QUANTUM LOOP ALGEBRA

Exercise 1 -
We consider the loop algebra $\mathcal{L}sl_2 = sl_2 \otimes \mathbb{C}[t^{\pm 1}]$.

(1) Let $a \in \mathbb{C}^*$. Recall how to define for V representation of sl_2 an evaluation representation of $\mathcal{L}sl_2$ denoted by $(V)_a$ (by specialization t to a).

(2) For any $a \in \mathbb{C}^*$, describe the evaluation representation $(V)_a$ of dimension 2 obtained from a simple representation V of sl_2 (that is, give the action on a basis of the representation).

In the next two section we work with these 2-dimensional representations $(V)_a$.

(3) Prove that for $a \in \mathbb{C}^*$, $(V)_a \otimes (V)_a$ is not simple. Is it semi-simple ?

(4) Prove that for any $a \neq b$ in $\mathbb{C}^*$, $(V)_a \otimes (V)_b$ is simple.

We consider the quantum analog of these questions. The quantum parameter $q \in \mathbb{C}^*$ is not a root of unity. We recall the $\mathcal{U}_q(sl_2)$ has generators X, Y, $K^{\pm 1}$ with relations $KE = q^2 EK$, $KF = q^{-2} FK$, $[E, F] = (K - K^{-1})(q - q^{-1})^{-1}$.

We admit there exists a quantum loop algebra $\mathcal{U}_q(\mathcal{L}sl_2)$ generated by certain X_1, X_0, Y_1, Y_0, $k^{\pm 1}$ with a coproduct satisfying

$$\Delta(X_i) = X_i \otimes 1 + K_i \otimes X_i \ , \ \Delta(K_i) = K_i \otimes K_i \ , \ \Delta(Y_i) = Y_i \otimes K_i^{-1} + 1 \otimes Y_i,$$

for $i = 0$ or $i = 1$, where $K_0 = k^{-1}$ and $K_1 = k$.

We admit that for any $a \in \mathbb{C}^*$, there is an evaluation morphism

$$ev_a : \mathcal{U}_q(\mathcal{L}sl_2) \to \mathcal{U}_q(sl_2)$$

such that the images of X_1, X_0, Y_1, Y_0, k are respectively X, aY, Y, $a^{-1}X$, K with X, Y, K, K^{-1} the standard generators of $\mathcal{U}_q(sl_2)$.

(5) Check that ev_a is an algebra morphism but not a Hopf algebra morphism.

(6) For any $a \in \mathbb{C}^*$, describe the evaluation representation $(V_q)_a$ of $\mathcal{U}_q(\mathcal{L}sl_2)$ with V_q simple representation of $\mathcal{U}_q(sl_2)$ of dimension 2.

In the next questions we work with representations of $\mathcal{U}_q(\mathcal{L}sl_2)$ and with the 2-dimensional representations V_q.

(7) For $a, b \in \mathbb{C}^*$, prove that $(V_q)_a \otimes (V_q)_b$ is simple if $ab^{-1} \notin \{q^2, q^{-2}\}$.

(8) Prove that if $ab^{-1} = q^2$ or q^{-2}, then $(V_q)_a \otimes (V_q)_b$ is not simple. Is it semi-simple ?

(9) Are $(V_q)_a \otimes (V_q)_{aq^2}$ and $(V_q)_{aq^2} \otimes (V_q)_a$ isomorphic ?

Exercise 2 -
Let H be a Hopf algebra. For V a finite-dimensional representation of H, we consider the dual representation V^* de V so that

$$(x.u)(v) = u(S^{-1}(x).v) \text{ for } x \in H, v \in V, u \in V^*,$$

where S is the antipode of H. We recall that

$$\mu \circ (S \otimes \mathrm{Id}) \circ \Delta = \mu \circ (\mathrm{Id} \otimes S) \circ \Delta = \eta \circ \epsilon : H \to H$$

with the appropriate notations (Δ is the coproduct, μ is the product, ϵ is the counit and η is the unit).

(1) Explain why this defines a structure of H-module on V^*.

(2) Prove that the assignement $v \otimes f \mapsto f(v)$ defines a morphism of H-module on $V \otimes V^*$.

(3) Prove that $V \otimes V^*$ has a quotient module of dimension 1.

Let $H = \mathcal{U}_q(\mathcal{L}sl_2)$ of the first problem with $S(X_i) = -K_i^{-1} X_i$, $S(Y_i) = -Y_i K_i$, $S(K_i) = K_i^{-1}$ $(i = 0, 1)$.

(4) In the case of $V = (V_q)_a$, prove that $V^* \simeq (V_q)_b$ for a certain complex b. Compute b in terms of a.