

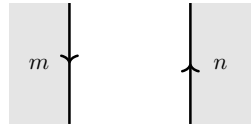
Long exercise

1. Determine the F -symbols of the Fibonacci fusion category Fib , which contains only two simple objects 1 and τ with non-trivial fusion rule $\tau \otimes \tau = 1 \oplus \tau$. Choosing the F -matrix to be real, this is completely determined up to a sign (which is a gauge choice: it is standard to choose the positive real root). Hint: you need to consider only the case where $a = b = c = d = \tau$.
2. For $\mathcal{C} = \text{Fib}$, there exists only the regular module category $\mathcal{M}_{\text{reg}} \cong \text{Fib}$. Determine the vacua and the symmetry action on them.
3. Consider tricritical Ising deformed by the operator we denoted by σ' , which commutes only with the Fib symmetry. The theory flows to a gapped phase which spontaneously breaks the full Fib symmetry (with two vacua v_1, v_τ). To study particles, we usually quantize the theory on the spatial line \mathbb{R} . We model \mathbb{R} by taking the infinite limit of an interval with boundary conditions specified by elements of the module category \mathcal{M}_{reg} , which are in 1-1 correspondence with the vacua v_1, v_τ . The Hilbert space splits accordingly into sectors \mathcal{H}_{ij} , with i the vacuum at $-\infty$ and j the vacuum at $+\infty$. In this case

$$\mathcal{H} = \mathcal{H}_{11} \oplus \mathcal{H}_{1\tau} \oplus \mathcal{H}_{\tau 1} \oplus \mathcal{H}_{\tau\tau}. \quad (5.24)$$

In the literature, it was observed the massive spectrum contains stable particle states in $\mathcal{H}_{1\tau}$ and $\mathcal{H}_{\tau 1}$ (solitons), interpolating between distinct vacua. These can be viewed as a particle/anti-particle pair, so naturally they have the same mass $m = m_s$. More surprisingly, it turns out there is a particle (breather) in $\mathcal{H}_{\tau\tau}$ with the same mass $m = m_p = m_s$! We want to explain this mass degeneracy from the presence of the non-invertible symmetry Fib .

4. Consider a general theory on an interval with topological boundaries labeled by elements $m, n \in \mathcal{M}$



$$(5.25)$$

This defines an Hilbert space \mathcal{H}_{mn} , capturing the states that interpolate from boundary condition m on the left to boundary condition n on the right. A symmetry line $a \in \mathcal{C}$

can act on this via $\mathcal{M} \times \mathcal{C} \rightarrow \mathcal{C}$

$$(5.26)$$

This defines a map $L_{mn}^{rs}(a) : \mathcal{H}_{mn} \rightarrow \mathcal{H}_{rs}$. Determine the composition of two lines $a, b \in \mathcal{C}$ into a single line $c \in \mathcal{C}$ (this is done in terms of the F -symbols and properties of line junctions that we introduced in the lectures):

$$(5.27)$$

This defines an algebra known as the *strip algebra*.

5. Because everything is topological, $L_{mn}^{rs}(a)$ defined above actually commutes with the Hamiltonian in each sector:

$$H_{rs}L_{mn}^{rs}(a) = L_{mn}^{rs}(a)H_{mn}, \quad (5.28)$$

with H_{mn} the Hamiltonian in the \mathcal{H}_{mn} Hilbert space. Take $|\psi\rangle \in \mathcal{H}_{mn}$, an eigenstate of H_{mn} with energy E . Then

$$H_{rs}L_{mn}^{rs}(a)|\psi\rangle = L_{mn}^{rs}(a)H_{mn}|\psi\rangle = E(L_{mn}^{rs}(a)|\psi\rangle). \quad (5.29)$$

However, this does not immediately imply that there is a state of energy E in \mathcal{H}_{rs} , as one has to check that $L_{mn}^{rs}(a)|\psi\rangle \neq 0$, i.e. $|\psi\rangle \notin \ker L_{mn}^{rs}(a)$. At the same time, having an eigenstate $|\phi\rangle \in \mathcal{H}_{rs}$ of H_{rs} with energy E implies a state of the same energy in \mathcal{H}_{mn} only if there exist $|\varphi\rangle \in \mathcal{H}_{mn}$ such that $|\phi\rangle = L_{mn}^{rs}(a)|\varphi\rangle$, so if $|\phi\rangle \notin \text{coker}(L_{mn}^{rs}(a))$.

6. Apply this to the Fib case to show that the degeneracy $m_s = m_p$ is effectively enforced. Hint 0: first consider the allowed junctions (5.26). Hint 1: Consider e.g. $L_{1\tau}^{\tau\tau}(\tau) : \mathcal{H}_{1\tau} \rightarrow \mathcal{H}_{\tau\tau}$. Hint 2: To enforce the mass degeneracy between the solitons and the particle in $\mathcal{H}_{\tau\tau}$ you also need show that the morphism $L_{1\tau}^{\tau\tau}(\tau)$ has trivial kernel. To do so, you can explicitly construct the left (right) inverse to show that $L_{1\tau}^{\tau\tau}(\tau)$ ($L_{\tau\tau}^{1\tau}(\tau)$) is injective (surjective). This is done by considering $L_{\tau\tau}^{1\tau}(\tau)$ ($L_{1\tau}^{\tau\tau}(\tau)$) and computing $L_{\tau\tau}^{1\tau}(\tau) \circ L_{1\tau}^{\tau\tau}(\tau)$ using the algebra (5.27) you previously determined
7. Summarize the degeneracies in a quiver diagram, where the nodes are vacua and the arrows between them are excited particle and soliton states.