

MONOIDAL CATEGORIFICATION OF CLUSTER ALGEBRAS

Let \mathfrak{g} be a simple finite-dimensional complex Lie algebra and $q \in \mathbb{C}^*$ not root of 1.
 For a finite-dimensional representation V of $\mathcal{U}_q(\mathcal{L}\mathfrak{g})$ its q -character is

$$\chi_q(V) \in \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{1 \leq i \leq n, a \in \mathbb{C}^*}.$$

A monomial $m \in \mathcal{Y}$ is called dominant if it is a product of positive powers of $Y_{i,a}$. A representation is called zero (resp. trivial) if it has dimension 0 (resp. 1). Recall that if V is not zero, then $\chi_q(V)$ contains at least one dominant monomial.

- (1) Show that if V is non-zero and $\chi_q(V)$ contains a unique dominant monomial, with multiplicity 1, then V is simple.

Recall that for $\mathfrak{g} = \mathfrak{sl}_2$, a simple representation of dimension 2 has a q -character of the form

$$Y_{1,a} + Y_{1,aq^2}^{-1}$$

for some $a \in \mathbb{C}^*$. There exists a unique simple representation $V_1(a)$ with such a q -character.

- (2) Give an example of a non-zero, non-trivial simple representation V such that $V \otimes V$ is simple.

For the rest of this problem, assume $\mathfrak{g} = \mathfrak{sl}_3$. We have the fundamental representations $V_1(1)$, $V_1(q^2)$, $V_2(q)$ with Drinfeld polynomials $(1-z, 1)$, $(1-q^2z, 1)$, $(1, 1-qz)$, respectively. Consider \mathcal{C} the subcategory of finite-dimensional representations of $\mathcal{U}_q(\mathcal{L}\mathfrak{sl}_3)$ whose image in the Grothendieck ring is in the subring generated by the classes of $V_1(1)$, $V_1(q^2)$, and $V_2(q)$.

- (3) Show that \mathcal{C} is stable under extensions and tensor products.
 (4) Show that its Grothendieck ring $K(\mathcal{C})$ is a polynomial ring in 3 variables.

We admit that $\chi_q(V_1(1)) = Y_{1,1} + Y_{1,q^2}^{-1}Y_{2,q} + Y_{2,q^3}^{-1}$.

- (5) Deduce $\chi_q(V_1(q^2))$ and $\chi_q(V_2(q))$.
 (6) Show that if a simple module V is in \mathcal{C} , then the highest weight monomial in $\chi_q(V)$ is a monomial in $Y_{1,1}$, Y_{1,q^2} , $Y_{2,q}$.
 (7) Show that for a, b natural numbers, the following tensor products are simple :

$$(V_1(1))^{\otimes a} \otimes (V_2(q))^{\otimes b} \quad \text{and} \quad (V_1(q^2))^{\otimes a} \otimes (V_2(q))^{\otimes b}.$$

We admit that $V_1(1) \otimes V_1(q^2)$ has a simple submodule W of dimension 6.

- (8) Show that $V_1(1) \otimes V_1(q^2)$ is a module of length 2.
 (9) Show that W is in \mathcal{C} .

We admit that the tensor product of W with an arbitrary simple representation in \mathcal{C} is simple.

- (10) Give the basis of simple modules of $K(\mathcal{C})$.
 (11) Show that for V_1, \dots, V_N simple objects in \mathcal{C} , the tensor product $V_1 \otimes \dots \otimes V_N$ is simple if and only if for $i \neq j$, $V_i \otimes V_j$ is simple.
 (12) Show that \mathcal{C} is a monoidal categorification of a cluster algebra and show an initial seed.