Quantum transport theory for mixing neutrinos

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Theory of neutrino oscillations

- Fascinating features, e.g. flavor, particle-antiparticle and helicity coherence
- Multiple approaches: wave packets, density matrices, mean field equations, spin projection formalism ...
- From quantum field theoretical viewpoint: How to include
  - Coherence effects?
  - Collision term for coherent neutrino states?
  - Heavy neutrinos?
- Conceptual problems: decoherence effects (quantum entanglement), separation of oscillations scales, external magnetic fields...

Quantum kinetic equations

- Derive transport equations for mixing neutrinos from fundamental field theory formalism
    - Include all flavor and particle-antiparticle coherences
    - Only adiabatic background fields are assumed
    - Equations valid for heavy and light neutrinos (UR-limit is not assumed)
    - Generalized Feynman rules to compute collision terms which include coherently mixing species

Real time Kadanoff-Baym equations in Wigner space

$$\hat{\not{K}} S^p(k,x) - e^{-\frac{i}{2}\partial_x^\Sigma \cdot \partial_k}\left[\Sigma^p_{\rm out}(\hat{K},x) S^p(k,x)\right] = 1, \quad \text{(pole)}$$

$$\hat{\not{K}} S^s(k,x) - e^{-\frac{i}{2}\partial_x^\Sigma \cdot \partial_k}\left[\Sigma^r_{\rm out}(\hat{K},x) S^s(k,x)\right] = e^{-\frac{i}{2}\partial_x^\Sigma \cdot \partial_k}\left[\Sigma^s_{\rm out}(\hat{K},x) S^a(k,x)\right] \quad \text{(statistical)}$$

- Exact to a given approximation for the self-energy function
  $\Rightarrow$ information about flavor and all particle-antiparticle coherences
- Need for an approximation scheme which does not lose information about coherence

$$iS^<(u,v) \equiv \langle \bar\psi(v)\psi(u)\rangle$$
$$iS^>(u,v) \equiv \langle \psi(u)\bar\psi(v)\rangle$$
$$S^r(u,v) \equiv \theta(u_0 - v_0)(S^> + S^<)$$
$$S^a(u,v) \equiv -\theta(v_0 - u_0)(S^> + S^<)$$
$$\hat{\not{K}} = \not{k} + \tfrac{i}{2}\not{\partial}_x \qquad p = <,>, \qquad s = r, a$$
$$g(k,x) \equiv \int \mathrm{d}^4 r\, e^{ik\cdot r} g(x + \tfrac{1}{2}r, x - \tfrac{1}{2}r)$$
$$\Sigma_{\rm out}(k,x) \equiv e^{\frac{i}{2}\partial_x^\Sigma \cdot \partial_k^\Sigma}\Sigma(k,x)$$

Reduction of the KB equations

Decoupling

- Statistical functions splitted into background and perturbation parts

Localization

- Reduction of the infinite order gradient expansion: adiabatic background fields and local limit

Density matrix equations

- Projective basis

General transport equations

- Describes flavor and all particle-antiparticle coherence effects
- Holds for light and heavy neutrinos

$$\overbrace{\partial_t f^{<ee'}_{\boldsymbol{k}hij} + (\mathcal{V}^{e'e}_{\boldsymbol{k}hij})_{aa'}\,\hat{\boldsymbol{k}}\cdot\boldsymbol{\nabla} f^{<aa'}_{\boldsymbol{k}hij}}^{\text{Liouville term}} = \overbrace{-2i\Delta\omega^{ee'}_{\boldsymbol{k}ij} f^{<ee'}_{\boldsymbol{k}hij}}^{\text{Leading oscillation scale}} \overbrace{+\, i[(\mathcal{W}^{\mathrm{H}e'e}_{\boldsymbol{k}hji})^{l}_{a}]^{*} f^{<ea}_{\boldsymbol{k}hil} - i(\mathcal{W}^{\mathrm{H}ee'}_{\boldsymbol{k}hij})^{l}_{a} f^{<ae'}_{\boldsymbol{k}hlj}}^{\text{Generalized forward scattering terms}} + \overbrace{\bar{\mathcal{C}}^{<hee'}_{\boldsymbol{k}hij}}^{\text{Collision term}}$$

(Integrating particle-antiparticle coherence out) $\downarrow$

$$\partial_t f^{e}_{\boldsymbol{k}hij} + \bar{v}_{\boldsymbol{k}ij}\,\hat{\boldsymbol{k}}\cdot\boldsymbol{\nabla} f^{e}_{\boldsymbol{k}hij} = -2ie\Delta\omega_{\boldsymbol{k}ij} f^{e}_{\boldsymbol{k}hij} + f^{e}_{\boldsymbol{k}hil}\, i\sigma^{eee}_{\boldsymbol{k}hjil} V_{\boldsymbol{k}hlj}(e\omega_{\boldsymbol{k}l}) - i\sigma^{eee}_{\boldsymbol{k}hlji} V_{\boldsymbol{k}hil}(e\omega_{\boldsymbol{k}l}) f^{e}_{\boldsymbol{k}hlj} + \bar{\mathcal{C}}^{<hee'}_{\boldsymbol{k}hij}$$

(UR-limit) $\downarrow$

$$\partial_t f^{e}_{\boldsymbol{k}hij} + \bar{v}_{\boldsymbol{k}ij}\,\hat{\boldsymbol{k}}\cdot\boldsymbol{\nabla} f^{e}_{\boldsymbol{k}hij} = -i[H^{e}_{\boldsymbol{k}h}, f^{e}_{\boldsymbol{k}h}]_{ij} + \bar{\mathcal{C}}^{<ee'}_{\boldsymbol{k}hij} \qquad \text{with} \qquad \bar{v}_{\boldsymbol{k}ij} \equiv \frac{1}{2}(v_{\boldsymbol{k}i} + v_{\boldsymbol{k}j})$$

Collision terms

- How to compute collision terms?

- Cannot use the usual non-coherent Feynman rules
  - Neglect collisions between coherent particle species

- Necessary to understand neutrino evolution in dense environments
  - Role of particle-antiparticle coherence?
  - Role of flavor coherence (collisions between flavor off-diagonal particle species)?

Collision terms

- Can be written as

$$\bar{\mathcal{C}}^{<hee'}_{\mathrm{H},\boldsymbol{k}hij} = \sum_Y \frac{1}{2\bar{\omega}^{aa'}_{\boldsymbol{k}lj}} \int \mathrm{dPS}_3 \Big[\frac{1}{2}(\mathcal{M}^2)^{ee'}_{\boldsymbol{k}hij\{\boldsymbol{p}_i,Y\}} \Lambda_{\boldsymbol{k}hj\{\boldsymbol{p}_i,Y\},x} + (h.c.)^{e'e}_{ji}\Big].$$

- Here we defined

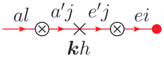
$$\int \mathrm{dPS}_3 \equiv \int \Big[\prod_{i=1,3} \frac{\mathrm{d}^3\boldsymbol{p}_i}{(2\pi)^3 2\bar{\omega}_{\boldsymbol{p}_i l_i l_i'}}\Big](2\pi)^4 \delta^4(k^a_l + p^{a_2}_{2\,l_2} - p^{a_1'}_{1\,l_1'} - p^{a_3'}_{3\,l_3'}),$$

$$\Lambda_{\boldsymbol{k}hj\{\boldsymbol{p}_i,\mathrm{Y}\}}(x) = f^{<aa'}_{\boldsymbol{k}hlj}(x)\, f^{>}_{\mathrm{X}_1\boldsymbol{p}_1}(x)\, f^{<}_{\mathrm{X}_2\boldsymbol{p}_2}(x)\, f^{>}_{\mathrm{X}_3\boldsymbol{p}_3}(x) \; - (\, > \leftrightarrow < \,).$$

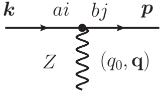
- All summed indices are in curly brackets $Y \equiv \{\mathrm{X}_i, h', a, a', l\}$
- Shorthand notation $f^{s\,a_i a_i'}_{\boldsymbol{p}_i h_i l_i l_i'}(x) \equiv f^s_{\boldsymbol{p}_i \mathrm{X}_i}(x)$, where h_i is the helicity, a_i energy sign index, and l_i flavor index

Feynman rules for the matrix element

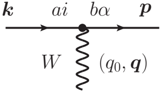
$\sim D^{ab}_{\boldsymbol{k}hij}$



$\sim \frac{1}{2\bar{\omega}^{e'e}_{\boldsymbol{k}ij}} D^{aa'}_{\boldsymbol{k}hlj}\gamma^0 D^{e'e}_{\boldsymbol{k}hji}$

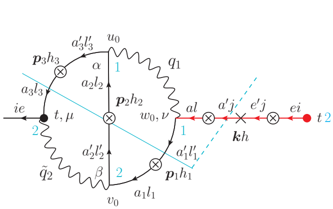


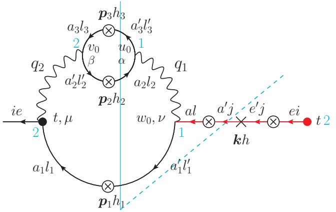
$\sim \frac{ig}{2c_w}\gamma^\mu P_{\mathrm{L}} \bar{U}_{ij}$



$\sim \frac{ig}{\sqrt{2}}\gamma^\mu P_{\mathrm{L}} U^*_{\alpha i}$

- Simple and straightforward rules to compute the matrix element
- $D^{ab}_{\boldsymbol{k}hij} \equiv ab\hat{N}^{ab}_{\boldsymbol{k}ij} P_{\boldsymbol{k}h}(\not{k}^a_i + m_i)(\not{k}^b_j + m_j)$





Results and Applications

- Particle-antiparticle coherence negligible
- Usual density matrix equations and collision terms reproduced at the UR-limit
- Useful to study decoherence effects, quantum entanglement ...
  - Flavor dependence in the velocity term
  - Controlled gradient expansion
- Quantitative way to study how observer-system interference affects the evolution of the system

Summary

- Developed general formalism to model neutrino evolution with all local coherences and the full collision term for coherent neutrino states
- Only flavor coherence is needed for most of the neutrino physics problems

Projective representation

- $\bar{S}^{<}_{\boldsymbol{k}ij}$ can be parametrized as

$$\bar{S}^{<}_{ij}(k,x) = \sum_{haa'} f^{<aa'}_{\boldsymbol{k}hij}(t,\boldsymbol{x}) P^{aa'}_{\boldsymbol{k}hij}$$

where

$$P^{ab}_{\boldsymbol{k}hij} = N^{ab}_{\boldsymbol{k}ij} P_{\boldsymbol{k}h} P^{a}_{\boldsymbol{k}i} \gamma^0 P^{b}_{\boldsymbol{k}j}$$

and the normalization factors are chosen as

$$N^{ab}_{\boldsymbol{k}ij} \equiv \left(\mathrm{Tr}\left[P_{\boldsymbol{k}h} P^{a}_{\boldsymbol{k}i}\gamma^0 P^{b}_{\boldsymbol{k}j}\gamma^0\right]\right)^{-1/2} = \left[\frac{2\omega_{\boldsymbol{k}i}\omega_{\boldsymbol{k}j}}{\omega_{\boldsymbol{k}i}\omega_{\boldsymbol{k}j} + ab(m_i m_j - |\boldsymbol{k}|^2)}\right]^{1/2}$$

- The energy and helicity projection operators are defined as

$$P_{\boldsymbol{k}h} \equiv \frac{1}{2}\left(\mathbb{1} + h\boldsymbol{\alpha}\cdot\hat{\boldsymbol{k}}\gamma^5\right) \qquad \text{and} \qquad P^{a}_{\boldsymbol{k}i} \equiv \frac{1}{2}\left(\mathbb{1} + a\frac{\mathcal{H}_{\boldsymbol{k}i}}{\omega_{\boldsymbol{k}i}}\right)$$

with

$$\mathcal{H}_{\boldsymbol{k}ij} = (\boldsymbol{\alpha}\cdot\boldsymbol{k})_i\delta_{ij} - \gamma^0 m_i\delta_{ij}, \quad \hat{h}_{\boldsymbol{k}} \equiv \boldsymbol{\alpha}\cdot\hat{\boldsymbol{k}}\gamma^5$$

Projective representation

- Normalization factors can be chosen freely, but the above ones simplify the dynamical equations the most
- Apparent singularities at $\omega_{\boldsymbol{k}_i}\omega_{\boldsymbol{k}_j} = ee'(|\boldsymbol{k}|^2 - m_i m_j)$ do not cause problems, since they cancel at the end

Transport equations

$$\partial_t f^{<ee'}_{\boldsymbol{k}hij} + (\mathcal{V}^{e'e}_{\boldsymbol{k}hij})_{aa'}\hat{\boldsymbol{k}}\cdot\boldsymbol{\nabla} f^{<aa'}_{\boldsymbol{k}hij} = -2i\Delta\omega^{ee'}_{\boldsymbol{k}ij} f^{<ee'}_{\boldsymbol{k}hij} + i[(\mathcal{W}^{\mathrm{H}e'e}_{\boldsymbol{k}hji})^l_a]^* f^{<ea}_{\boldsymbol{k}hil} - i(\mathcal{W}^{\mathrm{H}ee'}_{\boldsymbol{k}hij})^l_a f^{<ae'}_{\boldsymbol{k}hlj} + \bar{\mathcal{C}}^{<hee'}_{\boldsymbol{k}hij}$$

where

$$\begin{aligned}
(\mathcal{V}^{e'e}_{\boldsymbol{k}hij})_{aa'} &= \delta_{a'e'}\mathcal{V}^{eae'}_{\boldsymbol{k}hij} + \delta_{ae}\mathcal{V}^{a'e'e}_{\boldsymbol{k}hji},\\
\mathcal{V}^{abc}_{\boldsymbol{k}hij} &\equiv \frac{1}{2}N^{ac}_{\boldsymbol{k}ij}N^{bc}_{\boldsymbol{k}ij}\left(\frac{1}{\omega_{\boldsymbol{k}i}}\left[\frac{a}{(N^{bc}_{\boldsymbol{k}ij})^2} + \frac{b}{(N^{ac}_{\boldsymbol{k}ij})^2}\right] - \frac{c}{\omega_{\boldsymbol{k}j}}\delta_{a-b}\right),\\
2\Delta\omega^{ee'}_{\boldsymbol{k}ij} &\equiv e\omega_{\boldsymbol{k}i} - e'\omega_{\boldsymbol{k}j},\\
(\mathcal{W}^{ee'}_{\boldsymbol{k}hij})^l_a &\equiv \mathrm{Tr}\Big[P^{e'e}_{\boldsymbol{k}hji}\bar{\Sigma}^{\mathrm{H}}_{\mathrm{eff}\,\boldsymbol{k}il}P^{ae'}_{\boldsymbol{k}hlj}\Big].
\end{aligned}$$

Transport equations

- We consider vector-like gauge interactions (a_{ij} and b_{ij} are flavor matrices, and u is the plasma 4-velocity):

$$\begin{aligned}\bar{\Sigma}_{\mathrm{H},ij}(k,x) &= \gamma^0(a_{ij}\not{k} + b_{ij}\not{u})P_L \\ &= \big(k_0 a_{ij} + b_{ij}\big)P_L - a_{ij}\boldsymbol{\alpha}\cdot\boldsymbol{k}P_L, \\ &\to \Big((k_0 + h|\boldsymbol{k}|)a_{ij} + b_{ij}\Big)P_L \equiv V_{\boldsymbol{k}hij}(k_0,x)P_L.\end{aligned}$$

- In this case the forward scattering tensor coefficient reads

$$\begin{aligned}(\mathcal{W}^{e'e}_{\boldsymbol{k}hij})^l_a &= \sigma^{eae'}_{\boldsymbol{k}hlji}V_{\boldsymbol{k}hil}(a\omega_{\boldsymbol{k}l},x), \\ \sigma^{abc}_{\boldsymbol{k}hlji} &\equiv \frac{1}{2}N^{ca}_{\boldsymbol{k}il}N^{bc}_{\boldsymbol{k}ji}\Big(\frac{\hat{P}^a_{\boldsymbol{k}hl}}{(N^{bc}_{\boldsymbol{k}ji})^2} + \frac{\hat{P}^b_{\boldsymbol{k}hj}}{(N^{ca}_{\boldsymbol{k}il})^2} - \hat{P}^c_{\boldsymbol{k}hi}\Big(\frac{1}{(N^{ab}_{\boldsymbol{k}lj})^2} - ab\frac{m_l m_j}{\omega_{\boldsymbol{k}l}\omega_{\boldsymbol{k}j}}\Big)\Big)\end{aligned} \tag{1}$$

Transport equations in the UR-limit

- In the UR-limit tensors $\sigma^{haee'}_{\boldsymbol{k}ijl}$ and $\mathcal{V}^{haee'}_{\boldsymbol{k}ij}$ become very simple:

$$\mathcal{V}^{haee'}_{\boldsymbol{k}ij} = \frac{e'}{|\boldsymbol{k}|}\delta_{a,e'}(\delta_{e,e'} + \delta_{e,-e'})$$

$$\sigma^{haee'}_{\boldsymbol{k}ijl} = \delta_{a,e'}(\delta_{e,e'}\delta_{h,-e} + \delta_{e,-e'}\delta_{h,e})$$

- Particle-antiparticle coherence averages out at timescales relevant for neutrino oscillations
- According to above, the transport equations can be written in the UR-limit as

$$\partial_t f^e_{\boldsymbol{k}h} + \frac{1}{2}\{v_{\boldsymbol{k}}, \hat{\boldsymbol{k}} \cdot \boldsymbol{\nabla} f^e_{\boldsymbol{k}h}\} = -i[H^e_{\boldsymbol{k}h}, f^e_{\boldsymbol{k}h}] + \bar{\mathcal{C}}^e_{\boldsymbol{k}h}$$

Integrating out the particle-antiparticle coherence

- Leading time-dependence of the particle-antiparticle coherence functions:

  $f^{e-e}_{\boldsymbol{k}hij}(t) \sim \exp(-2ie\bar{\omega}_{\boldsymbol{k}ij}t) \qquad \text{with} \qquad 2\bar{\omega}_{\boldsymbol{k}ij} = \omega_{\boldsymbol{k}i} + \omega_{\boldsymbol{k}j},$

- Leading time-dependence of the flavour coherence functions:

  $f^{ee}_{\boldsymbol{k}hij}(t) \sim \exp(-2ie\Delta\omega_{\boldsymbol{k}ij}t) \qquad \text{with} \qquad 2\Delta\omega_{\boldsymbol{k}ij} = \omega_{\boldsymbol{k}i} - \omega_{\boldsymbol{k}j}$

- Weierstrass transform of equation: $\int \mathrm{d}t'\, W(t,t')\big[\mathrm{e.o.m.}(t')\big]$
- $W(t,t') \sim \exp\big(-(t-t')^2\big)/2\sigma^2$
- Given hierarchy $\Delta\omega_{\boldsymbol{k}} \ll \bar{\omega}_{\boldsymbol{k}}$, we can choose a σ such that $1/\Delta\omega_{\boldsymbol{k}} \gg \sigma \gg 1/\bar{\omega}_{\boldsymbol{k}}$

Integrating out the particle-antiparticle coherence

- Terms proportional to the coherence functions get exponentially suppressed:
  $\int \mathrm{d}t'\, W(t,t')c^e_{\boldsymbol{k}}(t')\delta f^{e-e}_{\boldsymbol{k}h}(t') \sim c^e_{\boldsymbol{k}}(t)\delta f^{e-e}_{\boldsymbol{k}h}(t)\exp\left(-2(\bar{\omega}_{\boldsymbol{k}}\sigma)^2\right)$
- $c^e_{\boldsymbol{k}h}$ stands for any coefficient of $\delta f^{e-e}_{\boldsymbol{k}h}$ in the equation of motion
- For a generic 2-2 scattering process:
  $\phi_\Lambda(t) = -\left(2\Delta\omega^{ee'}_{\boldsymbol{k}ij} + 2\Delta\omega^{a_1 a_1'}_{\boldsymbol{p}_1 l_1 l_1'} + 2\Delta\omega^{a_2 a_2'}_{\boldsymbol{p}_2 l_2 l_2'} + 2\Delta\omega^{a_3 a_3'}_{\boldsymbol{p}_3 l_3 l_3'}\right)t$
- In the flavor oscillation scales $\phi(t)$ causes fast oscillations if any of the terms corresponds to coherence terms $(a \neq a')$

  $\Rightarrow$ Collision term averages out in the flavor oscillation scales