Exercises for ALPS III

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I. HIGHER-FORM SPTS

In 2+1d, we have a $\mathbb{Z}_2$ classification of SPTs protected by $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(0)}$ symmetry. Here $\mathbb{Z}_2^{(1)}$ is a $\mathbb{Z}_2$ 1-form symmetry and $\mathbb{Z}_2^{(0)}$ is a $\mathbb{Z}_2$ 0-form symmetry.

1. Consider a square lattice with qubits on both the sites v and the edges e. The two $\mathbb{Z}_2$ symmetries are $\prod_v X_v$ and $\prod_{e\in\gamma} \tilde{X}_e$ where γ is a loop (here the $\tilde{X}$ is just to differentiate these degrees of freedom from the vertex ones). We can write one SPT (the trivial one) as $H_A = -\sum_v X_v - \sum_e \tilde{X}_e$. The other SPT takes the form $H_B = -\sum_v A_v - \sum_e B_e$. Here, A_v is the 5-body term $X_v \prod_e \tilde{X}_e$ where we include the four edges emanating from v and B_e is the 3-body term $Z_v \tilde{X}_e Z_{v'}$ where v, v' are the two endpoints of e. This SPT is also known as the 1-form/0-form cluster state

2. Consider H_B with a boundary. Derive the effect boundary representations of the $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(0)}$ symmetry. Show that it is anomalous (this is very similar to the 0+1d case).

3. Insert symmetry defects for the $\mathbb{Z}_2^{(1)}$ symmetry and $\mathbb{Z}_2^{(0)}$ symmetry. How do these defects compare in H_A vs H_B?

II. FILLING FRACTION LSM ANOMALIES WITH FINITE SYMMETRIES

We mentioned in lecture 4 that LSM anomalies are anomalies that involve both internal symmetries and spatial symmetries like translation symmetry. There are two types: projective rep LSM anomalies are related to the internal symmetries forming projective representations, while symmetry sector LSM anomalies cannot be detected by the symmetry defects alone but require knowledge about the particular symmetry sector. Here we will study a simple example of a symmetry sector LSM anomaly in pure states and mixed states.

1. Using the fact that any translation invariant state with nonzero momentum (i.e. $T|\psi\rangle = e^{ik}|\psi\rangle$ for $k \neq 0 \bmod 2\pi$) must be long-range entangled, prove that any state satisfying $\prod_r X_r|\psi\rangle = -|\psi\rangle$ on a periodic spin chain with an even number of qubits must be long-range entangled. *Hint*: try to construct a finite-depth circuit boost operator, which boosts the momentum by π.

2. Find some examples of pure and mixed translation invariant states in the -1 sector of $\prod_r X_r$ and check that your boost operator works as expected on them.

3. Note that there is no anomaly when the spin chain has an odd number of sites. In this case, we have $|\{X_r = +1\}\rangle$ and $|\{X_r = -1\}\rangle$ as SRE translation symmetric states in the ± 1 sector of the $\mathbb{Z}_2$ symmetry. Find a gapped, strongly symmetric parent Lindbladian with steady state space spanned by these two states. What happens when you have the same translation invariant dynamics but add another qubit, so you get an even number of sites? *hint:* think about how we constructed the parent Lindbladian for the GHZ state, and what I mentioned in lecture 3 about the sink.

The last point is interesting because it shows that the steady state space can suddenly change with the addition of one extra site.

III. PURIFICATIONS

We mentioned in lecture 3 that purifications can be useful for studying properties of mixed states. A purification of ρ is an extension to a pure state $|\psi\rangle\rangle$ on a larger Hilbert space $\mathcal{H} \otimes \mathcal{H}_a$ such that $\mathrm{Tr}_a(|\psi\rangle\rangle\langle\langle\psi|) = \rho$. Here we will take a look at the canonical purification $\sum_i \sqrt{p_i}|\psi_i\rangle|\psi_i^*\rangle$ and compare it to other purifications.

1. The strong-to-weak spontaneous symmetry breaking (SWSSB) state $\rho \propto 1 + \prod_r X_r$ demonstrates SW-SSB of the strong $\mathbb{Z}_2$ symmetry $\prod_r X_r$. Write down its canonical purification. What phase of matter (according to the equilibrium pure state classification of phases) does it realize? It should be long-range entangled.

2. However, note that the SWSSB state does have an SRE purification: the 1+1d cluster state. Check that if we consider the spins to be the even sites and the ancillas to be the odd sites, tracing out the ancillas indeed recovers the SWSSB state.

3. All purifications on $\mathcal{H} \otimes \mathcal{H}_a$, where $\mathcal{H}_a$ is the Hilbert space of the ancillas, can be related to each other by unitaries (that may not be locality preserving) acting only in $\mathcal{H}_a$. This is because they simply perform a basis transformation on the ancillas, which we trace out anyways. What is the unitary that relates the canonical purification of the SWSSB state to the cluster state purification described above? *Hint:* think about how to map between their respective phases. *Second hint:* you will need to go beyond finite-depth circuits. Consider sequential circuits, which you may google.

This example is neat because even though the canonical purification is LRE, we can come up with a different purification that is SRE!

IV. NOISE THRESHOLDS

A good proxy for the fidelity correlator is the Renyi-2 correlator $\lim_{|i-j|\to\infty} \mathrm{Tr}(Z_i Z_j \rho Z_i Z_j \rho)/\mathrm{Tr}(\rho^2)$ (i.e. we replace $\sqrt{\rho}$ with ρ) (here we drop the "one-point" functions because, as we showed in lecture 3, they vanish when the state has the strong symmetry). This matches the fidelity correlator for fixed point states like $\rho \propto 1 + \prod_r X_r$, and is usually a good indicator of the true fidelity correlator, but may have a transition at a slightly different point (it has a role similar to Renyi entanglement entropy). The benefit is that it is easier to compute.

1. Consider the decoherence channel $\mathcal{E}_r[\rho] = (1-p)\rho + p Z_r Z_{r+1} \rho Z_r Z_{r+1}$ for $p \in [0, 1/2]$. The full channel is $\bigotimes_r \mathcal{E}_r = \mathcal{E}_{L-1} \circ \mathcal{E}_{L-2} \circ \cdots \circ \mathcal{E}_0$ (i.e. $\mathcal{E}_1 \circ \mathcal{E}_0[\rho] = \mathcal{E}_1[\mathcal{E}_0[\rho]]$). Apply this channel to the product state $|\{X_r = +1\}\rangle$ in 1+1d. Is there a threshold where the Renyi-2 correlator becomes long-ranged and the state becomes SWSSB? *Hint:* go to the doubled Hilbert space and consider how the channel modifies the thermal field double state $|\rho\rangle\rangle = \sum_i p_i |\psi_i\rangle |\psi_i^*\rangle$. Then map the Renyi-2 correlator to a correlation function in a stat mech problem.

2. Now consider the same problem in 2+1d: decohere the 2+1d product state $|\{X_r = +1\}\rangle$ with the same noise channel, now acting on all nearest neighbor pairs of sites on the square lattice. Is there a threshold in this case where the Renyi-2 correlator becomes long-ranged and the state becomes SWSSB?

V. SEPARABILITY TRANSITIONS

Consider a 2 qubit Bell state $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$. We will show that under unital noise (noise taking the state to the infinite temperature state), the quantum entanglement in this state with experience a "sudden death" where it becomes exactly zero.

- Apply an X-noise channel on this state: $\mathcal{E}_r[\rho] = (1-p)\rho + p X_r \rho X_r$ on both sites for $p \in [0, 1/2]$. Compute the logarithmic negativity for this state: $E_N(\rho) = \log \|\rho^{T_1}\|_1$ where T_1 indicates a partial transpose on site 1 and $\|\cdot\|_1$ is the trace norm. The negativity is a good proxy for quantum entanglement, and upper bounds the distillable entanglement. You may do this numerically, i.e. in mathematica. How does it behave as a function of p?

- Now apply both X noise and Z noise at amplitude p: $\mathcal{E} = \mathcal{E}^X \circ \mathcal{E}^Z$. Plot the negativity as a function of p. How does it behave?

- Including both X noise and Z noise should give you a "critical point" where the negativity hits zero and stays at exactly zero. Can you come up with a separable decomposition of ρ, i.e. $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ where the $|\psi_i\rangle$ states are product states for $p \geq p_c$? If you can, you should also be able to obtain the exact expression for p_c. *Hint:* note that the product states need not be orthogonal to each other.