Relativistic fluid dynamics in relativistic heavy-ion collisions

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Hiukkasfysiikan päivä, 16.11.2021

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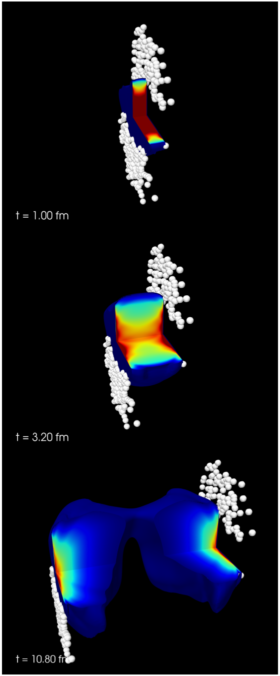


Search for QCD matter properties

- Matter properties (EoS, viscosity) direct input to fluid dynamics $\longrightarrow$ Very natural framework to study QCD matter properties.

Here: Brief summary of our efforts to derive equations of motion for relativistic fluid dynamics from kinetic theory

Fluid dynamics: Conservation laws & tensor decompositions

$$\partial_\mu N^\mu = 0$$
$$\partial_\mu T^{\mu\nu} = 0$$

Identify fluid dynamical variables by decomposing w.r.t. fluid velocity u^μ. Define projector $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ orthogonal to u^μ, $A^{\langle\mu\rangle} = \Delta^{\mu\nu} A_\nu$.

$$N^\mu = n u^\mu + n^\mu$$
$$T^{\mu\nu} = e u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu}$$

$n = u_\mu N^\mu$	LRF particle density
$n^\mu = \Delta^\mu_\alpha N^\alpha$	particle diffusion current
$e = u_\mu T^{\mu\nu} u_\nu$	LRF energy density
$W^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$	energy diffusion current
$p(e,n) + \Pi = -\frac{1}{3}\Delta_{\mu\nu} T^{\mu\nu}$	isotropic pressure ($p_{eq} + bulk$)
$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle}$	shear stress tensor

Small gradients & close to equilibrium

- ∇u small (gives macroscopic length scale $1/L$)
- microscopic length scale $\lambda_{\rm mfp}$ or thermalization time τ
- Knudsen number $\mathrm{Kn} = \frac{\ell_{\rm micr}}{L_{\rm macr}} \sim \lambda_{\rm mfp}\nabla u \ll 1$

Close to equilibrium (in equilibrium $\Pi = n^\mu = \pi^{\mu\nu} = 0$)

$$\mathrm{R}_\Pi^{-1} \sim \frac{|\Pi|}{P_0},\ \mathrm{R}_n^{-1} \sim \frac{|n^\mu|}{n_0},\ \mathrm{R}_\pi^{-1} \sim \frac{|\pi^{\mu\nu}|}{P_0}$$

Power counting in Kn and R^{-1}: Simplest solution (e.g. shear-stress tensor)

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu},\ \text{where } \sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle}$$

- Shear-stress tensor directly proportional to the gradients of velocity.
- Does not introduce any new dynamical variables compared to ideal fluids (only e, n and u^μ)

Transient fluid dynamics

- $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$ implies immediate reaction of momentum distribution to velocity gradients
- In reality takes time $\sim \lambda_{\rm mfp} \sim \tau_{\rm therm}$
- In HI collisions $\tau_{\rm therm} \sim$ (expansion rate)$^{-1}$ $\longrightarrow$ not possible to neglect $\tau_{\rm therm}$.

$\longrightarrow$ Use transient fluid dynamics (e.g. Israel-Stewart theory)

$$\frac{d}{d\tau}\pi^{\langle\mu\nu\rangle} = -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\sigma^{\mu\nu}\right) + \text{ (higher order terms)}$$

- $\pi^{\mu\nu}$ independent variable (approaches $2\eta\sigma^{\mu\nu}$ within τ_π)
- New transport coefficients: τ_π = relaxation time + other 2nd order coefficients
- Usually called "Second order theories"
- **Our goal is to derive all the terms and calculate coefficients for this type of theory**

Boltzmann Equation

- Evolution equation for the single particle distribution function $f_{\mathbf{k}}$

$$k^{\mu}\partial_{\mu}f_{\mathbf{k}} = C\left[f\right]$$

- Expand around local equilibrium $f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}}(1+\phi_{\mathrm{k}})$

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}\sum_{\ell=0}^{\infty}\sum_{n=0}^{N_{\ell}}\mathcal{H}_{\mathbf{k}}^{(n\ell)}\rho_{(n)}^{\mu_1\cdots\mu_\ell}k_{\langle\mu_1}\ldots k_{\mu_\ell\rangle}$$

- $\mathcal{H}_{\mathbf{k}}^{(n\ell)}$ is polynomial in $E_{\mathbf{k}} = u\cdot k$ (energy of the particle in LRF).
- $1\,,\ k^{\langle\mu\rangle}\,,\ k^{\langle\mu}k^{\nu\rangle}\,,\ k^{\langle\mu}k^{\nu}k^{\lambda\rangle}\,,\ \ldots$, traceless and otrhogonal to u^{μ}

Expansion coefficients are moments of $\delta f_{\mathbf{k}}$:

$$\rho_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \int dK\left(E_{\mathbf{k}}\right)^{n}k^{\langle\mu_1}\ldots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}}$$

These can be taken as dynamical variables (instead of $f_{\mathbf{k}}$)

Some of them can be identified:

$$\rho_{(0)} = -\frac{3}{m^2}\Pi\,,\quad \rho_{(0)}^{\mu} = n^{\mu}\,,\quad \rho_{(0)}^{\mu\nu} = \pi^{\mu\nu}$$

General equations of motion

- Boltzmann equation give evolution equations for moments $\rho_{(n)}^{\langle\mu_1\cdots\mu_\ell\rangle}$
- For example, equations of motion for second rank tensors:

$$
\begin{aligned}
\dot{\rho}_{(r)}^{\langle\mu\nu\rangle} - C_{(r-1)}^{\langle\mu\nu\rangle} = {} & 2\beta_\eta^{(r)}\sigma^{\mu\nu} - \frac{2}{7}\left[(2r+5)\,\rho_{(r)}^{\rho\langle\mu} - m^2 2\,(r-1)\,\rho_{(r-2)}^{\rho\langle\mu}\right]\sigma_\rho^{\ \nu\rangle} + 2\rho_{(r)\lambda}^{\langle\mu}\omega^{\ \nu\rangle\lambda} \\
& + \frac{2}{15}\left[(r+4)\,\rho_{(r+2)} - (2r+3)\,m^2\rho_{(r)} + (r-1)\,m^4\rho_{(r-2)}\right]\sigma^{\mu\nu} \\
& + \frac{2}{5}\nabla^{\langle\mu}\left(\rho_{(r+1)}^{\nu\rangle} - m^2\rho_{(r-1)}^{\nu\rangle}\right) \\
& - \frac{2}{5}\left[(r+5)\,\rho_{(r+1)}^{\langle\mu} - rm^2\rho_{(r-1)}^{\langle\mu}\right]\dot{u}^{\,\nu\rangle} - \frac{1}{3}\left[(r+4)\,\rho_{(r)}^{\mu\nu} - m^2\,(r-1)\,\rho_{(r-2)}^{\mu\nu}\right]\theta \\
& + (r-1)\,\rho_{(r-2)}^{\mu\nu\lambda\rho}\sigma_{\lambda\rho} - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{(r-1)}^{\alpha\beta\lambda} + r\rho_{(r-1)}^{\mu\nu\sigma}\dot{u}_\sigma .
\end{aligned}
$$

Moments of the collision integral:

$$
C_{(r)}^{\langle\mu_1\cdots\mu_\ell\rangle} = \int dK\,(E_{\mathbf{k}})^r\,k^{\langle\mu_1}\ldots k^{\mu_\ell\rangle}\,C[f] .
$$

Collision term

The moments of the collision integral can be written as

$$C_{(r-1)}^{\langle\mu_1\cdots\mu_\ell\rangle} = -\sum_{n=0}^{\infty} \mathcal{A}_\ell^{(rn)} \rho_{(n)}^{\mu_1\cdots\mu_\ell} + (\text{nonlinear terms}),$$

where the coefficients $\mathcal{A}_\ell^{(rn)}$ contain the information from the cross section
Then the general equations of motion are of the form

$$\dot{\rho}_{(r)}^{\langle\mu\nu\rangle} + \sum_{n=0}^{\infty} \mathcal{A}_2^{(rn)} \rho_{(n)}^{\mu\nu} \quad = \quad 2\beta_\eta^{(r)} \sigma^{\mu\nu} + \cdots$$

Even the linear part is coupled set of equations for all the moments $\rho_{(n)}^{\mu\nu}$. Remember: We want the equations of motion for dissipative quantities, e.g. $\pi^{\mu\nu} = \rho_{(0)}^{\mu\nu}$.

How to reduce the infinite degrees of freedom to fluid dynamics?

Truncations

- Truncate the expansion of $f_{\mathbf{k}}$ such that only 14 variables remain (Israel, Stewart 1978 and Denicol, Koide, Rischke 2010):

$$\delta f_{\mathbf{k}} = C_\pi f_{0\mathbf{k}} \pi^{\mu\nu} k_\mu k_\nu$$

- Every moment $\rho_{(i)}^{\mu\cdots}$ becomes proportional to the dissipative currents

$$\rho_{(i)}^{\mu\nu} = \gamma_i \pi^{\mu\nu} \qquad \text{(14-moment approximation)}$$

- Not expansion in Knudsen number In fact it neglects infinitely many terms first order in Kn (Navier-Stokes terms)

- (Denicol, HN, Molnar, Rischke 2012): Identify the slowest time scale from the Boltzmann equation as thermalization time. $\longrightarrow$ expansion in Kn and R^{-1}

$$\rho_{(i)}^{\mu\nu} \simeq \Omega_i \pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}) \qquad \text{(DNMR truncation)}$$

Equations of fluid dynamics

$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2\frac{\eta}{\tau_\pi}\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu} + \mathcal{K}^{\mu\nu}$$

Terms of order $\mathrm{Kn} \times \mathrm{R}^{-1}$ (only ones in 14-moment approximation (Israel and Stewart):

$$\mathcal{J}^{\mu\nu} = 2\pi_\alpha^{\langle\mu}\,\omega^{\,\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\,\sigma^{\,\nu\rangle\alpha} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi n}n^{\langle\mu}F^{\,\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\,\nu\rangle} + \lambda_{\pi n}n^{\langle\mu}I^{\,\nu\rangle}$$

Terms of order $\mathrm{R}^{-1} \times \mathrm{R}^{-1}$ (come from the non-linear parts of the collision integral):

$$\mathcal{R}^{\mu\nu} = \varphi_6\Pi\pi^{\mu\nu} + \varphi_7\pi^{\lambda\langle\mu}\,\pi_\lambda^{\,\nu\rangle} + \varphi_8 n^{\langle\mu}n^{\,\nu\rangle}$$

Terms of order Kn^2 (break the causality... contain $(\partial_x)^2$, $(\partial_t)^2$):

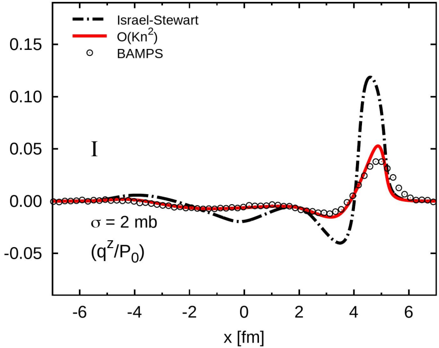
$$\mathcal{K}^{\mu\nu} = \eta_1\omega_\lambda{}^{\langle\mu}\,\omega^{\,\nu\rangle\lambda} + \eta_2\theta\sigma^{\mu\nu} + \eta_3\sigma^{\lambda\langle\mu}\,\sigma_\lambda^{\,\nu\rangle} + \eta_4\sigma_\lambda^{\langle\mu}\,\omega^{\,\nu\rangle\lambda} + \eta_5 I^{\langle\mu}I^{\,\nu\rangle} + \eta_6 F^{\langle\mu}F^{\,\nu\rangle} + \eta_7 I^{\langle\mu}F^{\,\nu\rangle} + \eta_8\nabla^{\langle\mu}I^{\,\nu\rangle} + \eta_9\nabla^{\langle\mu}F^{\,\nu\rangle}$$

Equations of fluid dynamics: complete second order

- It is possible to get rid of Kn^2 terms by introducing new dynamical variable $\pi^{\mu\nu} \longrightarrow (\pi^{\mu\nu}, \rho^{\mu\nu}_{(1)})$

$$\rho^{\mu\nu}_{(i)} \simeq \Omega^{(0)}_i \pi^{\mu\nu} + \Omega^{(1)}_i \rho^{\mu\nu}_{(1)} + \mathcal{O}(\mathrm{Kn}^2)$$

- Agrees well with the solutions of the Boltzmann equation
(Denicol, HN, Bouras, Molnar, Xu, Rischke and Greiner, arXiv:1207.6811 [nucl-th])



Multicomponent systems

- So far everything for a single component fluid
- Heavy-Ion collisions: Strongly interacting matter is a multicomponent system
- hadronic matter: hundreds of different type of hadrons. Usually up to $m \sim 2$ GeV
- Straightforward application of above $\longrightarrow \partial_\mu N_i^\mu = R_i$ for each type of hadron

We want something simpler:

$$N_q^\mu = n_q u^\mu + V_q^\mu \qquad T^{\mu\nu} = e u^\mu u^\nu - (p + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

where N_q^μ are the conserved charge currents (net-baryon, charge and strangeness).

- New phenomena: coupling between diffusion currents:
- e.g. $V_{\rm B}^\mu \sim \kappa_{\rm BB}\nabla^\mu\alpha_{\rm B} + \kappa_{\rm BS}\nabla^\mu\alpha_{\rm S} + \kappa_{\rm BC}\nabla^\mu\alpha_{\rm C}$
  (Fotakis, Greif, Greiner, Denicol, HN [arXiv:1912.09103 [hep-ph]].)
- Possibly large bulk viscosity (due to off-equilibrium chemical composition)
- $\longrightarrow$ reliable computation of $\mathcal{O}(2)$ coupling terms e.g. between shear and bulk: $\Pi\sigma^{\mu\nu}$, $\pi^{\mu\nu}\sigma_{\mu\nu}$

Multicomponent systems: Order of magnitude approximation

- Goal: Derive Israel-Stewart type of fluid dynamics, i.e. without the problematic Kn^2 terms and in terms of $\pi^{\mu\nu}$ alone (Fotakis, Molnar, HN, Rischke, in preparation)
- Order-of-magnitude approximation. For each hadron h:

$$\rho_{i,h}^{\mu\nu} = 2\eta_{i,h}\sigma^{\mu\nu} + O(2)$$

$$\longrightarrow$$

$$\pi^{\mu\nu} = \sum_h \rho_{0,h}^{\mu\nu} = 2\eta\sigma^{\mu\nu} + O(2)$$

$$\longrightarrow$$

$$\rho_{i,h}^{\mu\nu} = \frac{\eta_{i,h}}{\eta}\pi^{\mu\nu} + O(2)$$

$$\longrightarrow$$

- equations of motion for $\pi^{\mu\nu}$, V_q^μ, and Π.
- $\delta f_h = F_h(\pi^{\mu\nu}, V_q^\mu, \Pi)$
- τ_π loses the meaning as real microscopic time.
- Kn^2 terms are absent

Summary

- Heavy-ion collisions: Rapidly expanding multicomponent fluid $\longrightarrow$ transient second-order fluid dynamics
- Second-order fluid dynamics not uniquely defined, but there are different approaches
- 14-moment approximation, DNMR-truncation (identify microscopic thermalization time), Order-of-magnitude approximation (assume system close to gradient expansion)
- Can be generalized to e.g. magnetohydrodynamics, anisotropic fluid dynamics
- Multicomponent fluid dynamics using Order-of-magnitude approximation