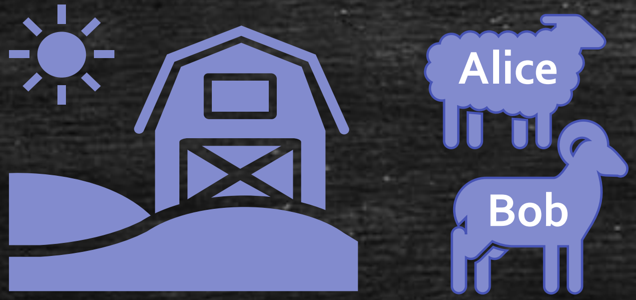

Computing secret key rates

Les Diablerets QKD Summer School 2026

Peter Brown

The key-rate problem

The key-rate problem

1001101…

1001111…

1. They generate raw keys using their devices.

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1. They generate raw keys using their devices.
2. They correct errors (*information reconciliation*) Cost: $leak_{\mathrm{EC}}$
3. They extract randomness (*privacy amplification*) Gain: $H(Z|E)$ E unknown

Key rate: $\min_{\text{attacks}} H(Z|E) - leak_{\mathrm{EC}}$ bits per round

The entropy optimization problem

Want to study the following problem

$$\min_{\text{attacks}} H(Z|E)$$

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Set of "attacks" depends on security model and protocol:

<u>Device dependent</u>

Characterized system

E.g. Known POVMs / state

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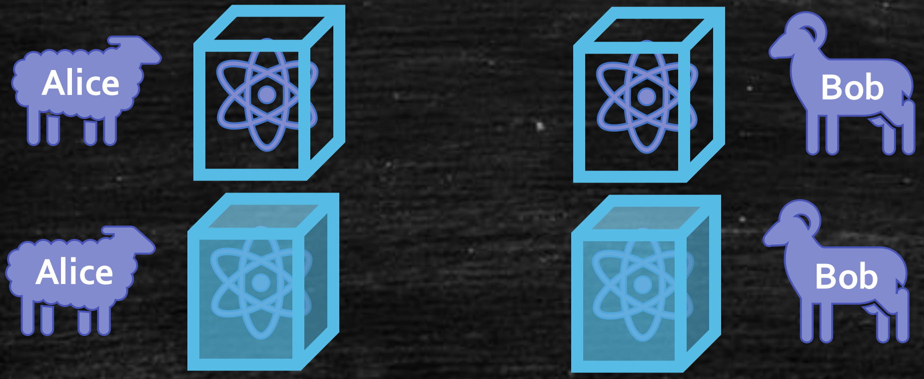

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E.g. Known dimension

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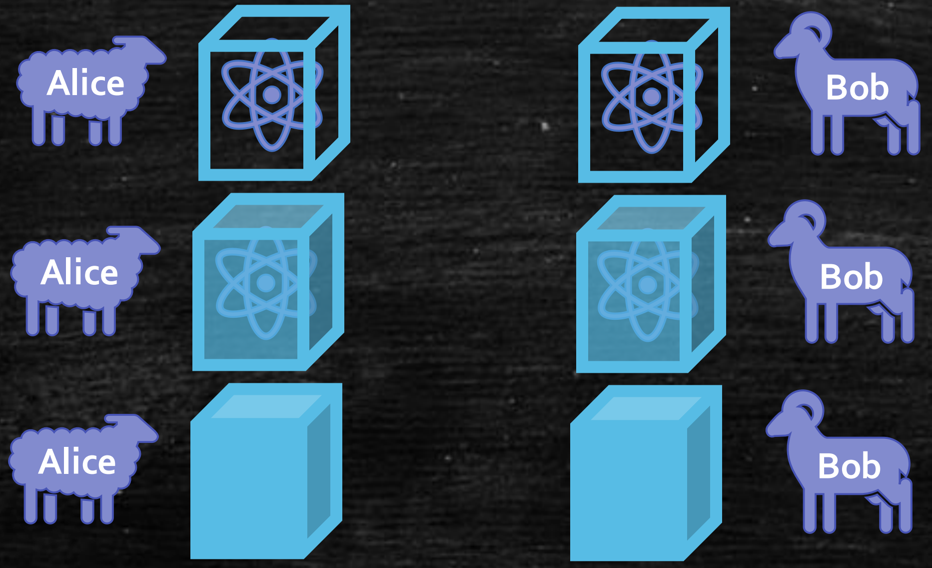

Device dependent — Characterized system
E.g. Known POVMs / state

Semi-device independent — Partially characterized system
E.g. Known dimension

Device independent — Fully uncharacterized system
E.g. We know it is quantum

The entropy optimization problem II

Things look difficult

$$\min_{\text{attacks}} H(Z|E)$$

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Our computers work with limited memory / time…

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Our computers work with limited memory / time…

- Conditional entropy is concave

Not nice (for minimizing)

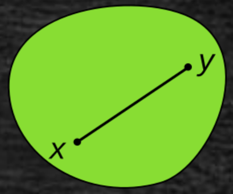
A Tool: Convex Optimisation

Convexity I

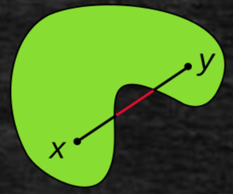
Definition:

A set X is convex if for all $x, y \in X$ and $\lambda \in [0,1]$ we have

$$\lambda x + (1-\lambda) y \in X \, .$$



Convex



Nonconvex

Examples:

- Quantum states

$$\mathcal{S} = \{\rho \;\mid\; \rho \succeq 0, \;\; , \mathrm{Tr}[\rho] = 1\}$$

- Quantum measurements

$$\mathcal{M} = \left\{ (M_1, \ldots, M_n) \;\mid\; \forall i \; M_i \succeq 0, \;\; , \sum_k M_k = \mathbb{I} \right\}$$

Convexity II

Convexity II

Definition:

A function $f : X \to \mathbb{R}$ is convex if for all $x, y \in X$ and $\lambda \in [0,1]$ we have

$$f\left(\lambda x + (1-\lambda)y\right) \leq \lambda f(x) + (1-\lambda)f(y)\,.$$

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Examples:

- Surprisal $-\log(x)$
- Trace norm $\|X\|_1$
- Negative entropy $-H(A|B)$

Convex Optimisation

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

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Theorem:

If $f : X \to \mathbb{R}$ is convex and X is convex, then any *local minimum* is also a global minimum.

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✓

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If $f : X \to \mathbb{R}$ is convex and X is convex, then any *local minimum* is also a global minimum.

Convexity does not imply efficiency though!

Semidefinite programming

Efficiently solvable convex optimisation problems exist:

$$\begin{aligned} \min_X \quad & \mathrm{Tr}[CX] \\ \text{s.t.} \quad & \mathrm{Tr}[F_i X] = b_i \quad \forall i \\ & X \succeq 0 \end{aligned}$$

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<u>Example (Min-entropy):</u>
Given a cq-state

$$\rho_{AE} = \sum_a p(a)|a\rangle\langle a| \otimes \rho_a$$

we have

$$H_{\min}(A|E) = -\log P_{\mathrm{guess}}(A|E)$$

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$$\begin{aligned} P_{\text{guess}}(A|E) = \max_{\{M_a\}_a} \quad & \sum_a p(a)\mathrm{Tr}[M_a \rho_a] \\ \text{s.t.} \quad & \sum_a M_a = \mathbb{I} \\ & M_a \succeq 0 \quad \forall a \end{aligned}$$

Core idea

Try to transform to a convex problem

$$\min_{\text{attacks}} H(Z|E) \quad \Longrightarrow \quad \begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

Core idea

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$$\min_{\text{attacks}} \; H(Z|E) \quad \Longrightarrow \quad \begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

- Guaranteed optima
- Efficient (hopefully)
- Existing numerical software

Device-dependent rates

Hidden convexity

Device-dependent QKD – Eve's attacks

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Eve can manipulate the transmitted state

$$\rho_{Q_A Q_B E}$$

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State at end of round

$$\rho_{ABPE} = (\mathcal{K} \otimes \mathcal{I}_E)(\rho_{Q_A Q_B E})$$

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$$\mathrm{Tr}[(M_{ax} \otimes N_{by})\rho_{Q_A Q_B}] = p(a, b, x, y)$$

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Statistical

$$\mathrm{Tr}[(M_{ax} \otimes N_{by})\rho_{Q_A Q_B}] = p(a,b,x,y)$$

Prepare and measure

$$\mathrm{Tr}_{Q_B E}[\rho_{Q_A Q_B E}] = \rho_{Q_A}$$

DD-QKD key rate problem

How much certifiable randomness does Alice generate in one-round? $\min_{\text{Attacks}} H(A|PE)$

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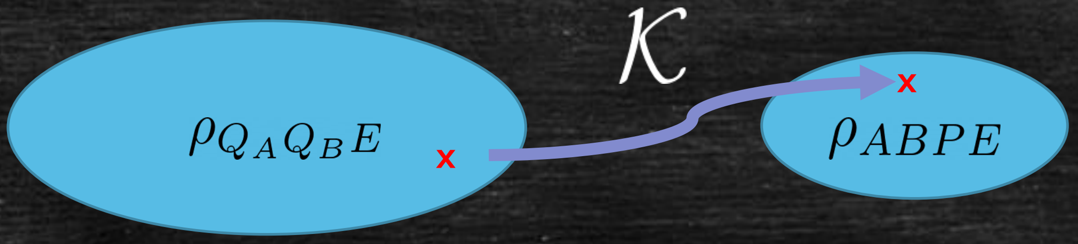

Eve's attacks

Post-measurement states

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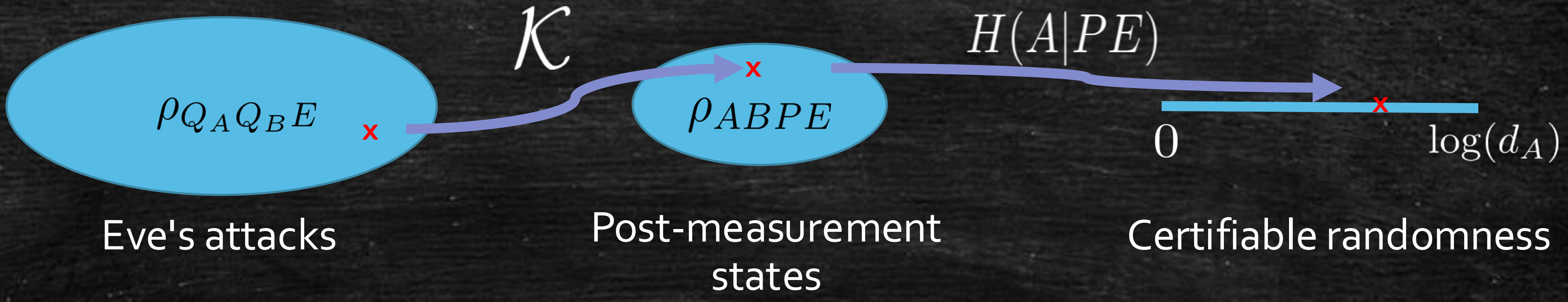

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$$\begin{aligned} \min \quad & H(A|PE) \\ \text{s.t.} \quad & \rho_{ABPE} = \mathcal{K}(\rho_{Q_A Q_B E}) \\ & \rho_{Q_A Q_B E} \succeq 0, \quad \mathrm{Tr}[\rho_{Q_A Q_B E}] = 1 \end{aligned}$$

Post-measurement state

Valid quantum state

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We have more constraints on Eve's attacks!

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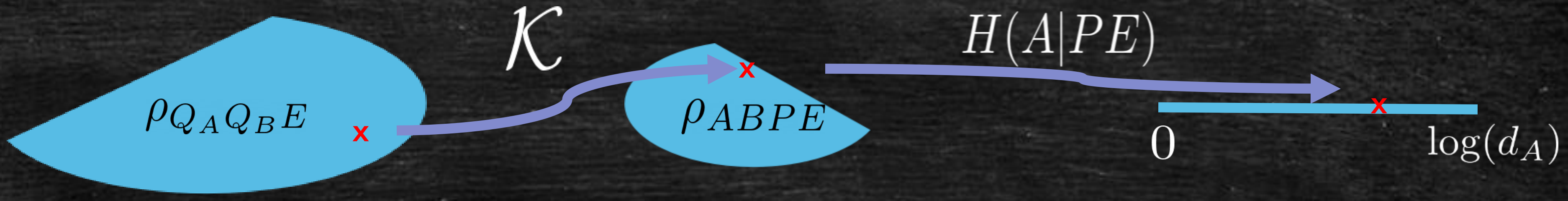

Eve's attacks

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$$\text{s.t.} \quad \rho_{ABPE} = \mathcal{K}(\rho_{Q_A Q_B E})$$ Post-measurement state

$$\rho_{Q_A Q_B E} \succeq 0, \quad \mathrm{Tr}[\rho_{Q_A Q_B E}] = 1$$ Valid quantum state

$$\mathrm{Tr}[(M_{ax} \otimes N_{by})\rho_{Q_A Q_B}] = p(a,b,x,y)$$ Linear constraints on P(a,b,x,y)

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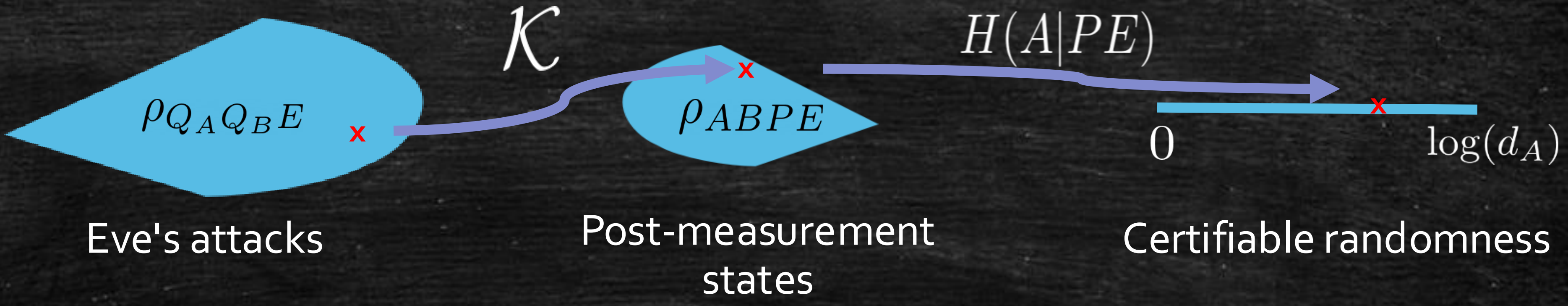

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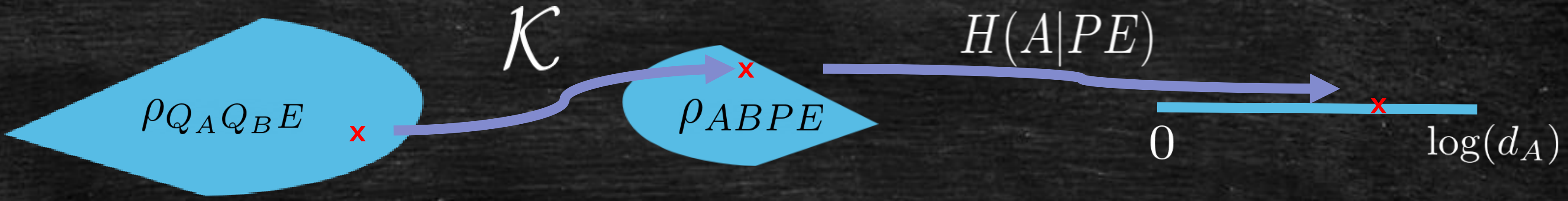

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E dimension a priori unbounded

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Simplification 1:

Eve can always purify!

$$\rho_{Q_A Q_B E} \mapsto |\psi\rangle_{Q_A Q_B EF}$$

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(strong subadditivity)

$$H(A|PEF) \leq H(A|PE)$$

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E dimension a priori unbounded

Minimizing a concave function

<u>Tool: Isometric equivalence of purifications</u>

If $|\psi\rangle_{QE}$ and $|\phi\rangle_{QF}$ are both purifications of ρ_Q. Then there exists an isometry $V: E \to F$ or $V: F \to E$ such that

$$(I_Q \otimes V)|\psi\rangle = |\phi\rangle \quad \text{or} \quad (I_Q \otimes V)|\phi\rangle = |\psi\rangle$$

Simplifying the problem

$$\begin{aligned} \min \quad & H(A|PE) \\ \text{s.t.} \quad & \rho_{ABPE} = \mathcal{K}(|\psi\rangle\langle\psi|_{Q_A Q_B E}) \\ & \mathrm{Tr}[(M_{ax} \otimes N_{by})\rho_{Q_A Q_B}] = p(a,b,x,y) \\ & \mathrm{Tr}_{Q_B E}[|\psi\rangle\langle\psi|_{Q_A Q_B E}] = \sigma \\ & \mathrm{Tr}[|\psi\rangle\langle\psi|_{Q_A Q_B E}] = 1 \end{aligned}$$

E dimension a priori unbounded

Minimizing a concave function

Isometric invariance of entropy $H(A|PE) = H(A|PF)$

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Simplifying the problem

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E dimension a priori unbounded

Minimizing a concave function

Simplification 2: All purifications are equivalent!

Simplifying the problem

$$\min \quad H(A|PE)$$

$$\text{s.t.} \quad \rho_{ABPE} = \mathcal{K}(|\psi\rangle\langle\psi|_{Q_A Q_B E})$$

$$\mathrm{Tr}[(M_{ax} \otimes N_{by})\rho_{Q_A Q_B}] = p(a,b,x,y)$$

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E dimension bounded

Minimizing a concave function

Simplification 2: All purifications are equivalent!

$$|\psi\rangle_{Q_A Q_B E} = \sum_k \sqrt{\rho_{Q_A Q_B}} |k\rangle_{Q_A Q_B} \otimes |k\rangle_E$$

E system completely fixed now

Simplifying the problem

$$\min \quad H(A|PE)$$

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The simpler problem

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Equivalent to a **convex optimization problem**.

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Current approaches

Computing DD key-rates is easy…

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…because some nice people have done the hard work for us.

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Ref.	Method	Language	Code
[OpenQKD]	Franke-Wolfe / Interior point	Matlab	https://openqkdsecurity.wordpress.com/
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Specify protocol → Construct key channel

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Specify protocol → Construct key channel → Pass to a package

Example code (QICS – BB84)

```python
import numpy as np

import picos

qx = 0.25
qz = 0.75

X0 = np.array([[.5,  .5], [ .5, .5]])
X1 = np.array([[.5, -.5], [-.5, .5]])
Z0 = np.array([[1.,  0.], [ 0., 0.]])
Z1 = np.array([[0.,  0.], [ 0., 1.]])

Ax = np.kron(X0, X1) + np.kron(X1, X0)
Az = np.kron(Z0, Z1) + np.kron(Z1, Z0)

# Define problem
P = picos.Problem()
X = picos.SymmetricVariable("X", 4)

P.set_objective("min", picos.quantkeydist(X))
P.add_constraint(picos.trace(X) == 1)
P.add_constraint((X | Ax) == qx)
P.add_constraint((X | Az) == qz)

# Solve problem
P.solve(solver="qics")

analytic_rate = np.log(2) + (qx*np.log(qx) + qz*np.log(qz))
numerical_rate = P.value

print("Analytic key rate: ", analytic_rate)
print("Numerical key rate:", numerical_rate)
```

Device-independent rates

Making convexity

Device-independent QKD – Eve's attacks

Device-independent QKD – Eve's attacks

Eve can manipulate the transmitted state and measurements

$$\rho_{Q_A Q_B E} \qquad \{M_{a|x}\}_a \qquad \{N_{b|y}\}_b$$

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State at end of round

$$\rho_{ABXYE} = \sum_{abxy} p(xy)|abxy\rangle\langle abxy| \otimes \mathrm{Tr}_{Q_A Q_B}[(M_{a|x} \otimes N_{b|y} \otimes \mathbb{I})\rho_{Q_A Q_B E}]$$

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But she is still constrained...

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But she is still constrained…

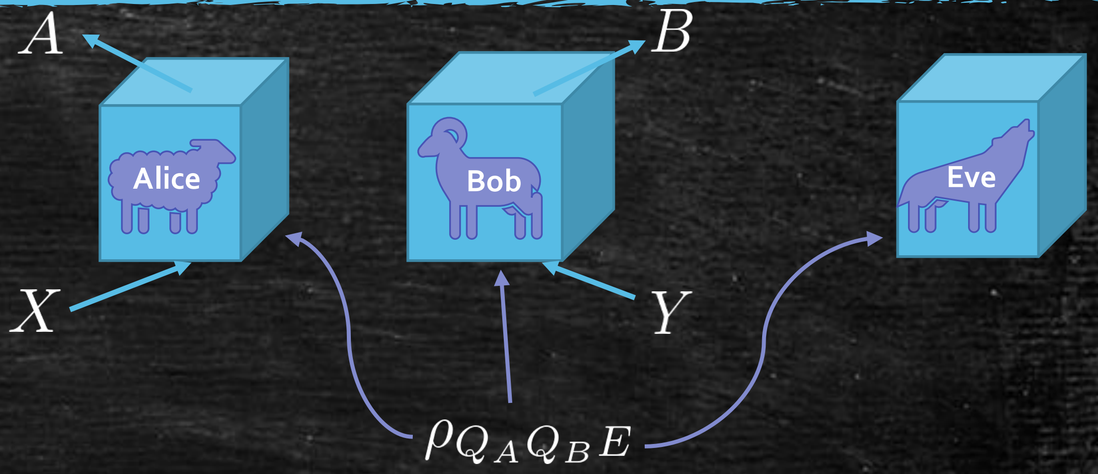
Statistical (e.g. Bell-inequality violation)

$$\mathrm{Tr}[(M_{a|x} \otimes N_{b|y} \otimes I_E)\rho_{Q_A Q_B E}] = p(a,b|x,y)$$

The DI randomness problem

$$\min_{\text{Attacks}} H(A|X=0,E)$$

How much randomness is generated from Alice's device when x=0?



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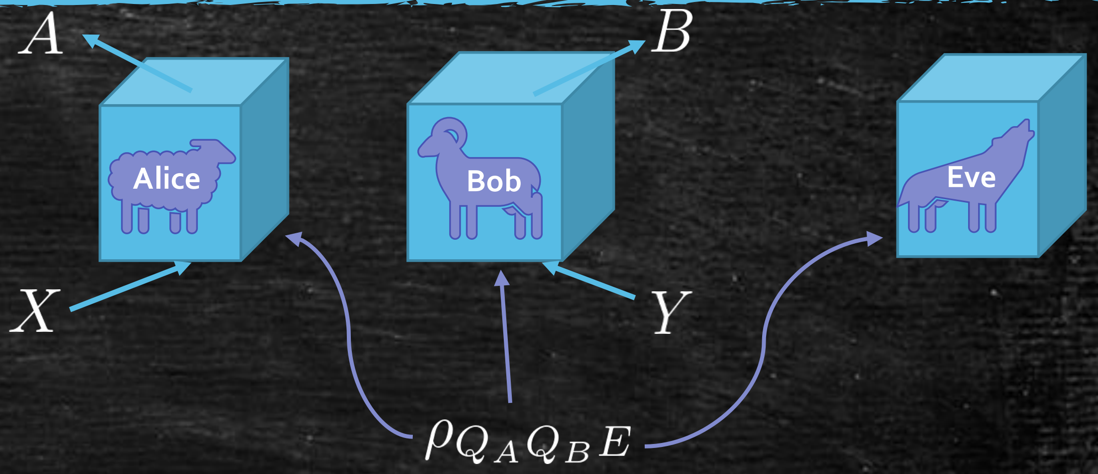

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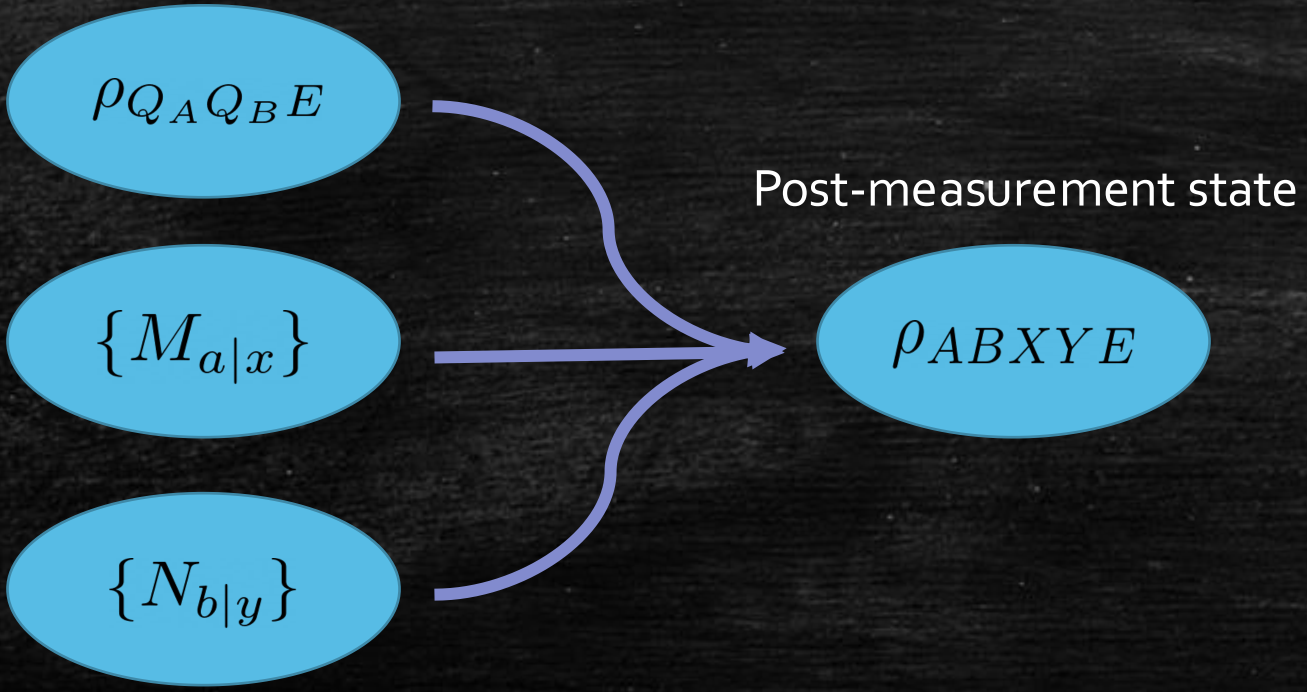

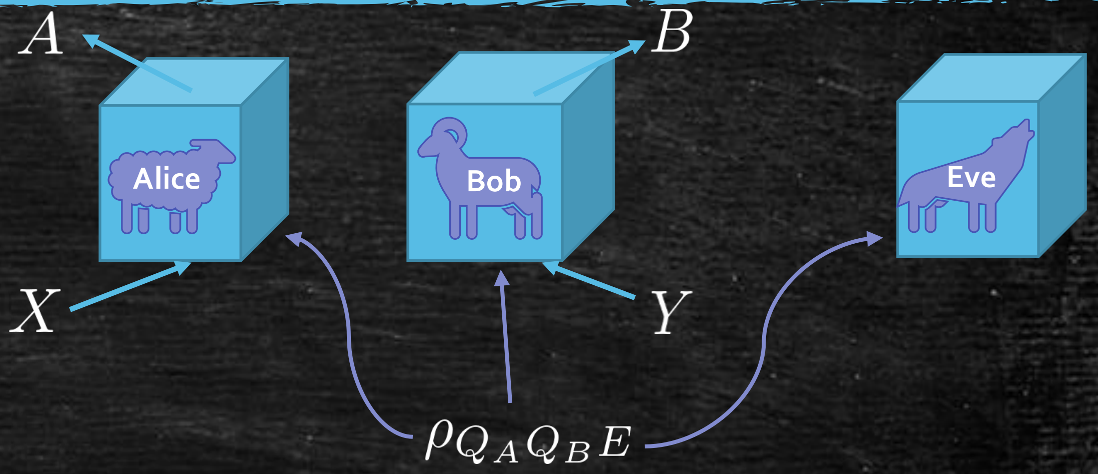

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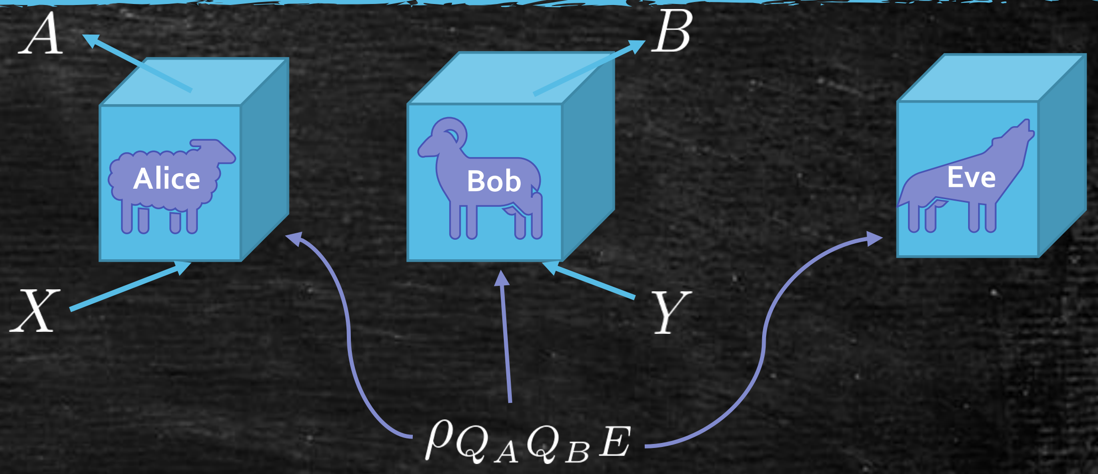

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$\rho_{Q_A Q_B E}$

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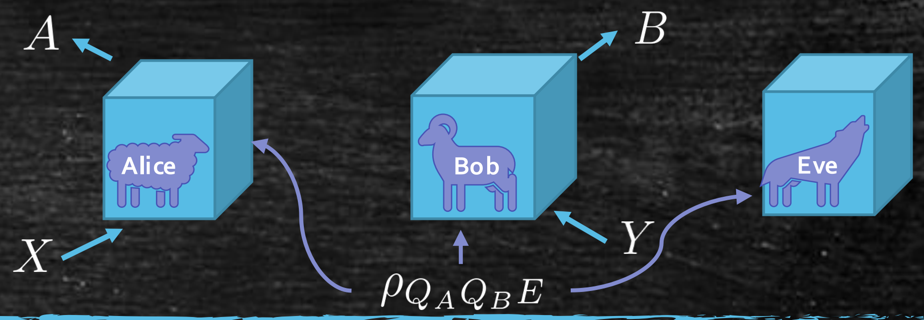
Post-measurement state

ρ_{ABXYE} $\xrightarrow{H(A|X=0,E)}$

0 $\quad$ $\log(d_A)$

The DI randomness problem



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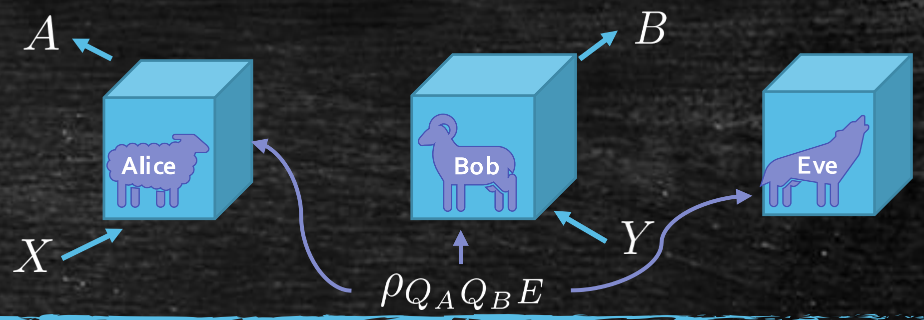

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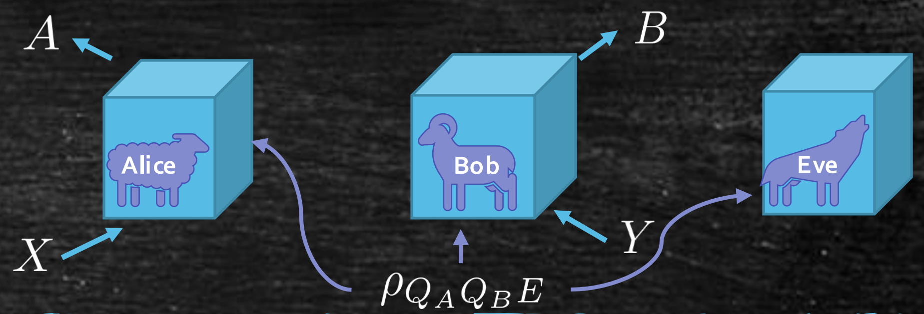

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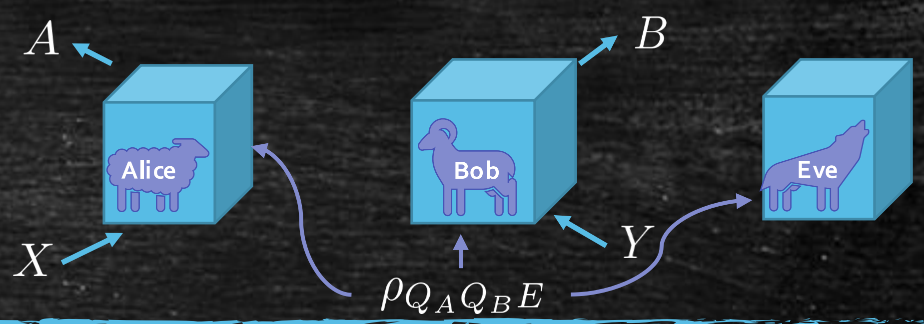

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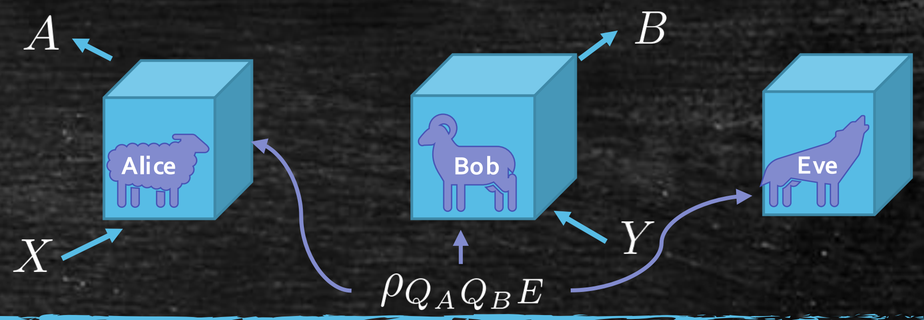

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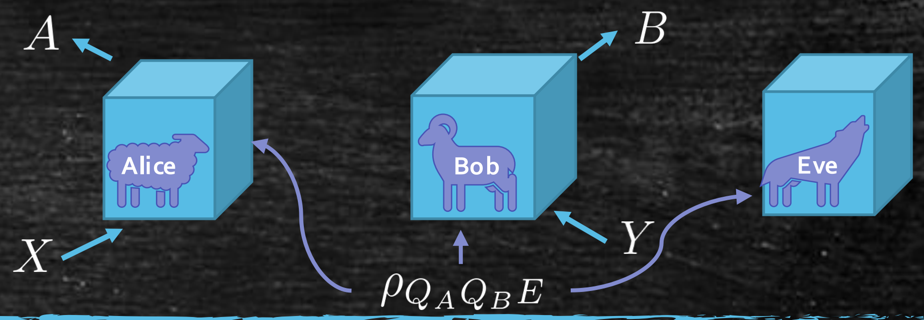

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Can be replaced with *any* linear constraints on p(a,b|x,y).

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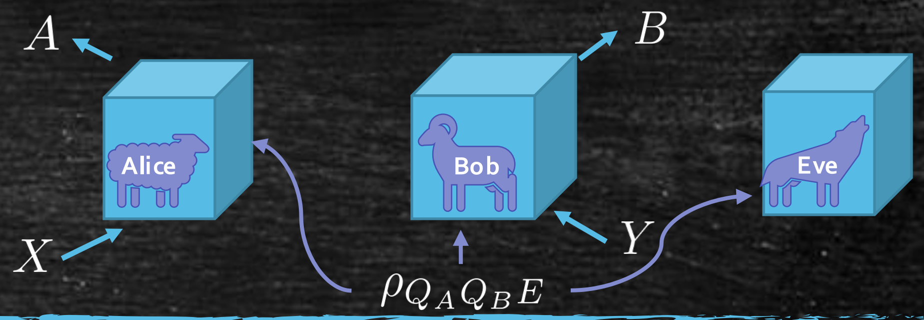

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Dimension of all quantum systems is unbounded…

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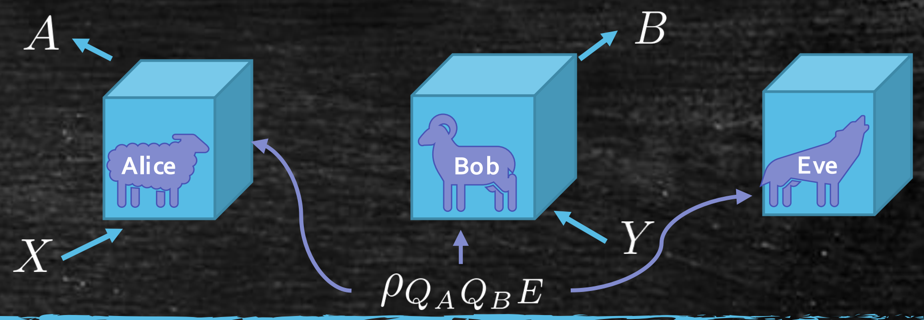

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Dimension of all quantum systems is unbounded…

…need a new perspective

Toolbox: Polynomial optimization [NPA07, PNA10]

Let's look at a simpler problem:

$$
\begin{aligned}
\max \quad & \mathrm{Tr}[\rho(X_1 X_2 + X_2 X_1)] \\
\text{s.t.} \quad & X_1^2 = I, \qquad X_2^2 = I \\
& \mathrm{Tr}[\rho X_1] = \alpha, \qquad \mathrm{Tr}[\rho X_2] = 0 \\
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Need a dimension-free compressed representation of a system...

Toolbox: Polynomial optimization [NPA07, PNA10]

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Need a dimension-free compressed representation of a system…

… re-express everything in terms of moments! $\mathrm{Tr}[\rho X^m]$

Toolbox: Polynomial optimization

Building a moment matrix:

1. Gather the operators of your problem

$$\{I, X_1, X_2\}$$

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Compressed representation of the system
Infinite dimensions -> 3x3

$$\begin{array}{c c} & \begin{array}{ccc} I & X_1 & X_2 \end{array} \\ \begin{array}{c} I \\ X_1 \\ X_2 \end{array} & \begin{pmatrix} \mathrm{Tr}[\rho] & \mathrm{Tr}[\rho X_1] & \mathrm{Tr}[\rho X_2] \\ \mathrm{Tr}[\rho X_1^\dagger] & \mathrm{Tr}[\rho X_1^\dagger X_1] & \mathrm{Tr}[\rho X_1^\dagger X_2] \\ \mathrm{Tr}[\rho X_2^\dagger] & \mathrm{Tr}[\rho X_2^\dagger X_1] & \mathrm{Tr}[\rho X_2^\dagger X_2] \end{pmatrix} \end{array}$$

Toolbox: Polynomial optimization

Building a moment matrix:

1. Gather the operators of your problem

$$\{I, X_1, X_2\}$$

2. Index a matrix with the operators, the (X,Y) entry represents

$$\mathrm{Tr}[\rho X^\dagger Y]$$

$$\begin{aligned} \max \quad & \mathrm{Tr}[\rho(X_1X_2 + X_2X_1)] \\ \text{s.t.} \quad & X_1^2 = I, \qquad X_2^2 = I \\ & \mathrm{Tr}[\rho X_1] = \alpha, \qquad \mathrm{Tr}[\rho X_2] = 0 \\ & \mathrm{Tr}[\rho] = 1, \qquad \rho \succeq 0 \end{aligned}$$

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Infinite dimensions -> 3x3

$$\begin{array}{c|ccc} & I & X_1 & X_2 \\ \hline I & \mathrm{Tr}[\rho] & \mathrm{Tr}[\rho X_1] & \mathrm{Tr}[\rho X_2] \\ X_1 & \mathrm{Tr}[\rho X_1] & \mathrm{Tr}[\rho X_1^2] & \mathrm{Tr}[\rho X_1 X_2] \\ X_2 & \mathrm{Tr}[\rho X_2] & \mathrm{Tr}[\rho X_2 X_1] & \mathrm{Tr}[\rho X_2^2] \end{array}$$

We can simplify!

- Hermitian X

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Compressed representation of the system

Infinite dimensions -> 3x3

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Toolbox: Polynomial optimization

Any **feasible** quantum system has a moment matrix of the form

$$\begin{pmatrix} 1 & \alpha & 0 \\ \alpha & 1 & \mathrm{Tr}[\rho X_1 X_2] \\ 0 & \mathrm{Tr}[\rho X_2 X_1] & 1 \end{pmatrix}$$

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Important: Moment matrices are ***always*** positive semidefinite.

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Not tight
in general

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Semidefinite program (SDP)
- Efficient algorithms
- Guaranteed optima

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Semidefinite program (SDP)
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$$u = \sqrt{1 - \alpha^2}$$

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Achievable!

$$\rho = |0\rangle\langle 0| \qquad X_1 = \alpha\sigma_Z + \sqrt{1-\alpha^2}\,\sigma_X$$
$$X_2 = \sigma_X$$

Semidefinite program (SDP)
- Efficient algorithms
- Guaranteed optima

$$u = \sqrt{1-\alpha^2}$$

Toolbox: Polynomial optimization

- Moment matrix method gives SDP relaxations.

Toolbox: Polynomial optimization

- Moment matrix method gives SDP relaxations.
- Better relaxations possible with larger indexing sets

$$\{I, X_1, X_2, X_1^2, X_1X_2, X_2X_1, X_2^2\}$$

"Level 2" relaxation

Toolbox: Polynomial optimization

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"Level 2" relaxation

- Relaxations converge to actual optima under mild conditions

Bounded operators sufficient

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Bounded operators sufficient

- Applies to very general polynomial problems

$$\begin{aligned} \max / \min \quad & \mathrm{Tr}[\rho\, P(X_1, \ldots, X_n)] \\ & \mathrm{Tr}[\rho\, Q_i(X_1, \ldots, X_n)] \leq q_i \\ & R_j(X_1, \ldots, X_n) \succeq 0 \\ & \rho \succeq 0 \end{aligned}$$

Expectation of polynomial

Equalities and inequalities

Operator constraints

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Expectation of polynomial

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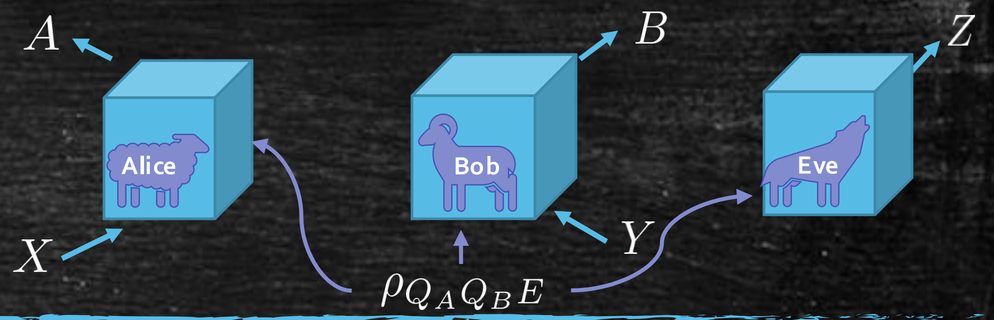
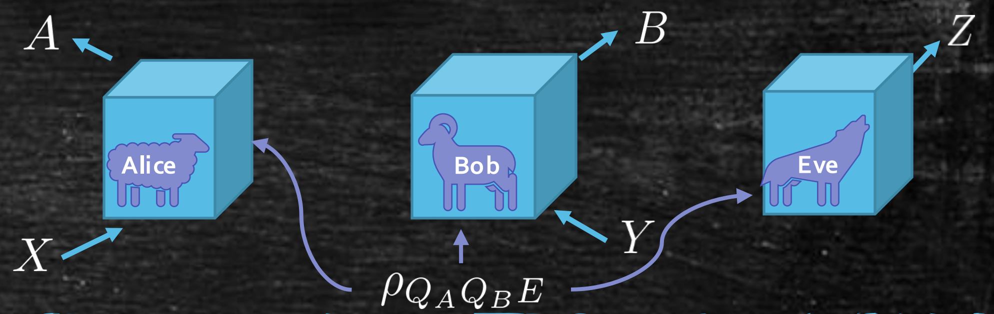
Lots of applications
beyond DI cryptography
Review [TPBA24]

Simple DI key-rate bounds



The von Neumann entropy is complicated, so let's make things easier

Simple DI key-rate bounds



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$$\min_{\text{Attacks}} H(A|X=0,E) \geq \min_{\text{attacks}} H_{\min}(A|X=0,E)$$

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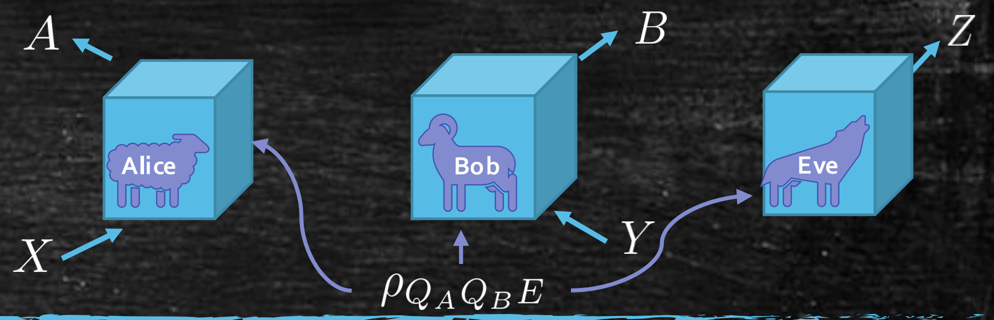

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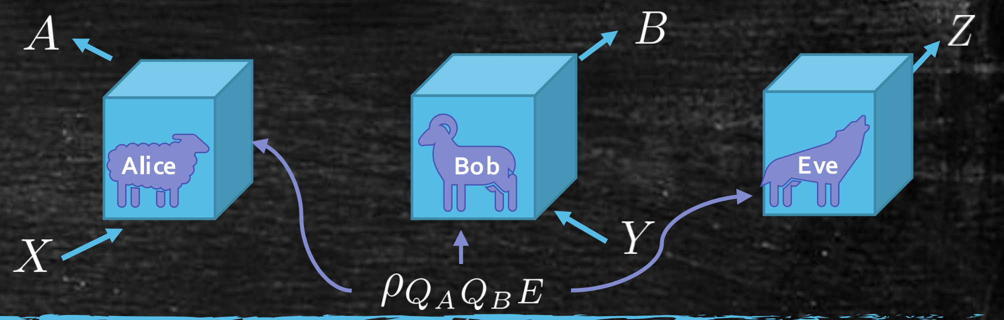

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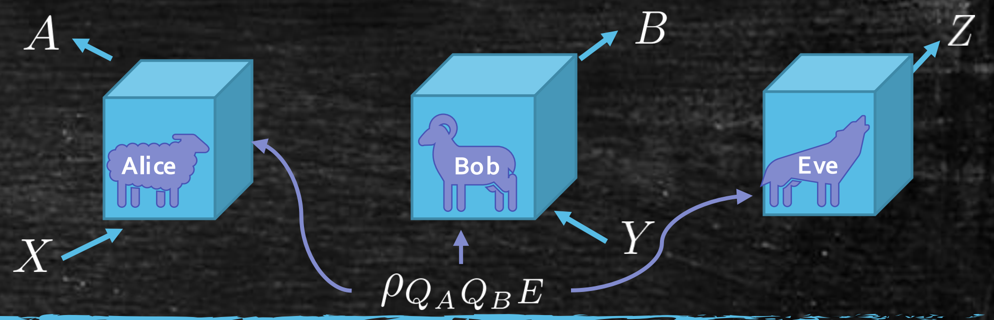

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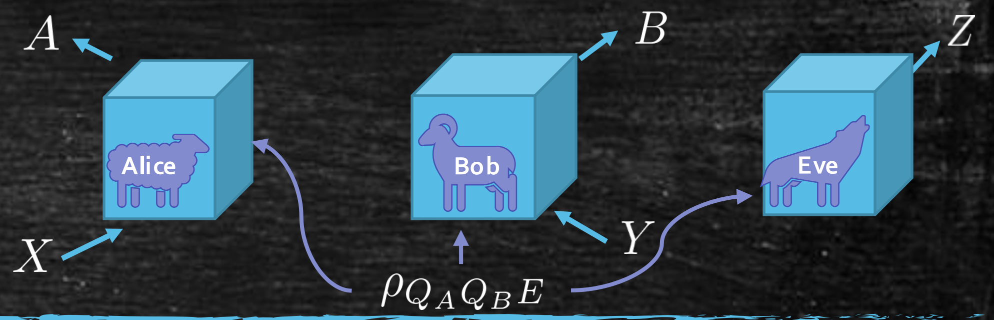
$$= -\log \max_{\text{attacks}} P_{\text{guess}}(A|X=0,E)$$

So we will try to solve

$$\max_{\text{attacks}} P_{\text{guess}}(A|X=0,E)$$

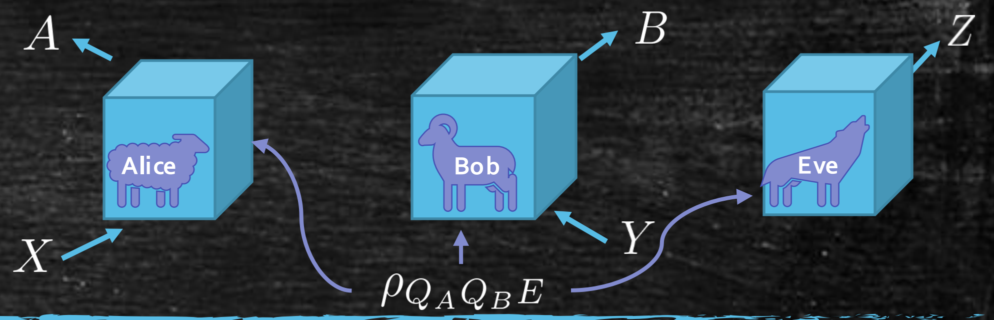
CHSH example



Guessing probability problem

$$
\begin{aligned}
g(\omega) := \max \quad & \sum_a \mathrm{Tr}[(M_{a|0} \otimes I_B \otimes Z_a)\rho_{Q_A Q_B E}] \\
\text{s.t.} \quad & \frac{1}{4} \sum_{a \oplus b = xy} \mathrm{Tr}[(M_{a|x} \otimes N_{b|y} \otimes I_E)\rho_{Q_A Q_B E}] = \omega \\
& \rho_{Q_A Q_B E} \succeq 0, \quad M_{a|x} \succeq 0, \quad N_{b|y} \succeq 0, \quad Z_c \succeq 0 \\
& \mathrm{Tr}[\rho_{Q_A Q_B E}] = 1, \quad \sum_a M_{a|x} = I_A, \quad \sum_b N_{b|y} = I_B, \quad \sum_c Z_c = I_E
\end{aligned}
$$

CHSH example



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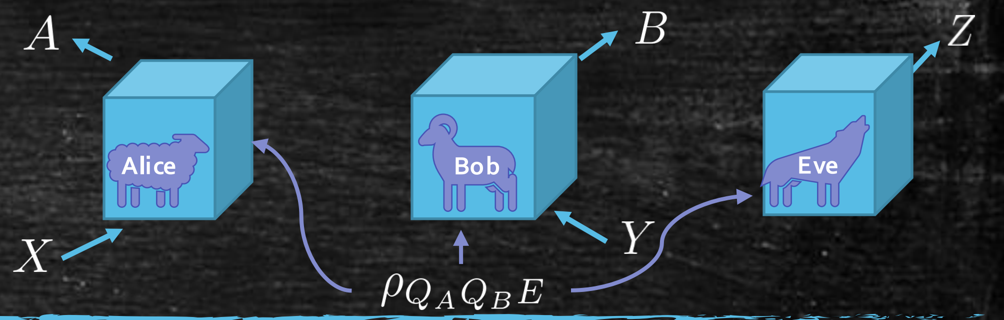
$$\rho_{Q_A Q_B E} \succeq 0, \quad M_{a|x} \succeq 0, \quad N_{b|y} \succeq 0, \quad Z_c \succeq 0$$

$$\mathrm{Tr}[\rho_{Q_A Q_B E}] = 1, \quad \sum_a M_{a|x} = I_A, \quad \sum_b N_{b|y} = I_B, \quad \sum_c Z_c = I_E$$

Not quite a polynomial problem yet

CHSH example



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$$\rho_{Q_A Q_B E} \succeq 0, \quad M_{a|x} \succeq 0, \quad N_{b|y} \succeq 0, \quad Z_c \succeq 0$$

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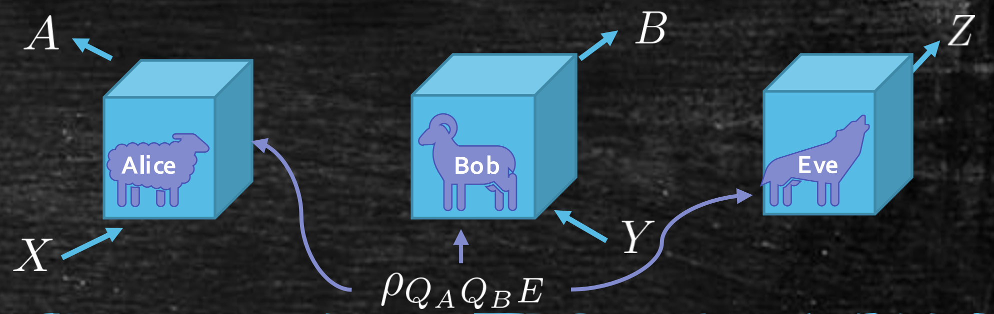
Not quite a polynomial problem yet

Need to "forget" the tensor product

$$\widehat{M}_{a|x} = M_{a|x} \otimes I_{Q_B} \otimes I_E \qquad \hat{N}_{b|y} = I_{Q_A} \otimes N_{b|y} \otimes I_E \qquad \hat{Z}_c = I_{Q_A} \otimes I_{Q_B} \otimes Z_c$$

CHSH example



Guessing probability problem

$$g(\omega) \leq \max \sum_a \mathrm{Tr}[\widehat{M}_{a|0} \hat{Z}_a \rho]$$

Not quite a polynomial problem yet

$$\text{s.t.} \quad \frac{1}{4} \sum_{a \oplus b = xy} \mathrm{Tr}[\widehat{M}_{a|x} \hat{N}_{b|y} \rho] = \omega$$

$$\rho \succeq 0, \quad \widehat{M}_{a|x} \succeq 0, \quad \hat{N}_{b|y} \succeq 0, \quad \hat{Z}_c \succeq 0$$

$$\mathrm{Tr}[\rho] = 1, \quad \sum_a \widehat{M}_{a|x} = I, \quad \sum_b \hat{N}_{b|y} = I, \quad \sum_c \hat{Z}_c = I$$

$$[\widehat{M}_{a|x}, \hat{N}_{b|y}] = 0, \quad [\widehat{M}_{a|x}, \hat{Z}_c] = 0, \quad [\hat{N}_{b|y}, Z_c] = 0$$

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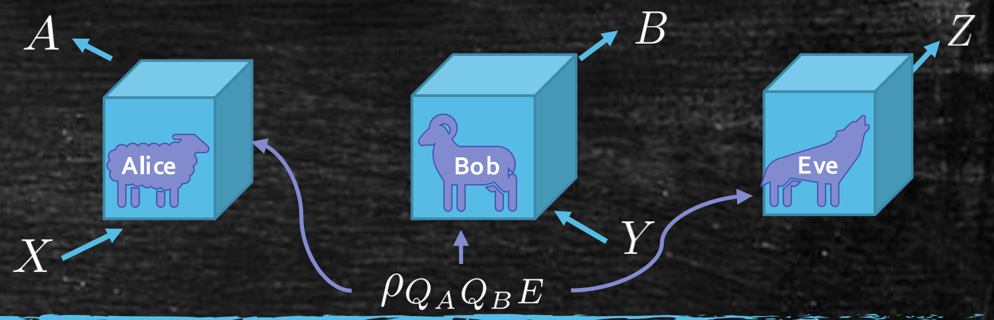
CHSH example



Guessing probability problem

$$g(\omega) \leq \max \sum_a \mathrm{Tr}[\widehat{M}_{a|0} \widehat{Z}_a \rho]$$

$$\text{s.t.} \quad \frac{1}{4} \sum_{a \oplus b = xy} \mathrm{Tr}[\widehat{M}_{a|x} \widehat{N}_{b|y} \rho] = \omega$$

Not quite a polynomial problem yet

$$\rho \succeq 0, \quad \widehat{M}_{a|x} \succeq 0, \quad \widehat{N}_{b|y} \succeq 0, \quad \widehat{Z}_c \succeq 0$$

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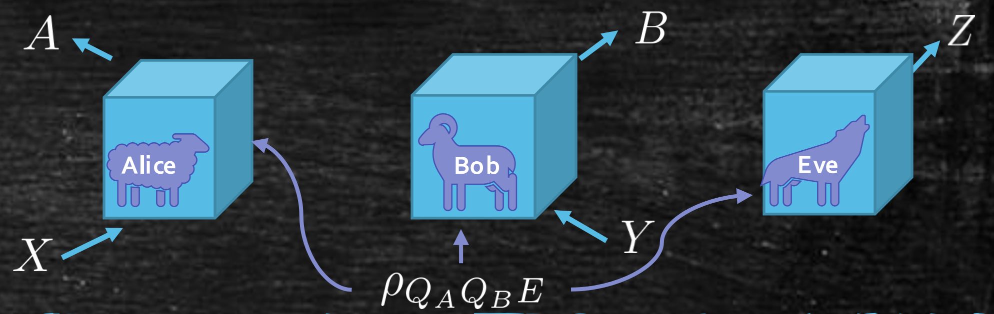

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Now we have a polynomial problem!

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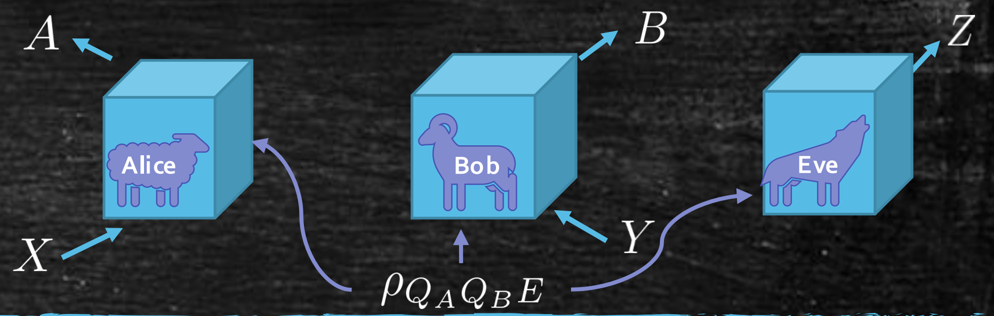

CHSH example

DI rates are also easy to compute

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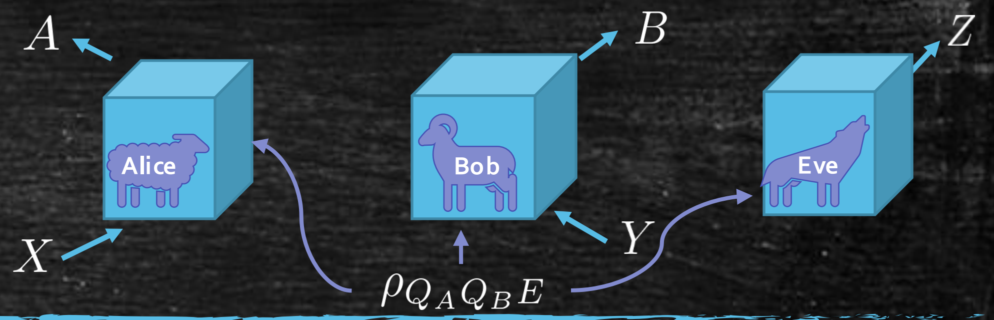

DI rates are also easy to compute

Name	Language	Code
PCPOP.jl	Julia	https://abh1mishra.github.io/PCPOP.jl/dev/
moment	Matlab	https://github.com/ajpgarner/moment
ncpoleon	Python	https://github.com/ncpoleon/ncpoleon

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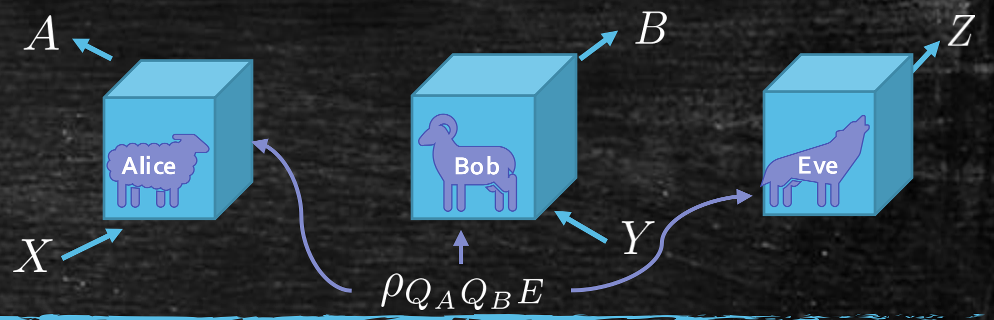

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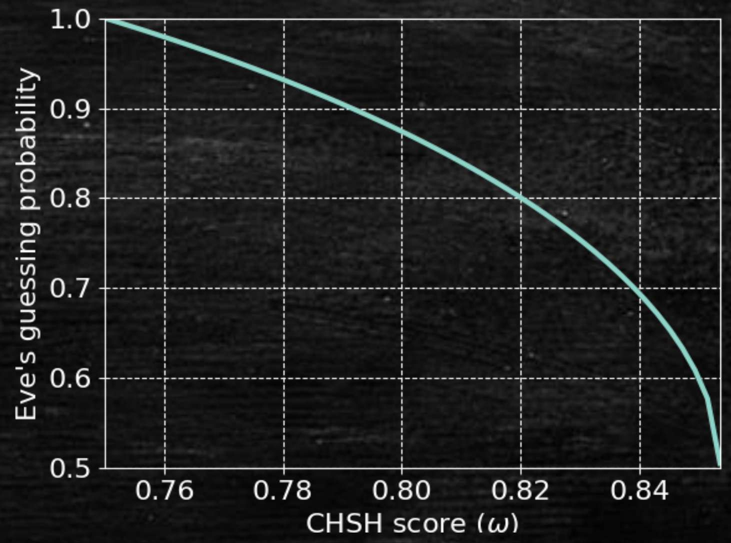
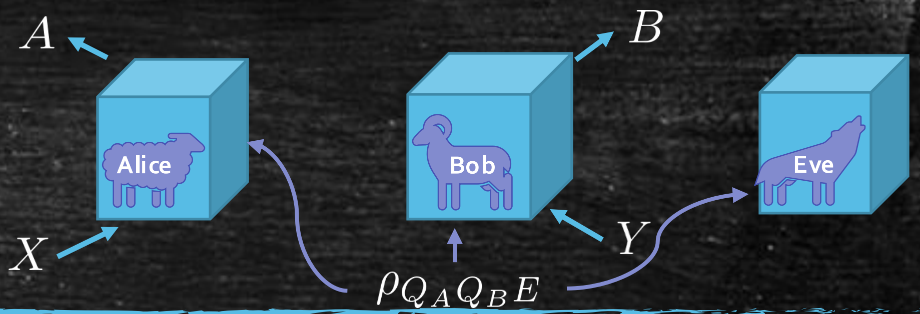
Running the numerics we find:

- Level 1 is not sufficient
- Level 2 gives optimal bound [MPA11]

$$g(\omega) = \frac{1 + \sqrt{2 - (8\omega - 4)^2/4}}{2}$$



Min vs. von Neumann entropy



Guessing probability gives us a min-entropy bound

$$\min_{\text{Attacks}} H_{\min}(A|X=0,E) \geq -\log g(\omega)$$

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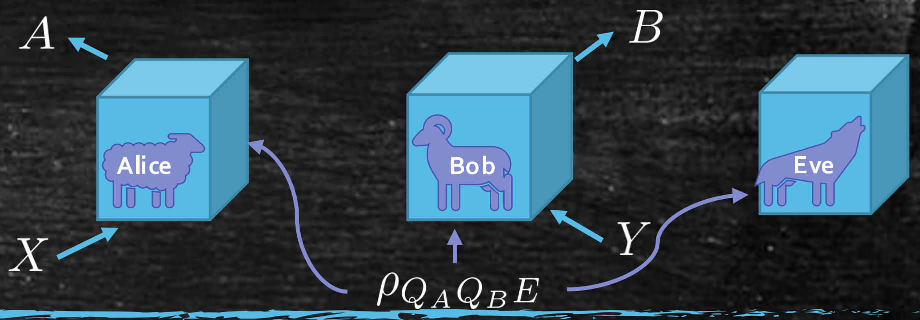

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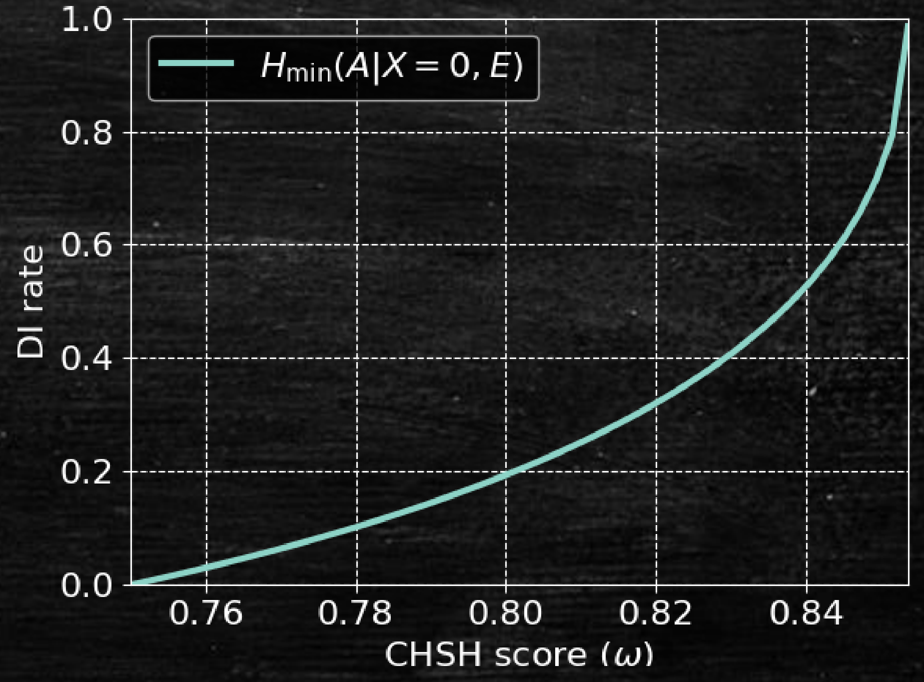

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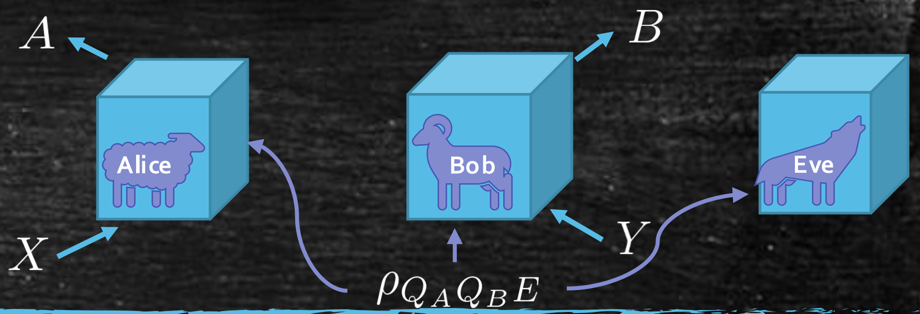

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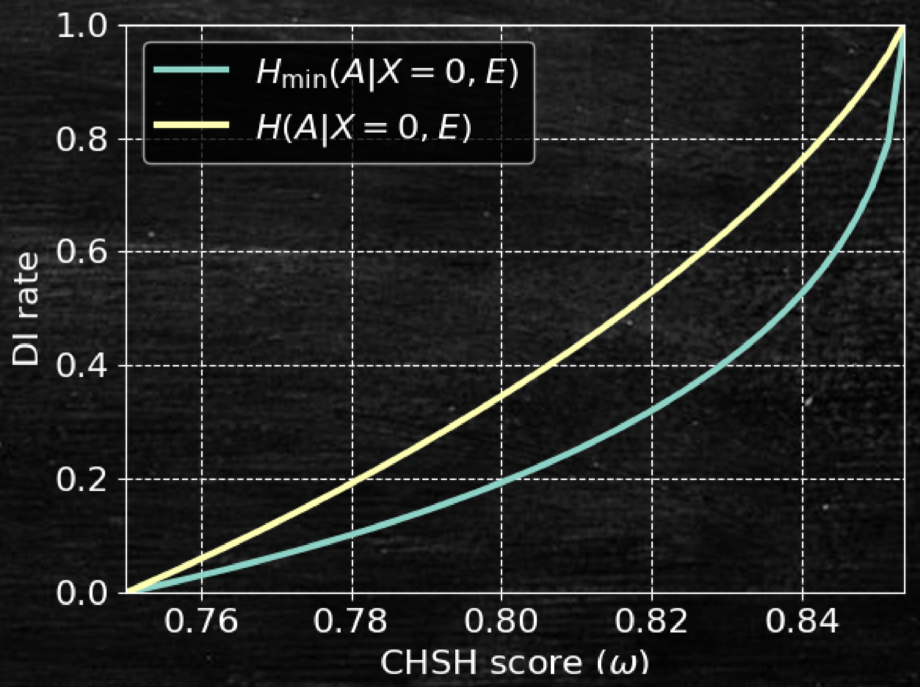
Valid lower bound on our randomness rates

Far from optimal…

$$\min_{\text{Attacks}} H(A|X=0,E) = 1 - h\left(\tfrac{1}{2} + \tfrac{1}{4}\sqrt{(8\omega-4)^2-4}\right)$$

[PABGMS09]



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Goal: Find better numerical bounds on H(A|E)…

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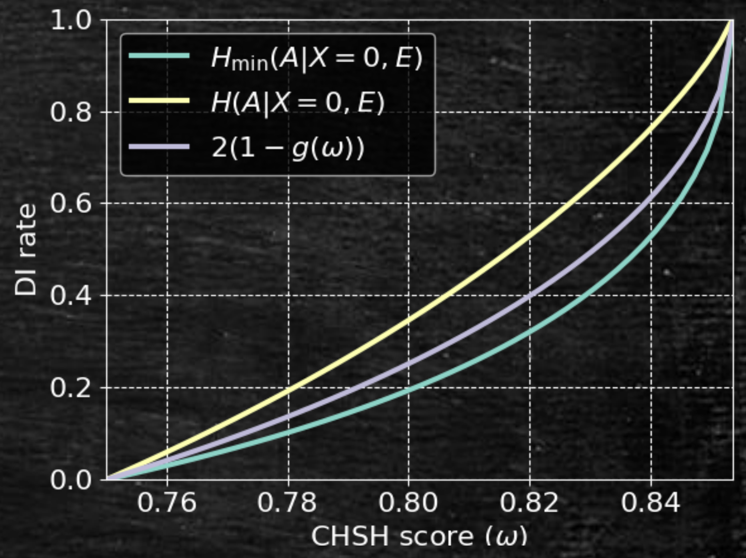
$$H(A|E) \geq 2(1 - P_{\text{guess}}(A|E)) \qquad \text{[BH09]}$$



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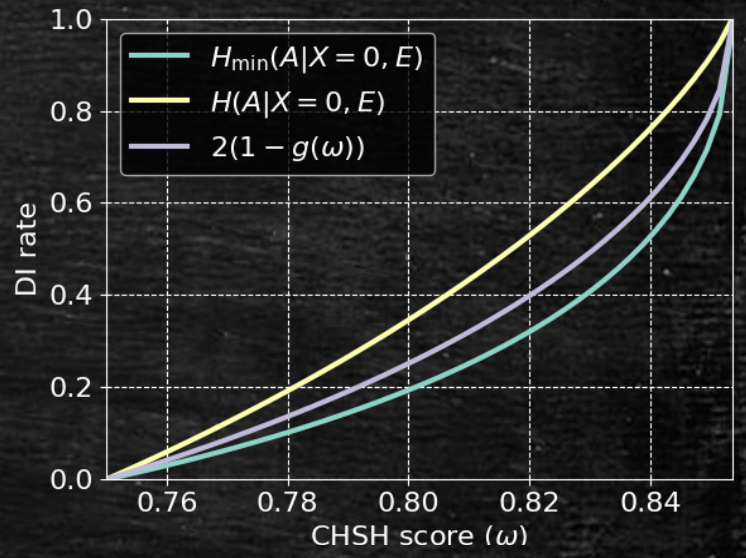

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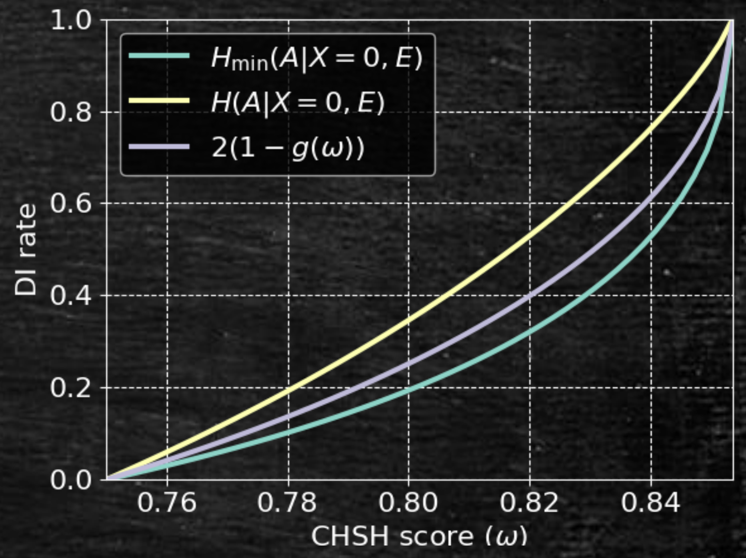

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$$H(A|E) \geq \min \quad \mathrm{Tr}[\rho P(M,N,Z)] \quad \text{s.t.} \quad \ldots$$ [TSGPL21, BFF21, BFF24, KS25]

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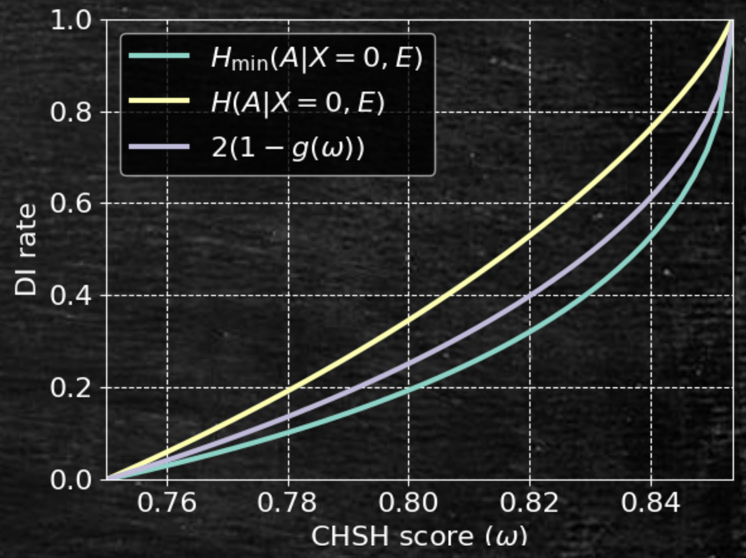

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Quickly too expensive

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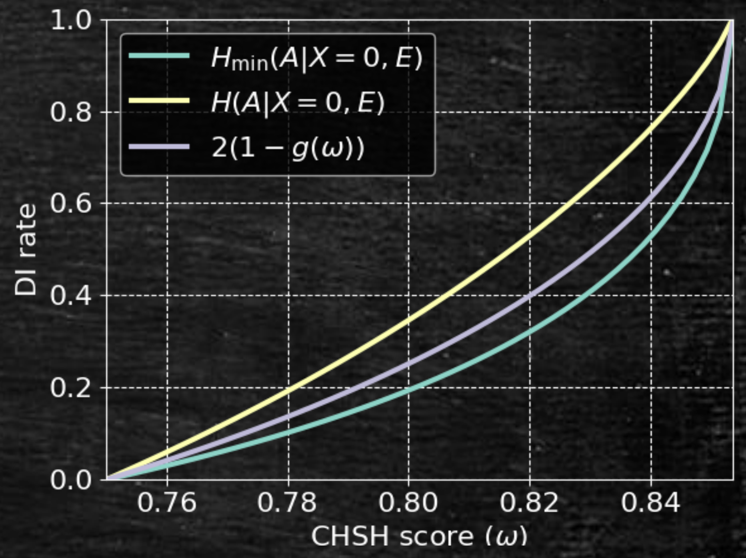

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Hybrid bounds: Analytical reductions + numerics

$$\min_{\text{Attacks}} H(A|E) \longmapsto \min_{\text{Qubit attacks}} H(A|E)$$ [MPW22]

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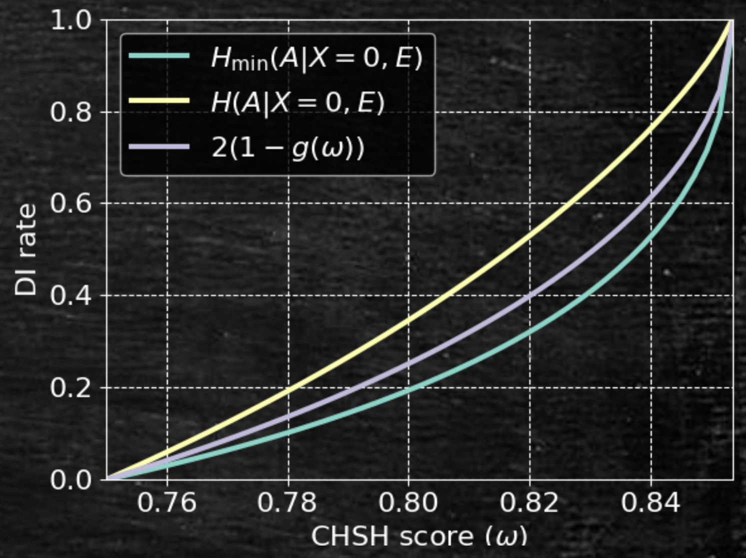

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[TSGPL21, BFF21, BFF24, KS25]

Hybrid bounds: Analytical reductions + numerics

Fast and tight

Limited scope

$$\min_{\text{Attacks}} H(A|E) \longmapsto \min_{\text{Qubit attacks}} H(A|E)$$

[MPW22]

What's next?

Further topics

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- Different assumptions / attacks
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