Improved lattice method for determining entanglement measures in SU(N) gauge theories

Tobias Rindlisbacher[1], Niko Jokela[2], Arttu Pönni[3], Kari Rummukainen[2] & Ahmed Salami[2]

[1] University of Bern, AEC & Institute for Theoretical Physics, Bern, Switzerland
[2] University of Helsinki, Department of Physics & Helsinki Institute of Physics, Helsinki, Finland
[3] Aalto University, Department of Electronics and Nanoengineering, Espoo, Finland

UNIVERSITÄT BERN
AEC
ALBERT EINSTEIN CENTER FOR FUNDAMENTAL PHYSICS

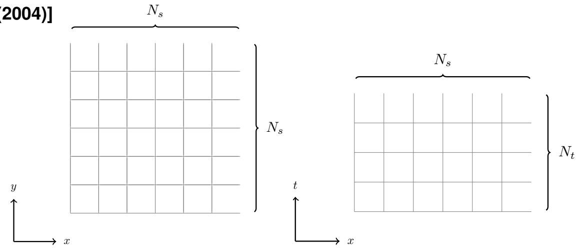
NOLA Summer 2022, August 22-25, 2022

HELSINGIN YLIOPISTO
HELSINGFORS UNIVERSITET
UNIVERSITY OF HELSINKI

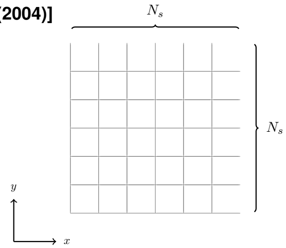
Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

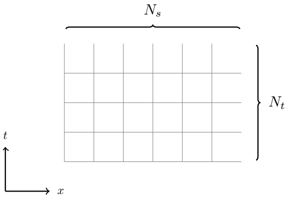
- SU(N) gauge theory on $N_s^{d-1} \times N_t$ lattice

  Partition function: $Z(N_t, N_s) = \int \mathcal{D}[U]\ \mathrm{e}^{-S_G[U]}$



Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

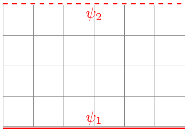




- SU(N) gauge theory on $N_s^{d-1} \times N_t$ lattice

  Partition function: $Z(N_t, N_s) = \int \mathcal{D}[U] \; e^{-S_G[U]}$

  ➔ Density matrix element:

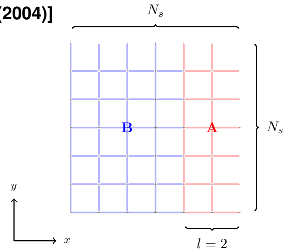
$\langle\psi_1|\rho|\psi_2\rangle =$ 

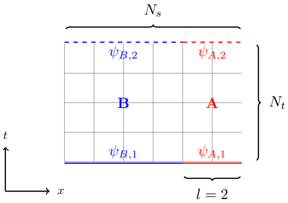
Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

- SU(N) gauge theory on $N_s^{d-1} \times N_t$ lattice

  Partition function: $Z(N_t, N_s) = \int \mathcal{D}[U]\; \mathrm{e}^{-S_G[U]}$

  ➔ Divide lattice into two parts (A, B)





Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

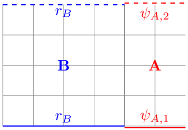
- $SU(N)$ gauge theory on $N_s^{d-1} \times N_t$ lattice

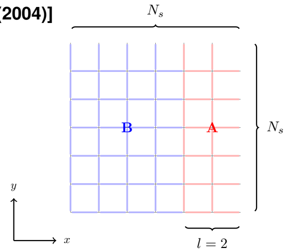
  Partition function: $Z(N_t, N_s) = \int \mathcal{D}[U]\, e^{-S_G[U]}$

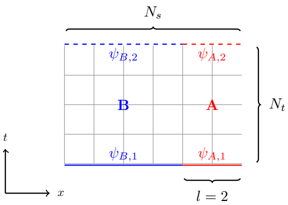
  ➔ Divide lattice into two parts (A, B)

  ➔ Reduced density matrix ρ_A for part A

$$\langle \psi_{A,1} | \rho_A | \psi_{A,2} \rangle =$$







Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

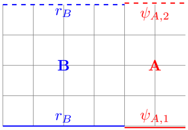
- SU(N) gauge theory on $N_s^{d-1} \times N_t$ lattice

  Partition function: $Z(N_t, N_s) = \int \mathcal{D}[U]\, \mathrm{e}^{-S_G[U]}$

  ➔ Divide lattice into two parts (A, B)

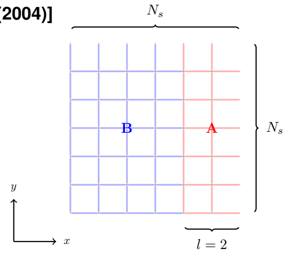
  ➔ Reduced density matrix ρ_A for part A

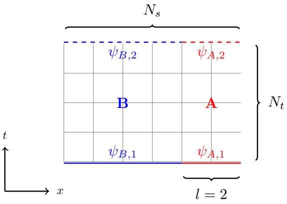
$$\langle \psi_{A,1} | \rho_A | \psi_{A,2} \rangle =$$



  ➔ Entanglement entropy:

$S_{EE} = -\,\mathrm{tr}_A(\rho_A \log \rho_A)$ (how ?)





Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

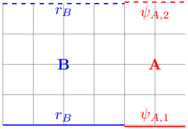
- SU(N) gauge theory on $N_s^{d-1} \times N_t$ lattice

  Partition function: $Z(N_t, N_s) = \int \mathcal{D}[U]\, e^{-S_G[U]}$

  ➔ Divide lattice into two parts (A, B)

  ➔ Reduced density matrix ρ_A for part A

$$\langle \psi_{A,1} | \rho_A | \psi_{A,2} \rangle =$$



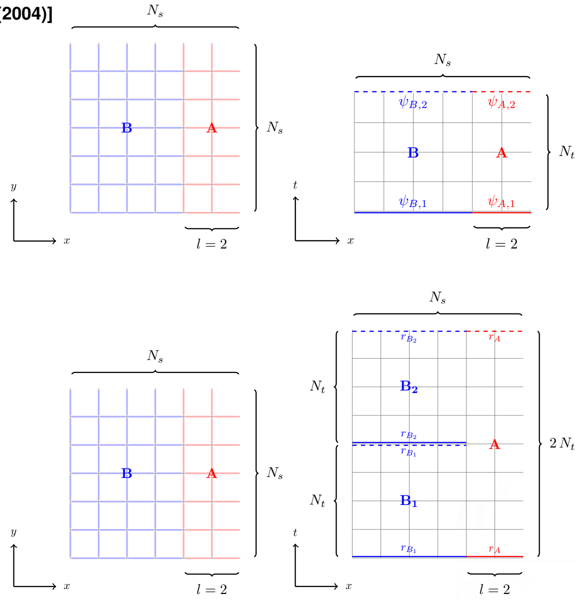
  ➔ Replica method for s-th Rényi entropy:

$$H_s(l, N_t, N_s) = \frac{1}{1-s} \log \mathrm{tr}(\rho_A^s) = \frac{Z_c(l, s, N_t, N_s)}{Z^s(N_t, N_s)}$$

with "cut partition function" $Z_c(l, s, N_t, N_s)$

$\rightarrow \quad Z_c(l = 0, s, N_t, N_s) = Z^s(N_t, N_s) \quad \forall s \in \mathbb{N}$

$\rightarrow \quad Z_c(l = N_s, s, N_t, N_s) = Z(s\, N_t, N_s) \quad \forall s \in \mathbb{N}$



Introduction

Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

➔ Entanglement entropy (EE):

$$S_{EE}(l, N_t, N_s) = -\lim_{s \to 1} \frac{\partial \log \mathrm{tr}\left(\rho_A^s\right)}{\partial s}$$

Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

➔ Entanglement entropy (EE):

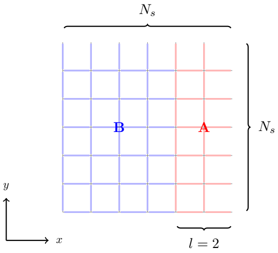
$$
\begin{aligned}
S_{EE}(l, N_t, N_s) &= -\lim_{s\to 1} \frac{\partial \log \operatorname{tr}\left(\rho_A^s\right)}{\partial s} \\
&= -\left(\lim_{s\to 1} \frac{\partial \log Z_c(l, s, N_t, N_s)}{\partial s} - \log Z(N_t, N_s)\right) \\
&\approx \; -\log Z_c(l, 2, N_t, N_s) \; - \; (-2 \, \log Z(N_t, N_s)) \\
&= -\log \operatorname{tr}\left(\rho_A^2\right) = H_2(l, N_t, N_s)
\end{aligned}
$$

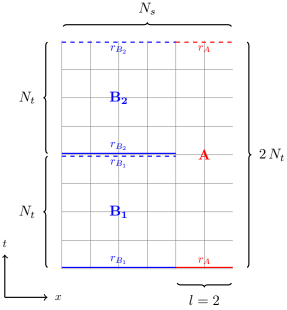
Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

➔ Entanglement entropy (EE):

$$
\begin{aligned}
S_{EE}(l, N_t, N_s) &= -\lim_{s\to 1} \frac{\partial \log \mathrm{tr}\left(\rho_A^s\right)}{\partial s} \\
&= -\left(\lim_{s\to 1} \frac{\partial \log Z_c(l, s, N_t, N_s)}{\partial s} - \log Z(N_t, N_s)\right) \\
&\approx \boxed{-\log Z_c(l, 2, N_t, N_s)} - \left(-2 \log Z(N_t, N_s)\right) \\
&= -\log \mathrm{tr}\left(\rho_A^2\right) = H_2(l, N_t, N_s)
\end{aligned}
$$

➔ measure free energy difference





Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

➔ Entanglement entropy (EE):

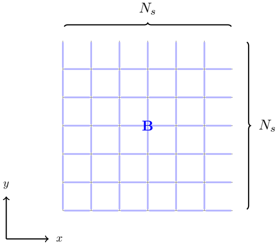
$$S_{EE}(l, N_t, N_s) = -\lim_{s\to 1} \frac{\partial \log \mathrm{tr}\left(\rho_A^s\right)}{\partial s}$$

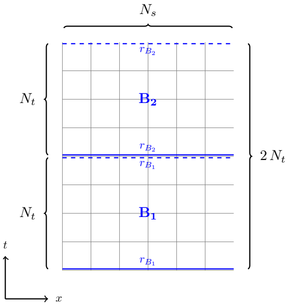
$$= -\left(\lim_{s\to 1} \frac{\partial \log Z_c(l, s, N_t, N_s)}{\partial s} - \log Z(N_t, N_s)\right)$$

$$\approx \; -\log Z_c(l, 2, N_t, N_s) \; - \; \boxed{(-2 \log Z(N_t, N_s))}$$

$$= -\log \mathrm{tr}\left(\rho_A^2\right) = H_2(l, N_t, N_s)$$

➔ measure free energy difference





Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

➔ Entanglement entropy (EE):

$$S_{EE}(l, N_t, N_s) = -\lim_{s\to 1} \frac{\partial \log \mathrm{tr}\left(\rho_A^s\right)}{\partial s}$$

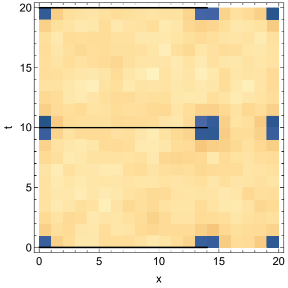
$$= -\left(\lim_{s\to 1} \frac{\partial \log Z_c(l, s, N_t, N_s)}{\partial s} - \log Z(N_t, N_s)\right)$$

$$\approx \; -\log Z_c(l, 2, N_t, N_s) \; - \; (-2 \log Z(N_t, N_s))$$

$$= -\log \mathrm{tr}\left(\rho_A^2\right) = H_2(l, N_t, N_s)$$

➔ measure free energy difference

Issue: UV-divergent piece $\frac{S_{EE}}{|\partial A|} = \boxed{\frac{C_0}{\epsilon^2}} - \frac{C}{l^q} + \text{(finite)}$



Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

➔ Entanglement entropy (EE):

$$S_{EE}(l, N_t, N_s) = -\lim_{s\to 1} \frac{\partial \log \mathrm{tr}\left(\rho_A^s\right)}{\partial s}$$

$$= -\left(\lim_{s\to 1} \frac{\partial \log Z_c(l, s, N_t, N_s)}{\partial s} - \log Z(N_t, N_s)\right)$$

$$\approx \; -\log Z_c(l, 2, N_t, N_s) \; - \; (-2 \, \log Z(N_t, N_s))$$

$$= -\log \mathrm{tr}\left(\rho_A^2\right) = H_2(l, N_t, N_s)$$

➔ measure free energy difference

Issue: UV-divergent piece $\frac{S_{EE}}{|\partial A|} = \frac{C_0}{\epsilon^2} - \frac{C}{l^q} + \text{(finite)}$

➔ Instead of EE, measure discrete derivative w.r.t. $l > 0$:

$$\left.\frac{\partial S_{EE}(l', N_t, N_s)}{\partial l'}\right|_{l'=l+1/2} \approx$$

$$-\log Z_c(l+1, 2, N_t, N_s) \; - \; (-\log Z_c(l, 2, N_t, N_s))$$

Introduction

Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

➔ Entanglement entropy (EE):

$$S_{EE}(l, N_t, N_s) = -\lim_{s\to 1} \frac{\partial \log \mathrm{tr}\left(\rho_A^s\right)}{\partial s}$$

$$= -\left(\lim_{s\to 1} \frac{\partial \log Z_c(l, s, N_t, N_s)}{\partial s} - \log Z(N_t, N_s)\right)$$

$$\approx \; -\log Z_c(l, 2, N_t, N_s) \; - \; (-2 \log Z(N_t, N_s))$$

$$= -\log \mathrm{tr}\left(\rho_A^2\right) = H_2(l, N_t, N_s)$$

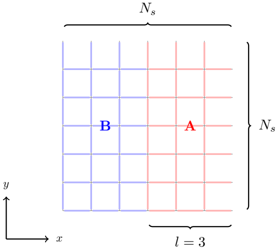
➔ measure free energy difference

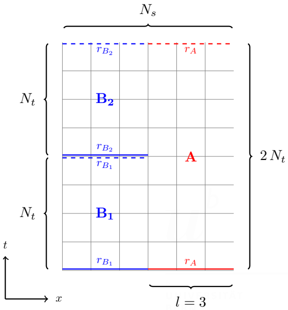
Issue: UV-divergent piece $\frac{S_{EE}}{|\partial A|} = \frac{C_0}{\epsilon^2} - \frac{C}{l^q} + \text{(finite)}$

➔ Instead of EE, measure discrete derivative w.r.t. $l > 0$:

$$\left.\frac{\partial S_{EE}(l', N_t, N_s)}{\partial l'}\right|_{l'=l+1/2} \approx$$

$$-\log Z_c(l+1, 2, N_t, N_s) \; - \; (-\log Z_c(l, 2, N_t, N_s))$$





Introduction

Entanglement entropy on the lattice **[P. Calabrese, J. Cardy (2004)]**

➔ Entanglement entropy (EE):

$$S_{EE}(l, N_t, N_s) = -\lim_{s\to 1} \frac{\partial \log \mathrm{tr}\left(\rho_A^s\right)}{\partial s}$$

$$= -\left(\lim_{s\to 1} \frac{\partial \log Z_c(l, s, N_t, N_s)}{\partial s} - \log Z(N_t, N_s)\right)$$

$$\approx \; -\log Z_c(l, 2, N_t, N_s) \; - \; (-2 \log Z(N_t, N_s))$$

$$= -\log \mathrm{tr}\left(\rho_A^2\right) = H_2(l, N_t, N_s)$$

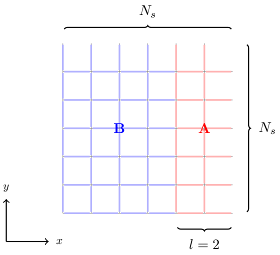
➔ measure free energy difference

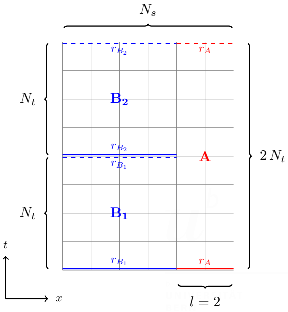
> Issue: UV-divergent piece $\frac{S_{EE}}{|\partial A|} = \frac{C_0}{\epsilon^2} - \frac{C}{l^q} + \text{(finite)}$

➔ Instead of EE, measure discrete derivative w.r.t. $l > 0$:

$$\left.\frac{\partial S_{EE}(l', N_t, N_s)}{\partial l'}\right|_{l'=l+1/2} \approx$$

$$-\log Z_c(l+1, 2, N_t, N_s) \; - \; \boxed{(-\log Z_c(l, 2, N_t, N_s))}$$





Entanglement entropy on the lattice

- Measuring free energy differences:

➔ $l \to l+1$ is non-local change $\to$ overlap problem

Entanglement entropy on the lattice

- Measuring free energy differences:
  - ➔ $l \to l+1$ is non-local change $\to$ overlap problem
- Approach from literature: [P. V. Buividovich, M. I. Polikarpov (2008)],[Y. Nakagawa et al. (2009)]
  - ➔ interpolating partition function:

$$Z_l^*(\alpha) = \int \mathcal{D}[U] \exp\Big(-(1-\alpha)\ S_l[\,U\,]\ -\alpha\ S_{l+1}[\,U\,]\ \Big), \text{ with } \alpha \in [\,0,1\,]$$

Entanglement entropy on the lattice

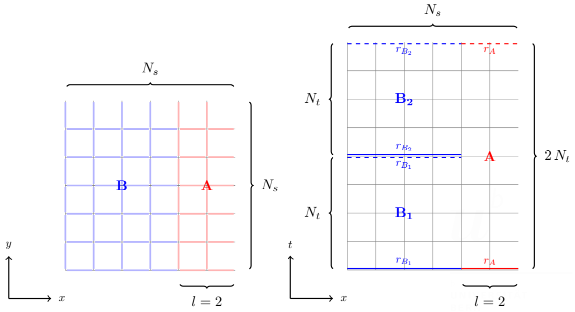
- Measuring free energy differences:

  ➔ $l \to l+1$ is non-local change $\to$ overlap problem

- Approach from literature: [P. V. Buividovich, M. I. Polikarpov (2008)],[Y. Nakagawa et al. (2009)]

  ➔ interpolating partition function:

$$Z_l^*(\alpha) = \int \mathcal{D}[U] \exp\Big(-(1-\alpha)\, S_l[U] - \alpha\, S_{l+1}[U]\Big), \text{ with } \alpha \in [0,1]$$



Entanglement entropy on the lattice

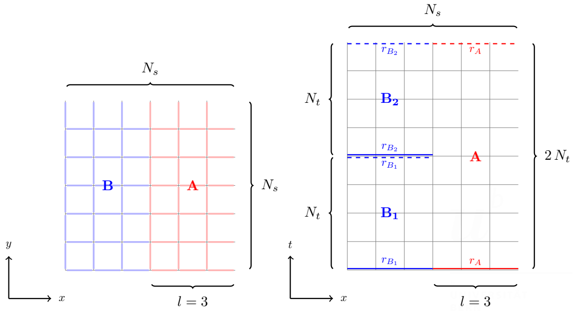
- Measuring free energy differences:

  ➔ $l \to l + 1$ is non-local change $\to$ overlap problem

- Approach from literature: [P. V. Buividovich, M. I. Polikarpov (2008)],[Y. Nakagawa et al. (2009)]

  ➔ interpolating partition function:

$$Z_l^*(\alpha) = \int \mathcal{D}[U] \, \exp\Big(-(1-\alpha) \; S_l[\,U\,] \; - \alpha \; \boxed{S_{l+1}[\,U\,]}\Big), \text{ with } \alpha \in [\,0, 1\,]$$



Entanglement entropy on the lattice

- Measuring free energy differences:
  - ➔ $l \to l+1$ is non-local change $\to$ overlap problem
- Approach from literature: [P. V. Buividovich, M. I. Polikarpov (2008)],[Y. Nakagawa et al. (2009)]
  - ➔ interpolating partition function:

$$Z_l^*(\alpha) = \int \mathcal{D}[U] \, \exp\Big(-(1-\alpha)\; S_l[\,U\,] \; - \alpha \; S_{l+1}[\,U\,]\;\Big), \text{ with } \alpha \in [\,0,1\,]$$

  - ➔ measure $\langle S_{l+1} - S_l \rangle_\alpha = -\frac{\partial \log Z_l^*(\alpha)}{\partial \alpha}$ for $\alpha \in [\,0,1\,]$

Entanglement entropy on the lattice

- Measuring free energy differences:
  - ➔ $l \to l + 1$ is non-local change $\to$ overlap problem
- Approach from literature: [P. V. Buividovich, M. I. Polikarpov (2008)],[Y. Nakagawa et al. (2009)]
  - ➔ interpolating partition function:

$$Z_l^*(\alpha) = \int \mathcal{D}[U] \exp\Big(-(1-\alpha)\ S_l[U]\ - \alpha\ S_{l+1}[U]\ \Big), \text{ with } \alpha \in [0,1]$$

  - ➔ measure $\langle S_{l+1} - S_l \rangle_\alpha = -\frac{\partial \log Z_l^*(\alpha)}{\partial \alpha}$ for $\alpha \in [0,1]$
  - ➔ interpolate and integrate:

$$\left.\frac{\partial S_{EE}(l', N_t, N_s)}{\partial l'}\right|_{l'=l+1/2} \approx -\int_0^1 \mathrm{d}\alpha \frac{\partial \log Z_l^*(\alpha)}{\partial \alpha} = \int_0^1 \mathrm{d}\alpha \langle S_{l+1} - S_l \rangle_\alpha$$

Entanglement entropy on the lattice

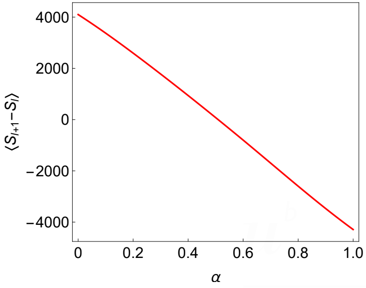
- Measuring free energy differences:
  - ➔ $l \to l+1$ is non-local change $\to$ overlap problem
- Approach from literature: [P. V. Buividovich, M. I. Polikarpov (2008)],[Y. Nakagawa et al. (2009)]
  - ➔ interpolating partition function:

$$Z_l^*(\alpha) = \int \mathcal{D}[U] \exp\Big(-(1-\alpha)\ S_l[U]\ - \alpha\ S_{l+1}[U]\ \Big), \text{ with } \alpha \in [0,1]$$

  - ➔ measure $\langle S_{l+1} - S_l \rangle_\alpha = -\dfrac{\partial \log Z_l^*(\alpha)}{\partial \alpha}$ for $\alpha \in [0,1]$
  - ➔ interpolate and integrate:

$$\left.\frac{\partial S_{EE}(l', N_t, N_s)}{\partial l'}\right|_{l'=l+1/2} \approx -\int_0^1 \mathrm{d}\alpha \frac{\partial \log Z_l^*(\alpha)}{\partial \alpha} = \int_0^1 \mathrm{d}\alpha \langle S_{l+1} - S_l \rangle_\alpha$$

Issue: huge free energy barrier $\to$ bad signal to noise ratio



data from [Y. Nakagawa et al. (2009)]

Introduction

Entanglement entropy on the lattice

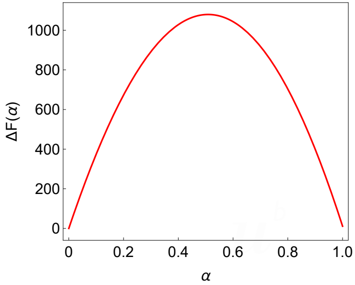
- Measuring free energy differences:
  - ➔ $l \to l+1$ is non-local change $\to$ overlap problem
- Approach from literature: [P. V. Buividovich, M. I. Polikarpov (2008)],[Y. Nakagawa et al. (2009)]
  - ➔ interpolating partition function:

$$Z_l^*(\alpha) = \int \mathcal{D}[U]\, \exp\Big(-(1-\alpha)\ S_l[\,U\,]\ -\alpha\ S_{l+1}[\,U\,]\ \Big), \text{ with } \alpha \in [\,0,1\,]$$

  - ➔ measure $\langle S_{l+1} - S_l \rangle_\alpha = -\dfrac{\partial \log Z_l^*(\alpha)}{\partial \alpha}$ for $\alpha \in [\,0,1\,]$
  - ➔ interpolate and integrate:

$$\left.\frac{\partial S_{EE}(l', N_t, N_s)}{\partial l'}\right|_{l'=l+1/2} \approx -\int_0^1 \mathrm{d}\alpha \frac{\partial \log Z_l^*(\alpha)}{\partial \alpha} = \int_0^1 \mathrm{d}\alpha \langle S_{l+1} - S_l \rangle_\alpha$$

Issue: huge free energy barrier $\to$ bad signal to noise ratio



data from [Y. Nakagawa et al. (2009)]

Entanglement entropy on the lattice

- Measuring free energy differences:

  ➔ $l \to l+1$ is non-local change $\to$ overlap problem

- Approach from literature: [P. V. Buividovich, M. I. Polikarpov (2008)],[Y. Nakagawa et al. (2009)]

  ➔ interpolating partition function:

$$Z_l^*(\alpha) = \int \mathcal{D}[U] \exp\Big(-(1-\alpha)\; S_l[\,U\,] \;-\alpha\; S_{l+1}[\,U\,]\;\Big), \text{ with } \alpha \in [\,0,1\,]$$

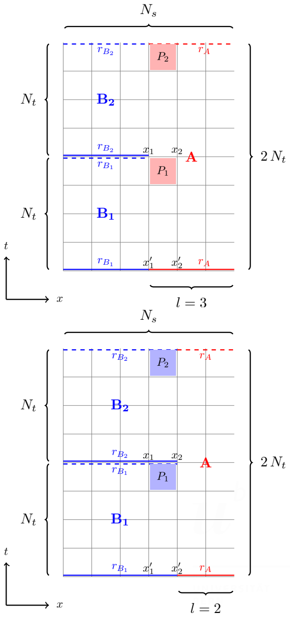
  ➔ measure $\quad \langle S_{l+1} - S_l \rangle_\alpha = -\dfrac{\partial \log Z_l^*(\alpha)}{\partial \alpha} \quad$ for $\alpha \in [\,0,1\,]$

  ➔ interpolate and integrate:

$$\left.\frac{\partial S_{EE}(l', N_t, N_s)}{\partial l'}\right|_{l'=l+1/2} \approx -\int_0^1 \mathrm{d}\alpha \frac{\partial \log Z_l^*(\alpha)}{\partial \alpha} = \int_0^1 \mathrm{d}\alpha \langle S_{l+1} - S_l \rangle_\alpha$$

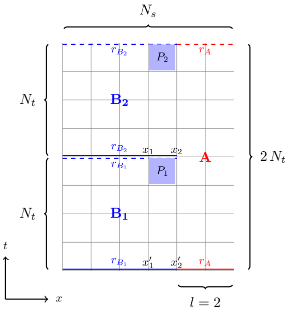
Issue: huge free energy barrier $\to$ bad signal to noise ratio

  ➔ $Z_l^*(\alpha)$ imposes simultaneously BC_A and BC_B on plaquettes P_1, P_2 if $\alpha \neq 0, 1$.

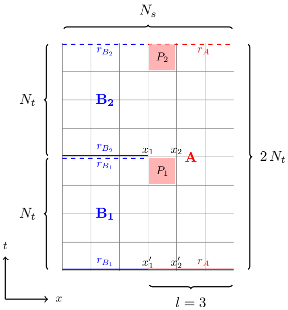


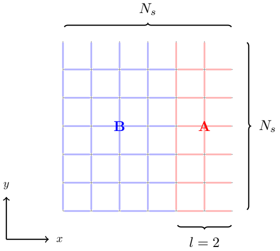
Entangling surface deformation method

How can we avoid (huge) free energy barriers?

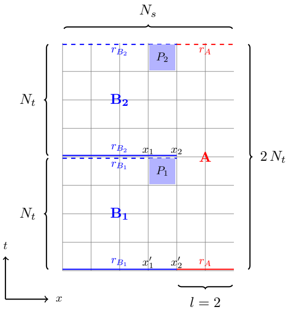


- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.

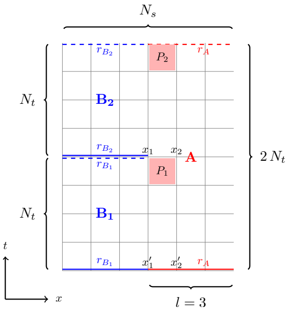


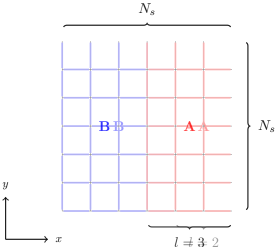


How can we avoid (huge) free energy barriers?



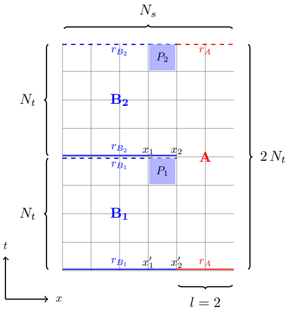
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.



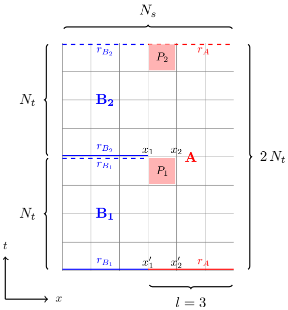


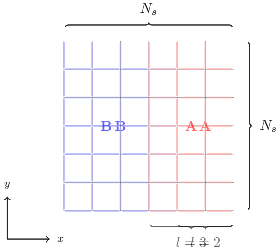
Entangling surface deformation method

How can we avoid (huge) free energy barriers?



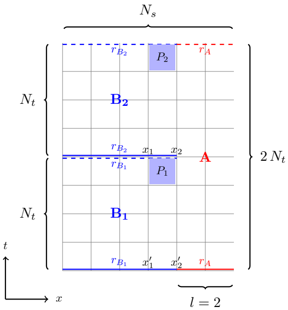
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.



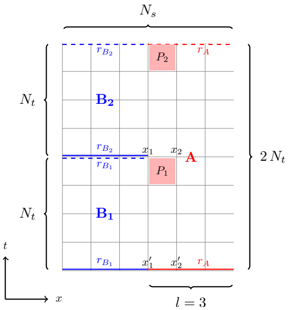


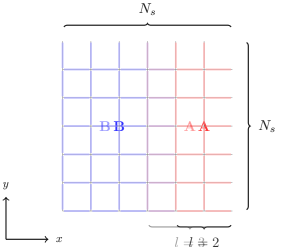
Entangling surface deformation method

How can we avoid (huge) free energy barriers?



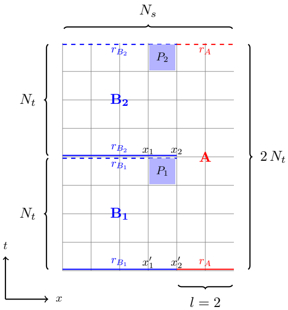
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.



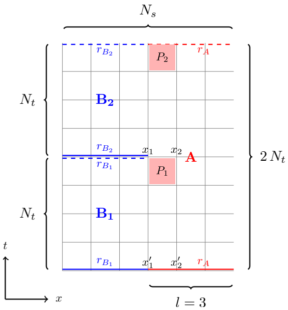


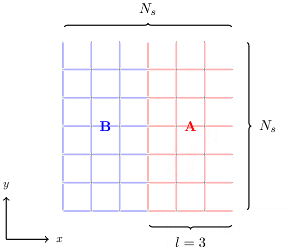
Entangling surface deformation method

How can we avoid (huge) free energy barriers?

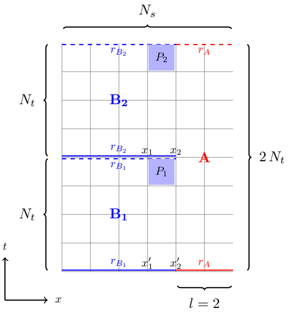


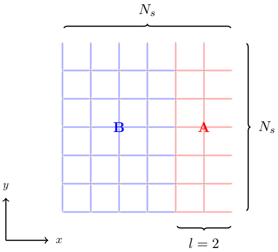
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.



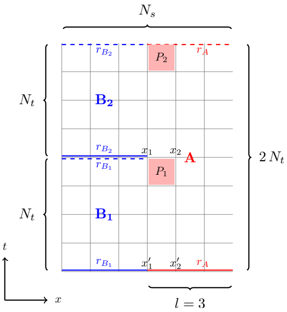


How can we avoid (huge) free energy barriers?



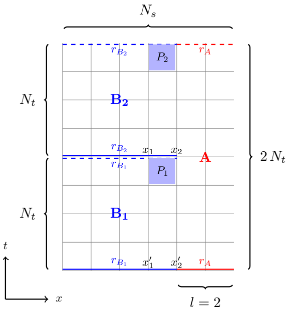


- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.

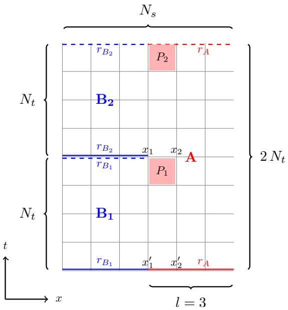


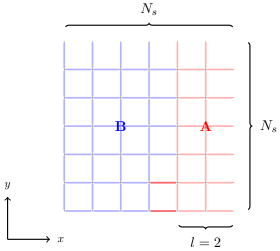
Entangling surface deformation method

How can we avoid (huge) free energy barriers?

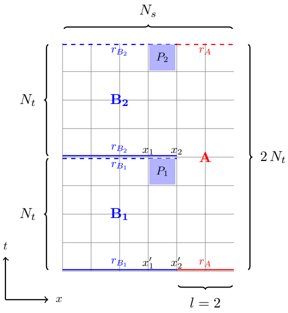


- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.

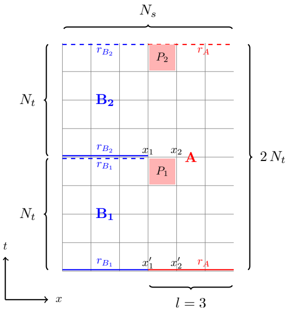


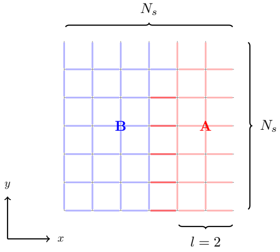


How can we avoid (huge) free energy barriers?



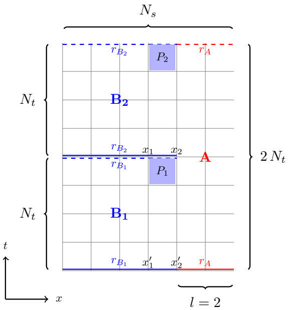
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.



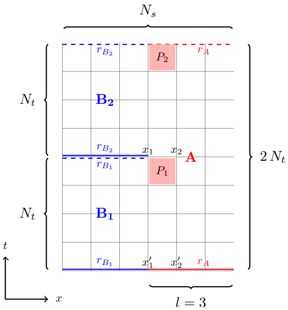


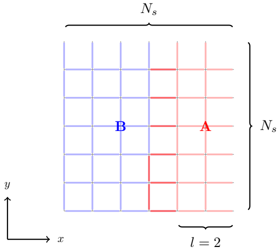
Entangling surface deformation method

How can we avoid (huge) free energy barriers?

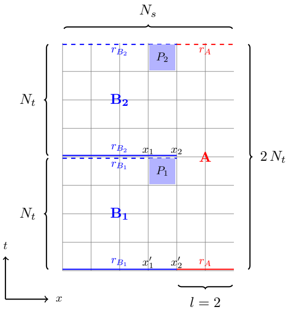


- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.

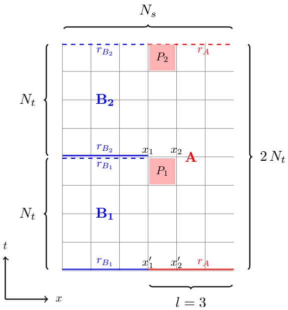


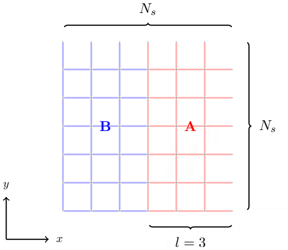


How can we avoid (huge) free energy barriers?



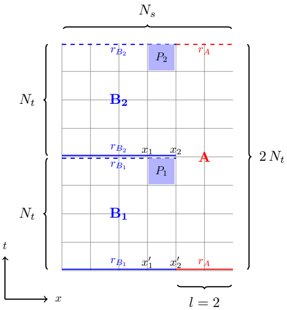
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.



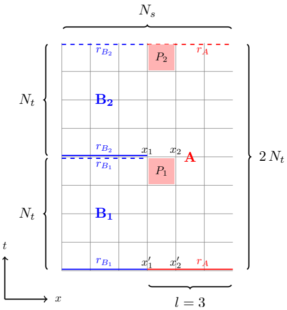


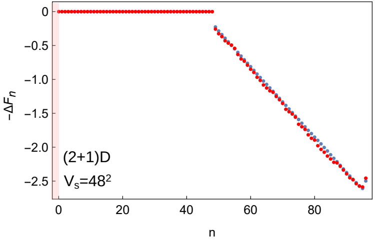
Entangling surface deformation method

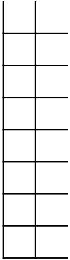
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (2+1) dimensions

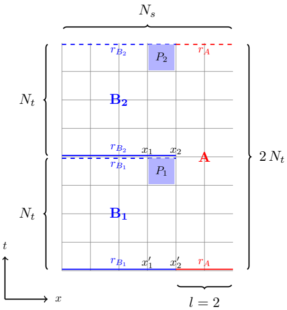




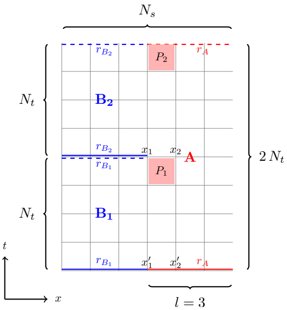


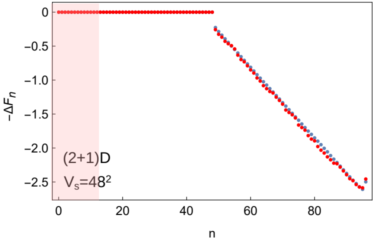
Entangling surface deformation method

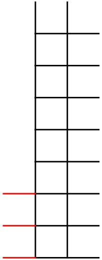
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (2+1) dimensions

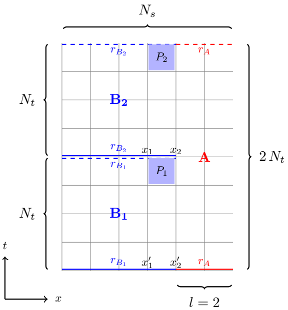




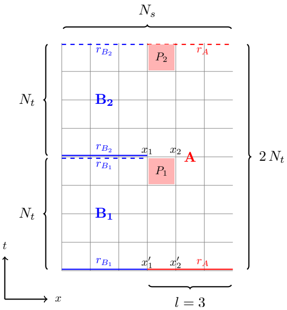


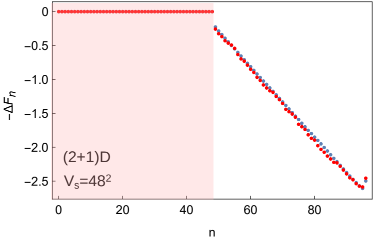
Entangling surface deformation method

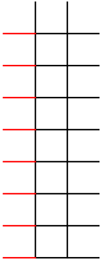
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (2+1) dimensions

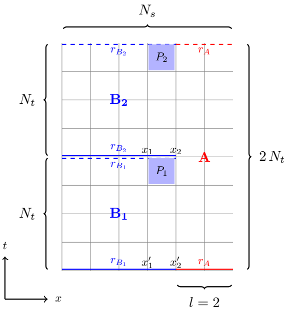




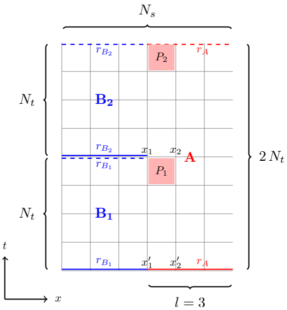


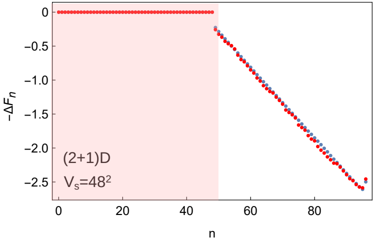
Entangling surface deformation method

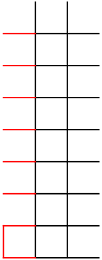
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (2+1) dimensions

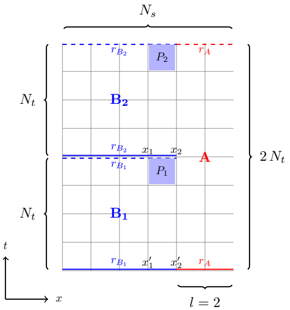




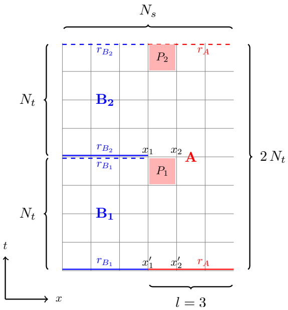


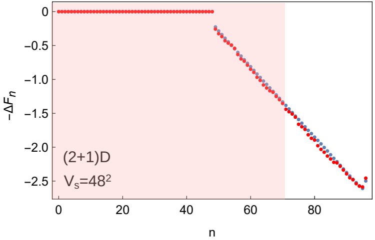
Entangling surface deformation method

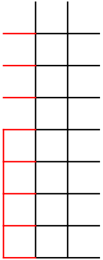
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (2+1) dimensions

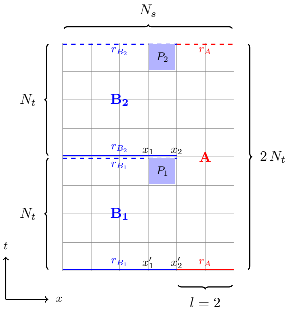




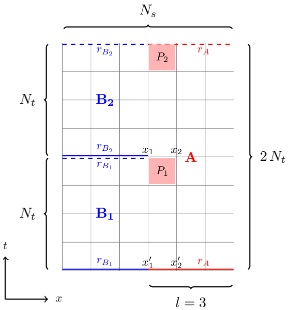


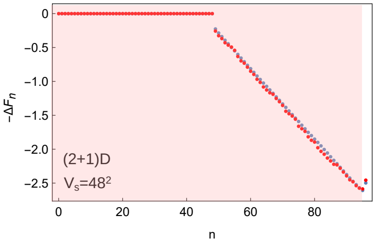
Entangling surface deformation method

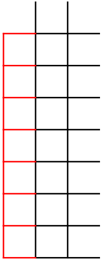
How can we avoid (huge) free energy barriers?



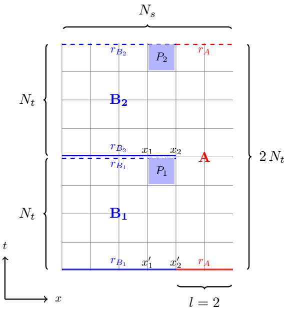
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (2+1) dimensions







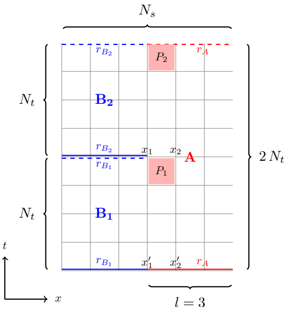
How can we avoid (huge) free energy barriers?

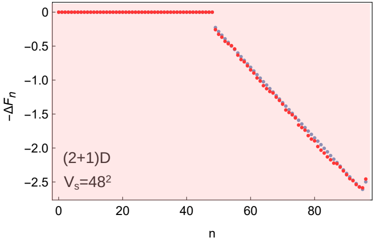


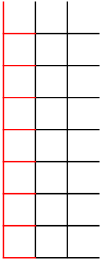
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.

➔ Examples for specific ordering:

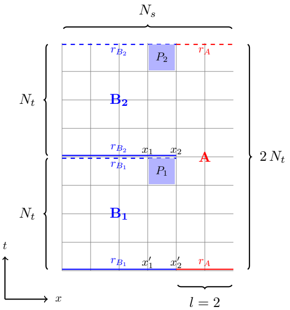
➔ in (2+1) dimensions



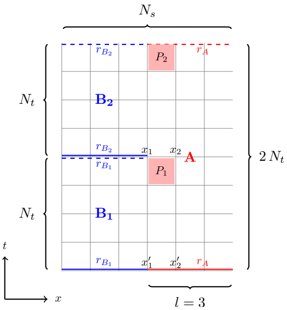


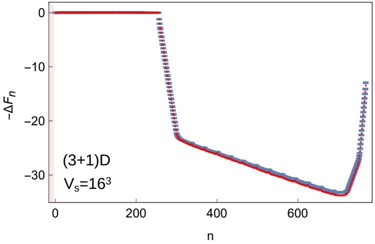


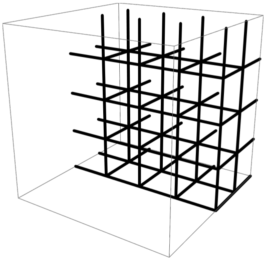
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions

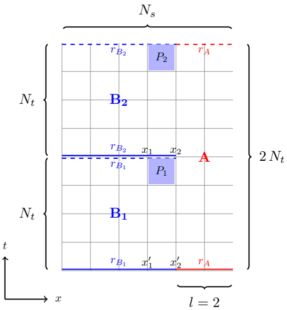




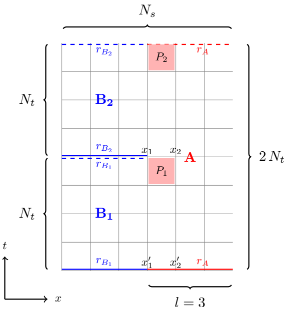


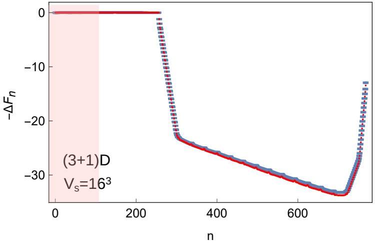
Entangling surface deformation method

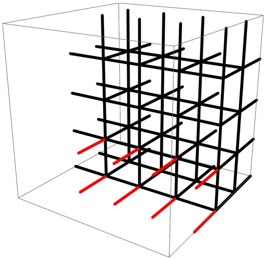
How can we avoid (huge) free energy barriers?



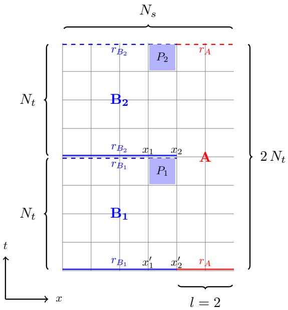
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions



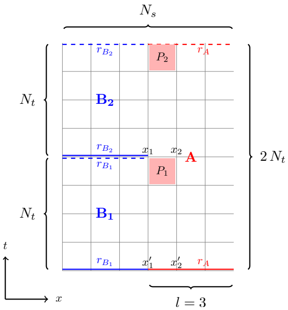


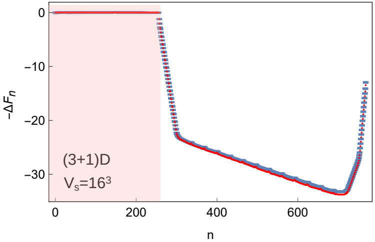


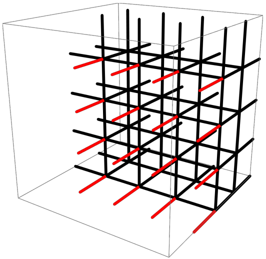
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions

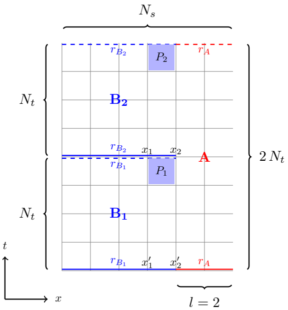




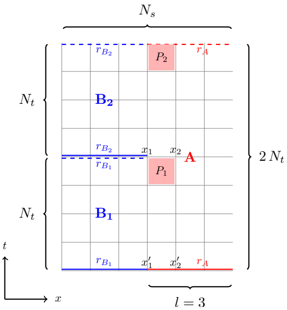


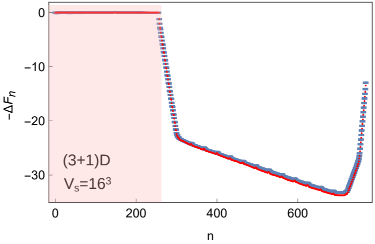
Entangling surface deformation method

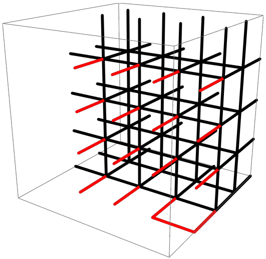
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions

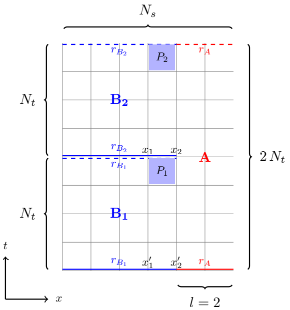




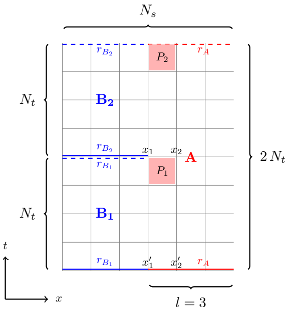


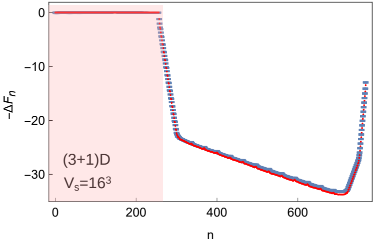
Entangling surface deformation method

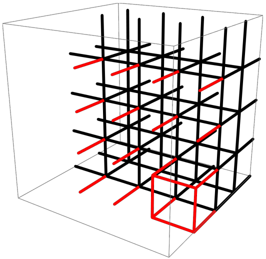
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions

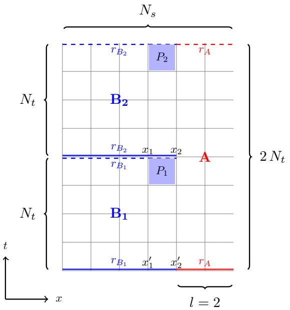




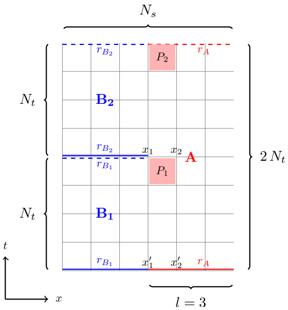


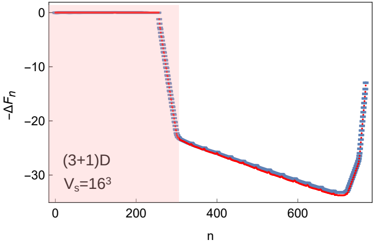
Entangling surface deformation method

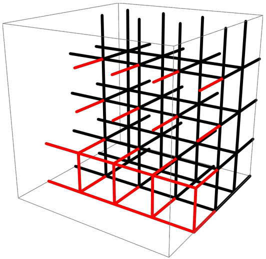
How can we avoid (huge) free energy barriers?



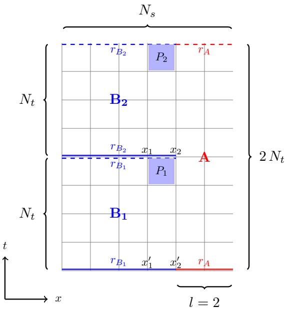
- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions



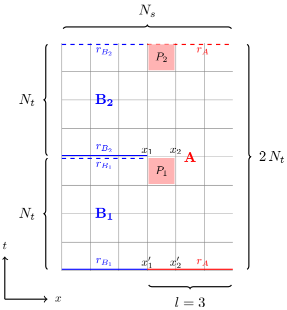


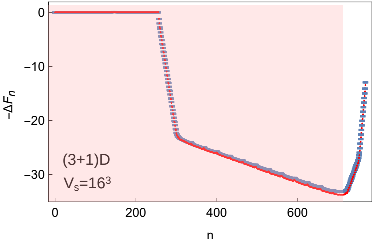


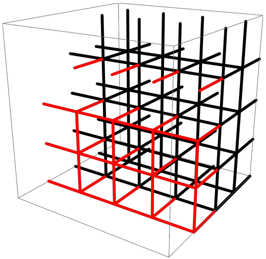
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions

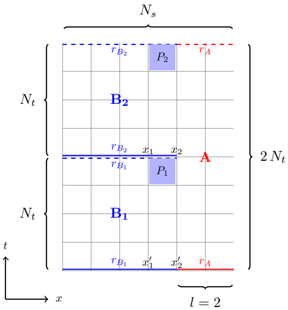




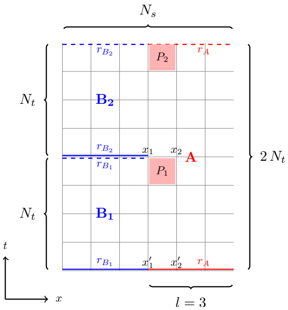


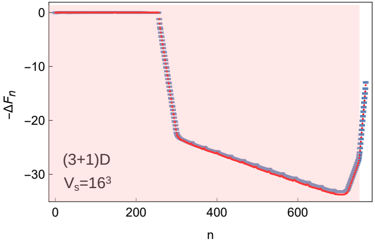
Entangling surface deformation method

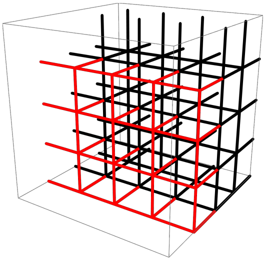
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions

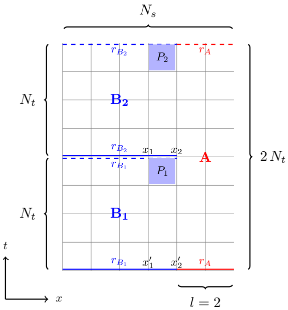




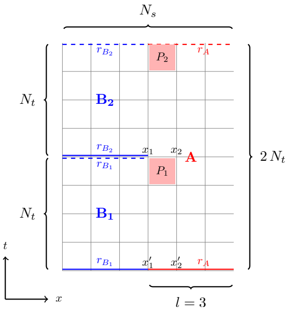


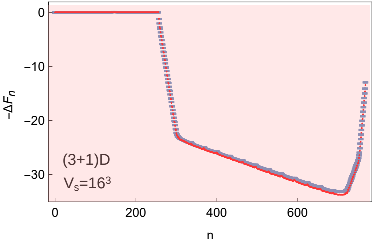
Entangling surface deformation method

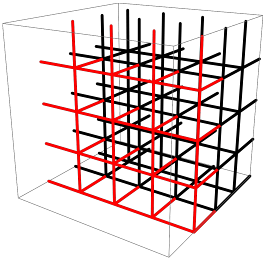
How can we avoid (huge) free energy barriers?



- Instead of "blending" from BC_B and BC_A for all plaquettes P_1, P_2.
- Interpolate by deforming entangling surface.
  - ➔ Examples for specific ordering:
    - ➔ in (3+1) dimensions



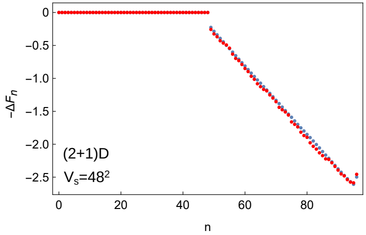


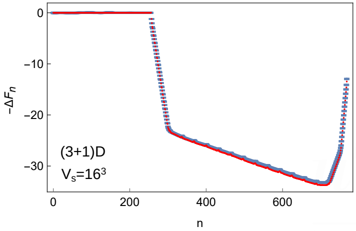


Entangling surface deformation method

Free-energy plateau

- Why does the free energy initially not change?

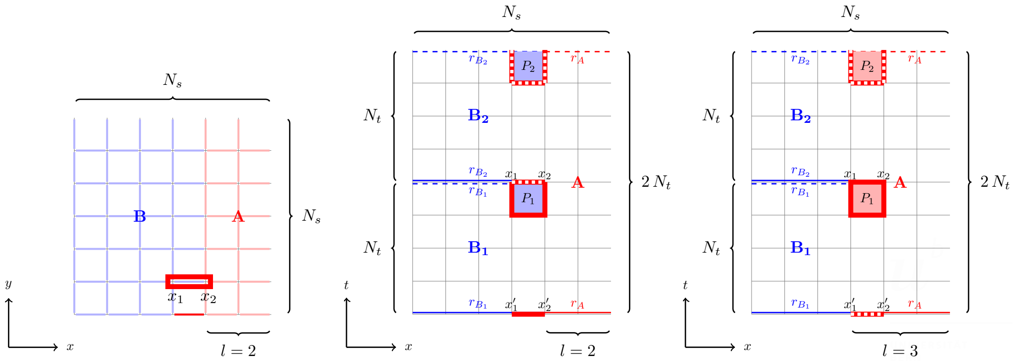




Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

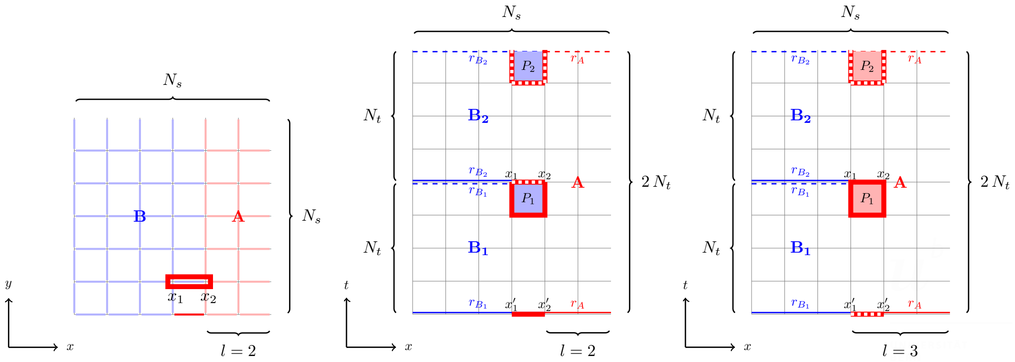


Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.



Entangling surface deformation method

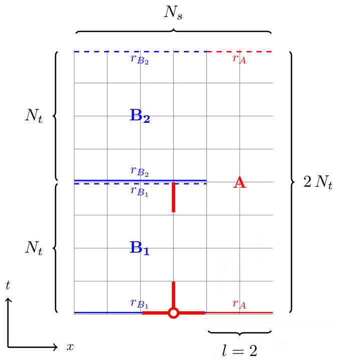
Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.

  ➔ When is this possible?



Entangling surface deformation method

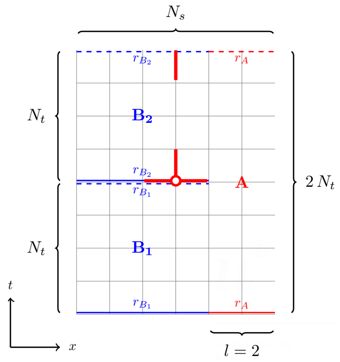
Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.

  ➔ When is this possible?



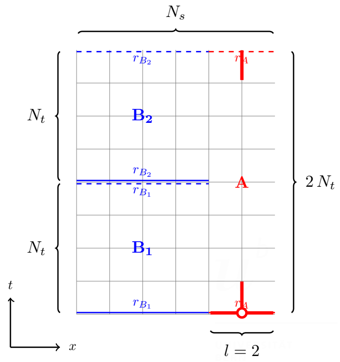
Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.

  ➔ When is this possible?



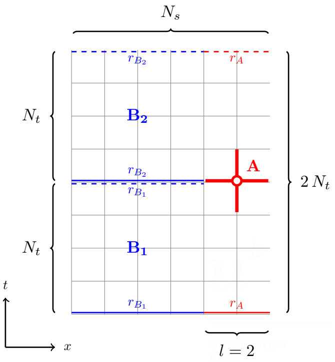
Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.

  ➔ When is this possible?



Entangling surface deformation method

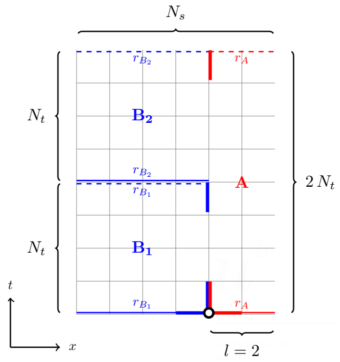
Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \rightarrow x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.

  ➔ When is this possible?



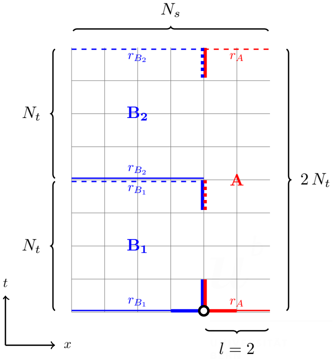
Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.

    ➔ When is this possible?



Entangling surface deformation method

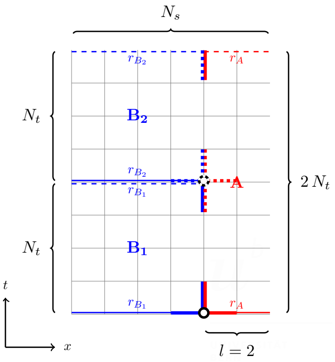
Free-energy plateau

- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.

  ➔ When is this possible?



Entangling surface deformation method

Free-energy plateau

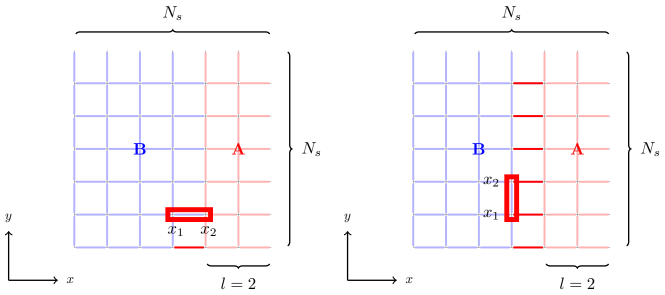
- Why does the free energy initially not change?

Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

➔ Trivial if to-be-swapped links can be gauge transformed individually.

➔ When is this possible?

Only if either for x_1 or x_2 all adjacent spatial link have same BC.



Entangling surface deformation method

Free-energy plateau

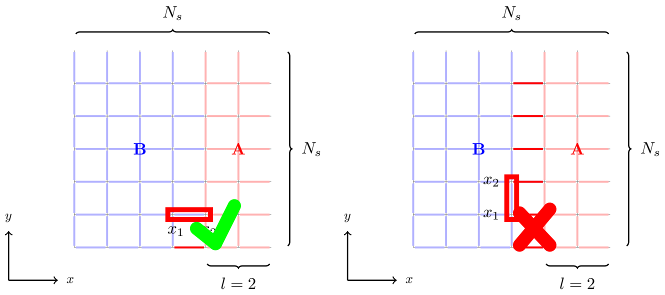
- Why does the free energy initially not change?

  Change of temp. BC over spatial link $(x_1 \to x_2) \Leftrightarrow P_1, P_2$ swap their upper links.

  ➔ Trivial if to-be-swapped links can be gauge transformed individually.

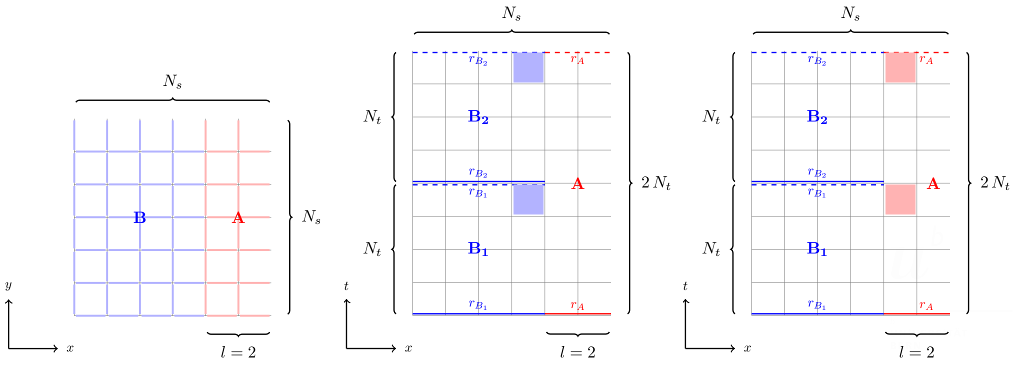
  ➔ When is this possible?

  Only if either for x_1 or x_2 all adjacent spatial link have same BC.



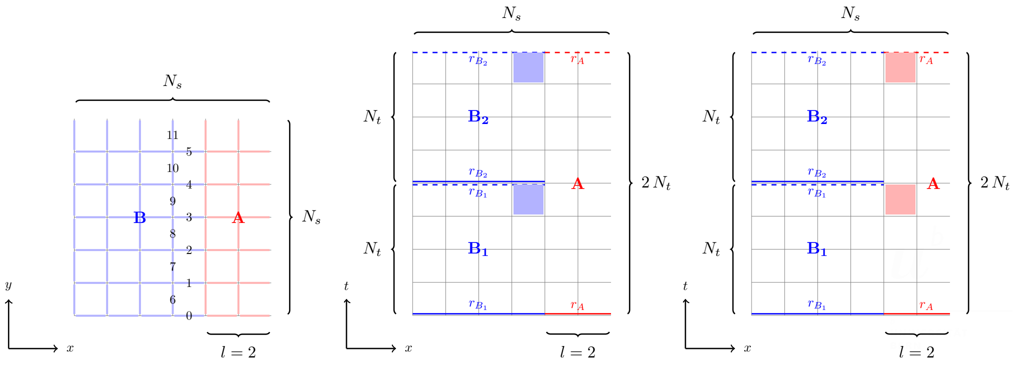
Simulation program

- Lattice setup:
  - ➔ SU(N) Wilson gauge action on $N_s^{d-1} \times s \cdot N_t$ lattice.
  - ➔ For each site, define nearest neighbor map for both types of temp. BCs.
  - ➔ Boolean variable for each spatial link to define which temp. BC applies $\rightarrow$ defines regions A, B.



Simulation program

- Lattice setup:
  - ➔ SU(N) Wilson gauge action on $N_s^{d-1} \times s \cdot N_t$ lattice.
  - ➔ For each site, define nearest neighbor map for both types of temp. BCs.
  - ➔ Boolean variable for each spatial link to define which temp. BC applies $\rightarrow$ defines regions A, B.
  - ➔ Define appropriately ordered list of spatial links that will be allowed to change BC $\rightarrow$ "boundary links".



Simulation program

- Lattice setup:
  - ➔ $\mathrm{SU}(N)$ Wilson gauge action on $N_s^{d-1} \times s \cdot N_t$ lattice.
  - ➔ For each site, define nearest neighbor map for both types of temp. BCs.
  - ➔ Boolean variable for each spatial link to define which temp. BC applies $\rightarrow$ defines regions A, B.
  - ➔ Define appropriately ordered list of spatial links that will be allowed to change BC $\rightarrow$ "boundary links".
- Simulation strategy:
  - ➔ normal Heatbath updates for bulk links. [A. D. Kennedy, B. J. Pendleton (1985)], [N. Cabibbo, E. Marinari]
  - ➔ modified Heatbath/Metropolis update for links which are part of temporal plaquettes that are affected by allowed change of BC $\rightarrow$ simultaneous update of BC and link variable.
  - ➔ Initial Wang-Landau (WL) sampling to achieve equally frequent temp. BC updates for all boundary links $\rightarrow$ WL weights provide initial estimator for free energy difference. [F. Wang, D. P. Landau (2001)]
  - ➔ Record histograms with fixed WL weights $\rightarrow$ improve free energy estimate with histogram ratios and to determine errors.

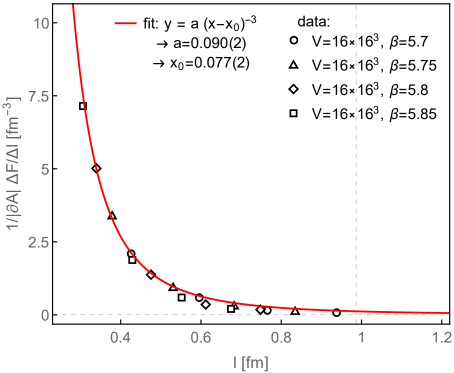
Entangling surface deformation method

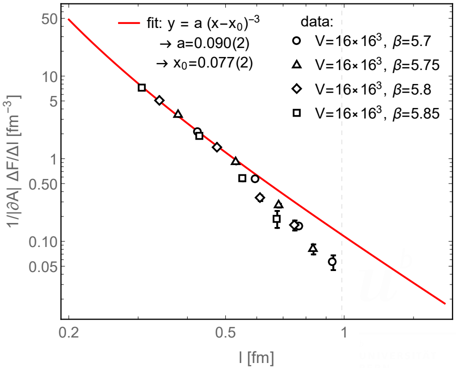
Simulation program

- Lattice setup:
  - ➔ SU(N) Wilson gauge action on $N_s^{d-1} \times s \cdot N_t$ lattice.
  - ➔ For each site, define nearest neighbor map for both types of temp. BCs.
  - ➔ Boolean variable for each spatial link to define which temp. BC applies $\rightarrow$ defines regions A, B.
  - ➔ Define appropriately ordered list of spatial links that will be allowed to change BC $\rightarrow$ "boundary links".
- Simulation strategy:
  - ➔ normal Heatbath updates for bulk links. [A. D. Kennedy, B. J. Pendleton (1985)], [N. Cabibbo, E. Marinari]
  - ➔ modified Heatbath/Metropolis update for links which are part of temporal plaquettes that are affected by allowed change of BC $\rightarrow$ simultaneous update of BC and link variable.
  - ➔ Initial Wang-Landau (WL) sampling to achieve equally frequent temp. BC updates for all boundary links $\rightarrow$ WL weights provide initial estimator for free energy difference. [F. Wang, D. P. Landau (2001)]
  - ➔ Record histograms with fixed WL weights $\rightarrow$ improve free energy estimate with histogram ratios and to determine errors.
- OpenMP (BC update non-local), MPI, WL + parallel tempering.

Results in 4D

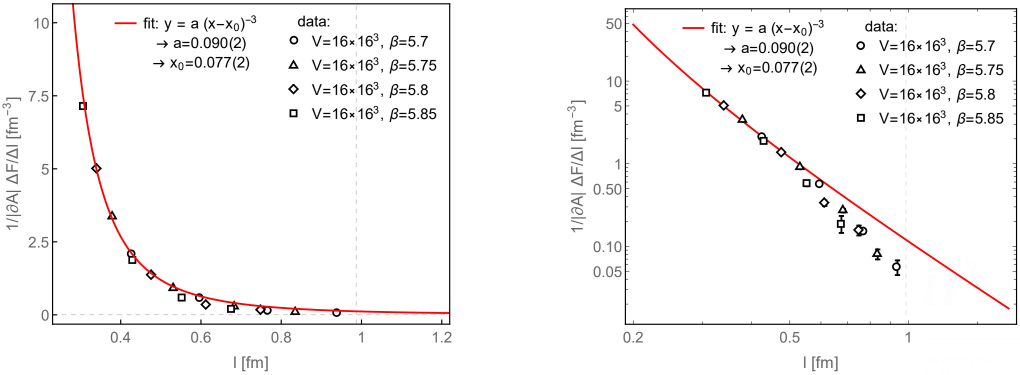
- Entanglement entropy change as function of entangling region width l for SU(3) on $N_s^3 \times 2 \cdot N_t$ lattice with $N_s = N_t = 16$, $\beta \in \{5.7, 5.75, 5.8, 5.85\}$.





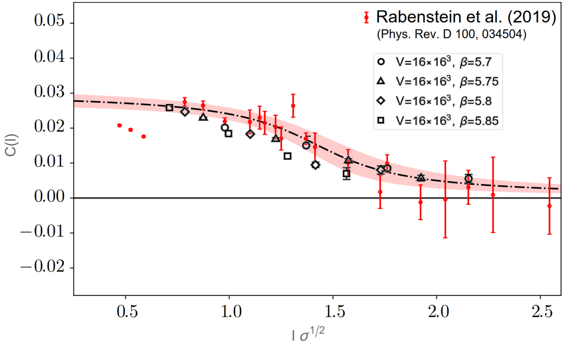
Results in 4D

- Entanglement entropy change as function of entangling region width l for SU(3) on $N_s^3 \times 2 \cdot N_t$ lattice with $N_s = N_t = 16$, $\beta \in \{5.7, 5.75, 5.8, 5.85\}$.
  - ➔ expected power law behavior $\sim l^{-3}$ (holography).
  - ➔ finite volume effects?



Results in 4D

- Entanglement entropy change as function of entangling region width l for SU(3) on $N_s^3 \times 2 \cdot N_t$ lattice with $N_s = N_t = 16$, $\beta \in \{5.7, 5.75, 5.8, 5.85\}$.
  - ➔ expected power law behavior $\sim l^{-3}$ (holography).
  - ➔ finite volume effects?
- Corresponding entropic C-function in comparison with results from literature.



Conclusions & outlook

Conclusions

- New method to determine entanglement measures (Rényi and entropies) in $\mathrm{SU}(N)$ lattice gauge theories.
  - quasi-absent free energy barriers in (2+1)d
  - significantly reduced free energy barriers in (3+1)d
    - ➔ significant error reduction possible.
  - Comparison with literature results promising.

Outlook

- total free energy difference during interpolation between l and $l+1$ is piecewise linear.
  - ➔ useful to speed up simulations in (3+1)d?
- Application to further cases:
  - $\mathrm{SU}(N)$, $N = 2, 3, 4, 5, \ldots, ?$, $d = 3, 4$, $T = 0$, $T \neq 0$
  - different entangling region shapes; alternative entropy measures?
  - "metric reconstruction" (holography) for $\mathrm{SU}(2)$, $\mathrm{SU}(3)$?
- "non-equilibrium work" approach [J. D'Emidio (2020)] applicable? More efficient?

Thank you!