HOMEWORK FOR LECTURE 3

*Problems marked * will be discussed during the Exercise session on Thursday.*

1. (a*) Verify that the map $\Phi\colon U_q^+(\mathfrak{g}) \to \mathcal{F}$ given by

$$\Phi(x) \;=\; \sum_{i_1,\dots,i_k \in I}^{k\in\mathbb{N}} \left[\prod_{a=1}^{k}(q_{i_a}^{-1} - q_{i_a})\right] \Big\langle x, f_{i_1}\dots f_{i_k}\Big\rangle \cdot [i_1 \dots i_k]$$

is indeed an algebra homomorphism, and deduce that it is actually an embedding.

(b) Show that the image of the embedding Φ consists of all

$$\sum_{i_1,\dots,i_k \in I}^{k\in\mathbb{N}} \gamma(i_1\dots i_k)\cdot [i_1\dots i_k]$$

where the constants $\gamma(i_1\dots i_k) \in \mathbb{C}(q)$ vanish for all but finitely many values of k and satisfy

$$\sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s}_i \gamma\left(w \underbrace{i\;\dots\;i}_{1-a_{ij}-s \text{ symbols}} \; j\; \underbrace{i\;\dots\;i}_{s \text{ symbols}} w'\right) = 0$$

for any $i \neq j \in I$ and any words w, w'.

2*. Verify that the map $\iota\colon \mathcal{S}^+ \to \mathcal{F}^L$ given by

$$\iota(R) \;=\; \sum_{\substack{i_1,\dots,i_k\in I\\ d_1,\dots,d_k\in\mathbb{Z}}}^{k\in\mathbb{N}} \left[\prod_{a=1}^{k}(q_{i_a}^{-1} - q_{i_a})\right] \Big\langle R, f_{i_1,-d_1}\dots f_{i_k,-d_k}\Big\rangle \cdot \left[i_1^{(d_1)}\dots i_k^{(d_k)}\right]$$

$$= \sum_{\substack{i_1,\dots,i_k\in I\\ d_1,\dots,d_k\in\mathbb{Z}}}^{k\in\mathbb{N}} \left[i_1^{(d_1)}\dots i_k^{(d_k)}\right] \cdot \int_{|z_1|\ll\dots\ll|z_k|} \frac{R(z_1,\dots,z_k)z_1^{-d_1}\dots z_k^{-d_k}}{\prod_{1\le a<b\le k}\zeta_{i_a i_b}(z_a/z_b)}$$

is indeed an algebra homomorphism, and deduce that it is actually an embedding.

3. (a) Compute standard Lyndon words for type A_n (as well as B_n, C_n, D_n) with the order $1 < 2 < \ldots < n$ on the underlying alphabet $I = \{1,\dots,n\}$.

(b) Compute standard Lyndon loop words for type $A_2^{(1)}$ (more generally $A_n^{(1)}$) with the order

$$i^{(d)} < j^{(e)} \quad\Longleftrightarrow\quad d > e \quad \text{or} \quad d = e \text{ and } i < j.$$

on the underlying alphabet $\mathcal{I} = \{1^{(d)}, 2^{(d)}\}_{d\in\mathbb{Z}}$ (resp. $\mathcal{I} = \{i^{(d)}\}_{1\le i\le n}^{d\in\mathbb{Z}}$).

(c) Compute standard Lyndon affine words for type $A_1^{(1)}$ (resp. $A_2^{(1)}$) with the order $1 < 0$ (resp. $1 < 2 < 0$) on the underlying alphabet $\widehat{I} = \{0,1\}$ (resp. $\widehat{I} = \{0,1,2\}$).