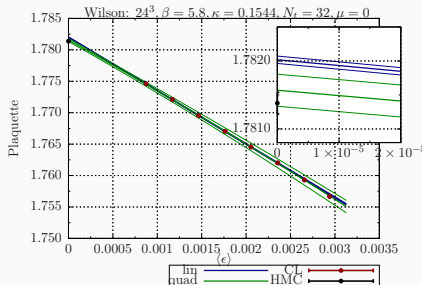
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Sanity check

Extrapolation of CL results and comparison with HMC (black data point)



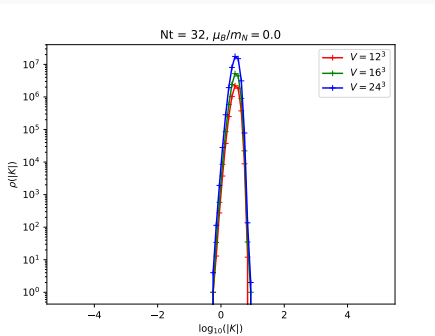
Deconfined phase



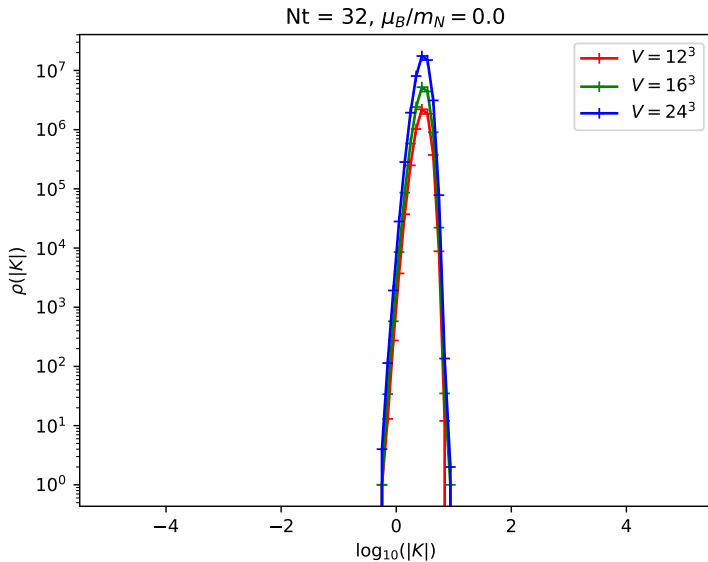
Confined phase

Checking the criteria of correctness

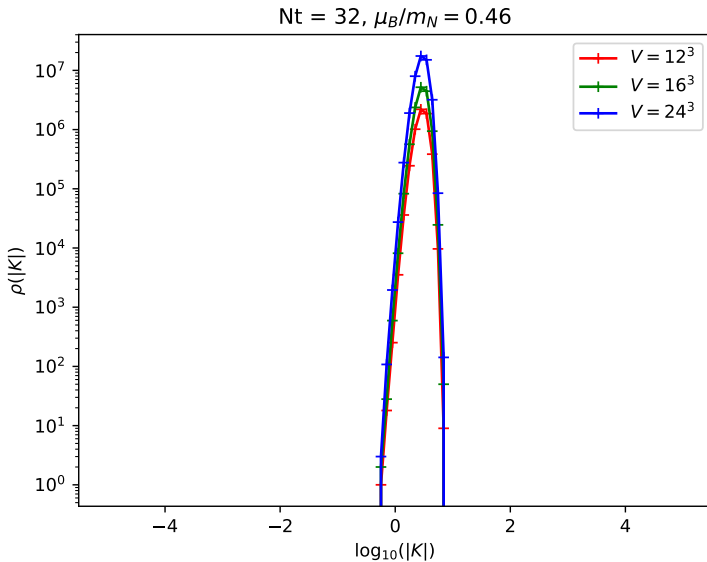
- exponentially decaying histogram of the drift K indicates that CL method works correctly
Nishimura et.al., Phys. Rev. D 92 (2015)
- We find well localized distributions, **also at large** μ_B .



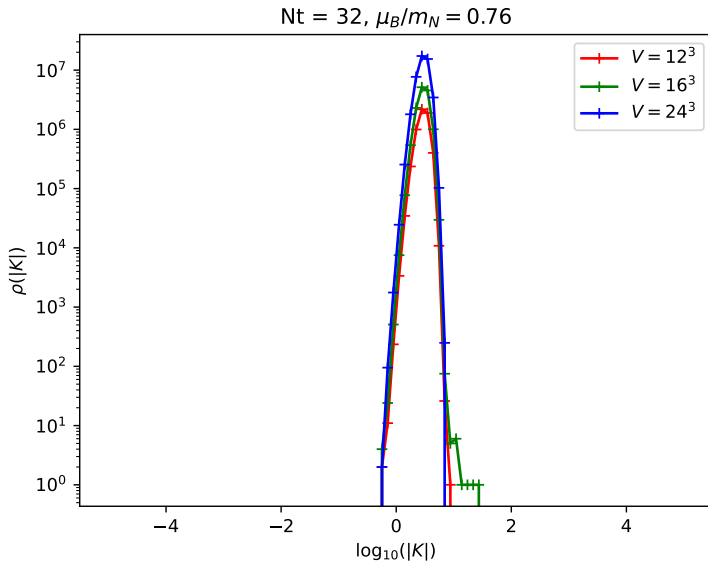
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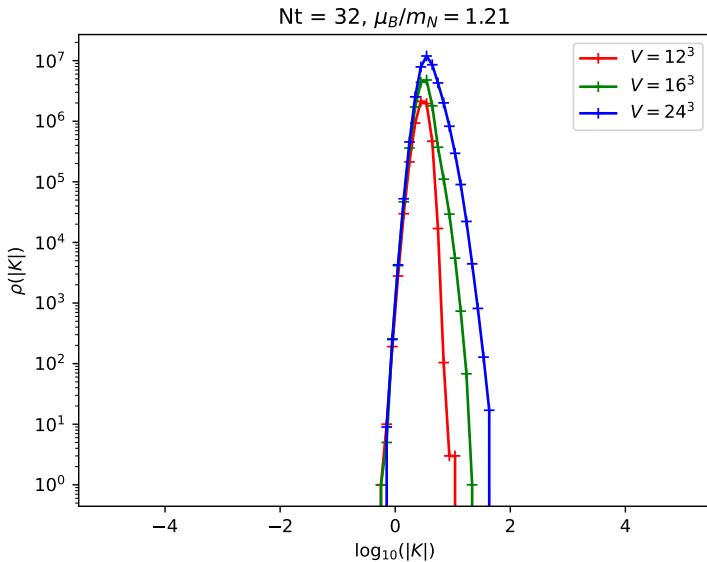
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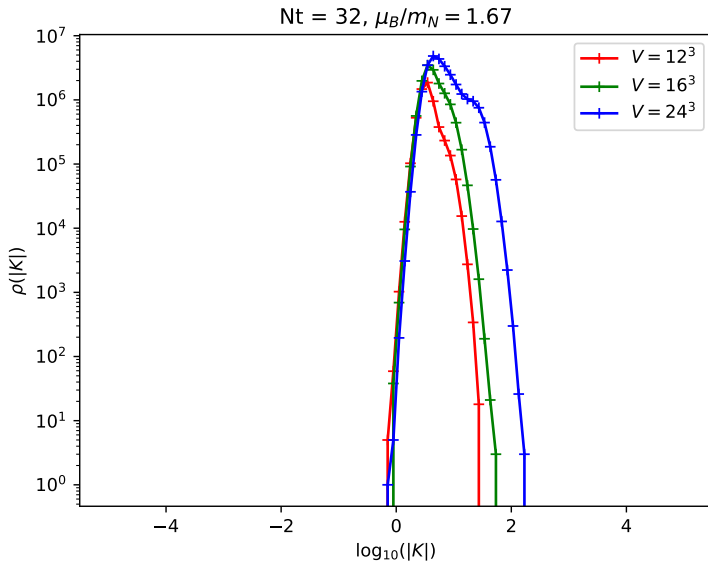
Checking the criteria of correctness



Checking the criteria of correctness



Checking the criteria of correctness



Observables for the phase diagram

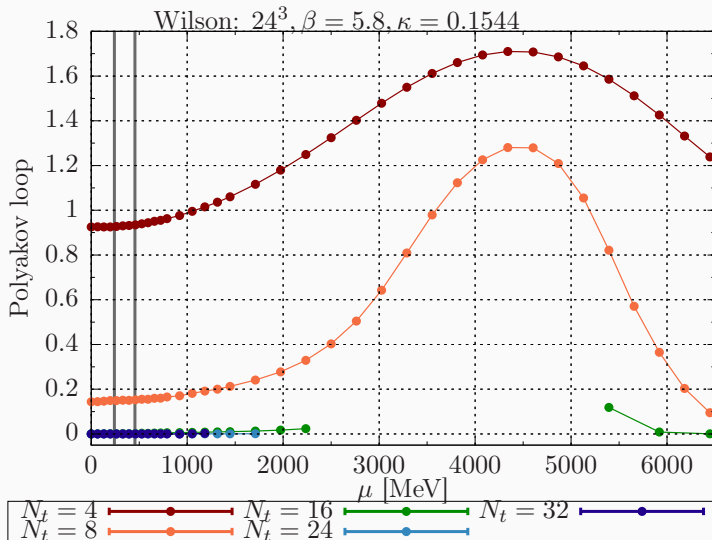
Polyakov loop (confinement - deconfinement transition)

$$P = \frac{1}{3V} \sum_{\vec{x}} \text{Tr} \left\langle \prod_{\tau} U_{(\vec{x}, \tau), \hat{0}} \right\rangle$$

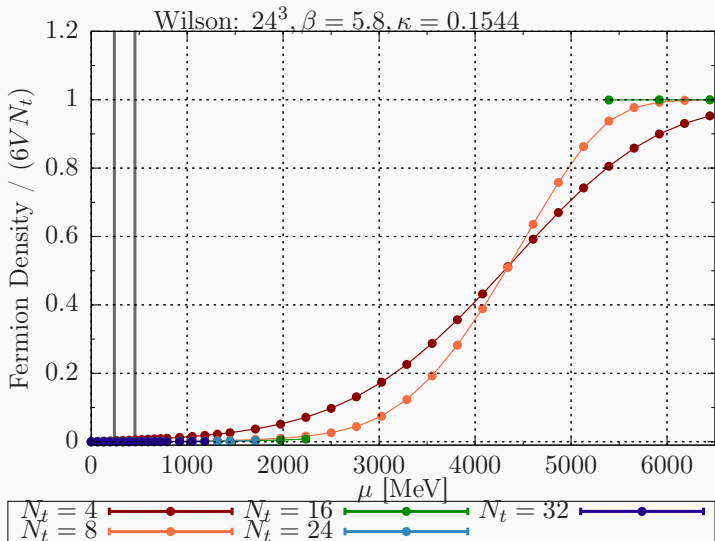
Quark density

$$\langle n \rangle = \frac{1}{\Omega} \frac{\partial \log Z}{\partial \mu}$$

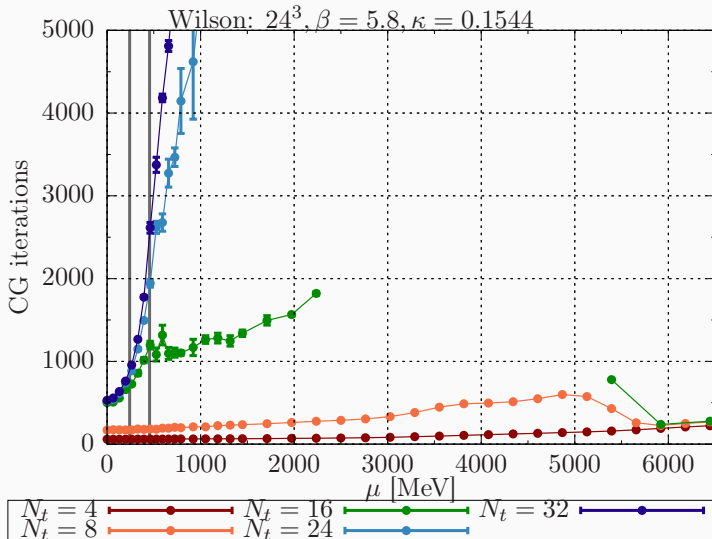
Phase diagram on large scales



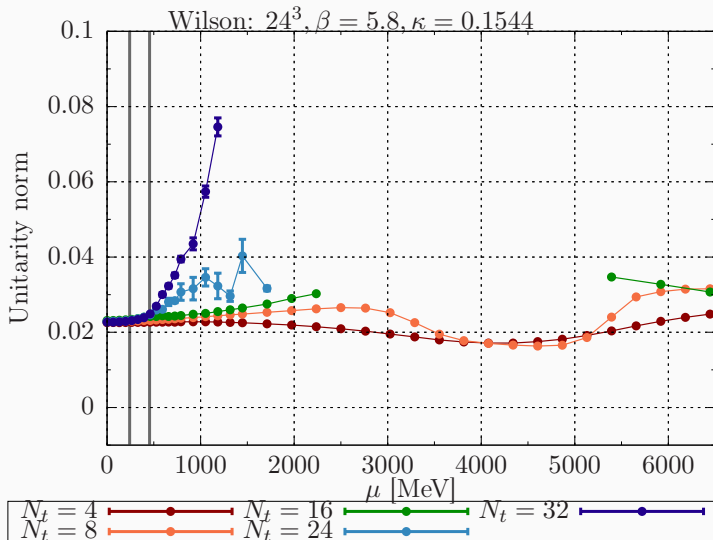
Phase diagram on large scales



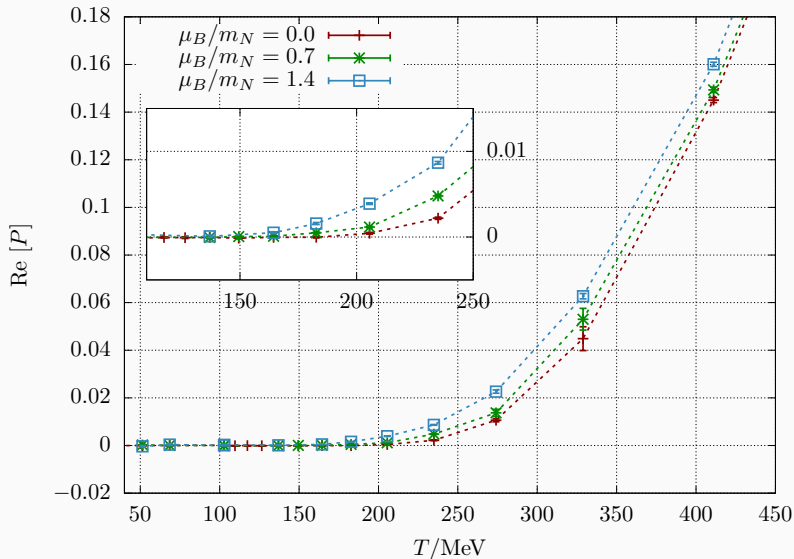
CG iterations



Unitarity norm



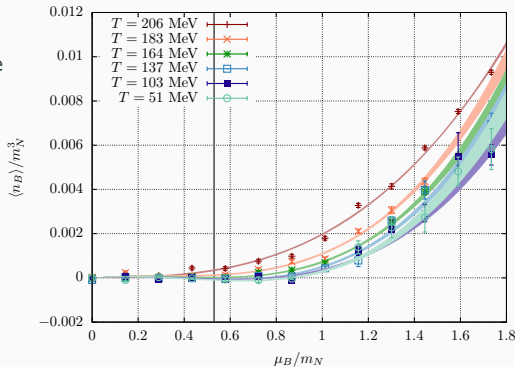
Confinement - deconfinement transition



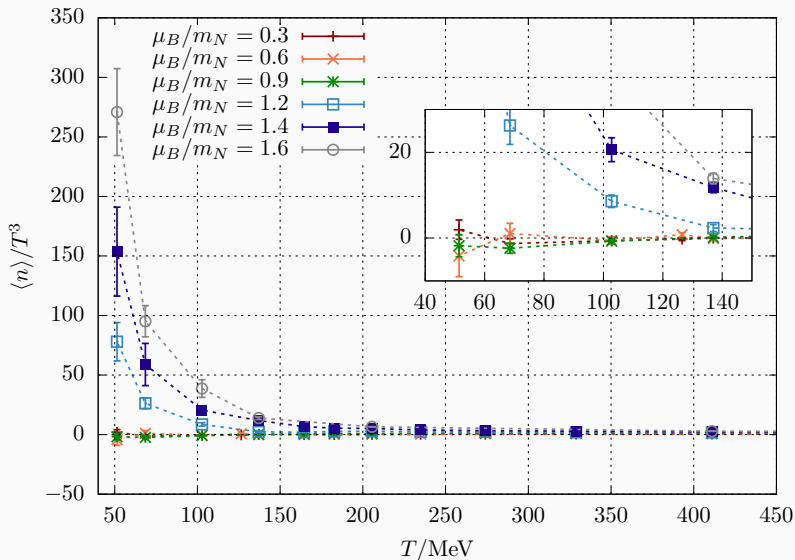
Phase structure at hadronic scales - baryon density

Silver Blaze phenomenon, T. Cohen,
Phys. Rev. Lett. 91 (2003)

- at $T = 0$ expect no μ dependence in thermodynamic observables for $0 \leq \mu \leq m_N/3$
- grey line indicates $\mu = m_\pi/2$
- Phase-quenched and full theory very different in range $m_\pi/2 < \mu < m_N/3$, severe cancelations of the integrand.

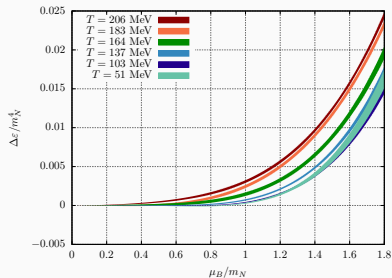
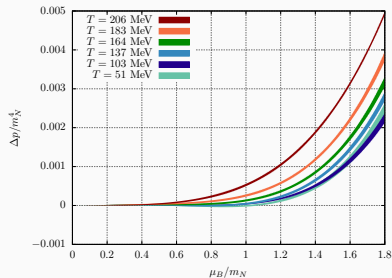


Phase structure at hadronic scales - baryon density



Silver Blaze phenomenon, see T. Cohen, Phys. Rev. Lett. 91 (2003)

Equation of State



Pressure equation of state $\Delta p(\mu_B, T)$ (left) and energy density (right),

$$\Delta p(\mu_B, T) = \int_0^{\mu_B} d\mu' \langle n(\mu', T) \rangle$$

Conclusion and plans

Result

- first step towards low temperatures and physical pion mass:
indications of the Silver Blaze phenomenon found.
- predictions for the QCD equation of state at densities $n \sim 15n_0$ where n_0 is the nuclear density
- Found that EoS gets stiffer as T decreases.

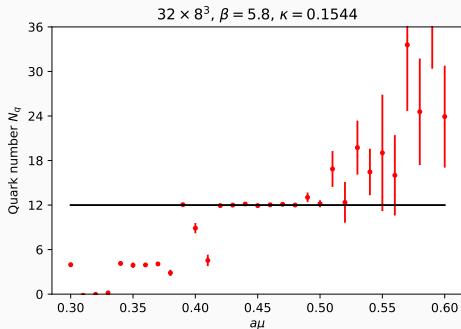
Future plans

- refined mapping of the confinement and chiral transitions,
finite size scaling \rightarrow critical endpoint (?)
- finer lattices and improved actions (Wilson clover)
- including the strange quark (2+1 flavour simulations)
- **better solvers for the fermionic inversion**
- improvements of systematics related to the CL method (step size extrapolation), criteria of correctness, boundary terms, ...

Thank you very much!

Quarks in a small box

- $V = 8^3$, $T \approx 100\text{MeV}$
- on the lattice expect plateau behaviour of the quark number (H. Matsuoka and M. Stone, Phys. Lett. 136B (1984))
- quantitative agreement between our dynamically stabilized simulations and such based on gauge cooling only, see PhD thesis by M. Scherzer, Heidelberg, 2019



PRELIMINARY