HOMEWORK FOR LECTURE 4

*Problems marked * will be discussed during the Exercise session on Thursday.*

1*. Let $\mathcal{V}$ be the big shuffle algebra from Lecture 2 with the shuffle product governed by $\zeta_{ij}(z) = \frac{z-q^{-d_{ij}}}{z-1}$. Define similarly an *integral* big shuffle algebra consisting of color-symmetric Laurent polynomials $\mathcal{V}' = \bigoplus_{\underline{k}\in\mathbb{N}^I} \mathcal{V}'_{\underline{k}}$ with $\mathcal{V}'_{\underline{k}} = \mathbb{C}[\{x_{ia}^{\pm 1}\}_{i\in I}^{1\leq a\leq k_i}]^{sym}$ and shuffle product governed by $\zeta'_{ij}(z) = \frac{1-zq^{d_{ij}}}{(1-z)^{\delta_{ij}}}(-z)^{\delta_{i>j}}$ (to define $\delta_{i>j}$, we pick an order $<$ on I from the start). Construct a graded algebra isomorphism $\mathcal{V}' \simeq \mathcal{V}$.

2. Pick any order on I. Recall the procedure from Lecture 4, where to each ordered monomial $\mu = z_{i_1,\bullet_1}^{-\ell_1} \dots z_{i_n,\bullet_n}^{-\ell_n}$ we assigned the loop word $w_\mu = [i_1^{(k_1)} \dots i_n^{(k_n)}]$ with

$$k_c = \ell_c - \#\{a < c \colon i_a > i_c\} + \#\{b > c \colon i_b < i_c\}.$$

Verify that the lexicographically largest of these w_μ (with respect to all orderings of the given monomial) is a *non-increasing word*, and show that all other w_μ are not non-increasing.

3. Verify that each connected component of graphs G_2, G_3 from Lecture 4 is indeed finite.