HOMEWORK FOR LECTURE 1

*Problems marked * will be discussed during the Exercise session on Tuesday.*

1. (a*) Let A and B be bialgebras over a field $\mathbf{k}$ endowed with a bialgebra pairing

$$\langle\cdot,\cdot\rangle\colon A\otimes B \longrightarrow \mathbf{k}. \tag{1}$$

Recall the definition of the double $D(A,B)$ from Lecture 1 with the defining relations

$$b_2a_2\langle a_1,b_1\rangle = a_1b_1\langle a_2,b_2\rangle \qquad \forall\, a\in A, b\in B. \tag{2}$$

Verify that this definition is equivalent to the more classical one with the defining relations

$$ba = \langle S(a_1),b_1\rangle a_2b_2\langle a_3,b_3\rangle \qquad \forall\, a\in A, b\in B,$$

when A,B are Hopf algebras, whereas $(\Delta\otimes \mathrm{Id})\Delta(a)=a_1\otimes a_2\otimes a_3$ in Sweedler notation.

(b) In the setup of part (a), let $A*B$ denote the free product of A and B. Define elements

$$u_{a,b} = b_2a_2\langle a_1,b_1\rangle - a_1b_1\langle a_2,b_2\rangle \in A*B \qquad \forall\, a\in A, b\in B.$$

Verify the following equalities for all $a,a'\in A$ and $b,b'\in B$:

$$u_{a,bb'} = \langle a_1,b_1\rangle b_2u_{a_2,b'} + \langle a_2,b_2'\rangle u_{a_1,b}b_1', \qquad u_{aa',b} = \langle a_1',b_1\rangle u_{a,b_2}a_2' + \langle a_2,b_2\rangle a_1u_{a',b_1}$$

(*this often allows to impose* (2) *only for generators* $a\in A$ *and* $b\in B$).

2. (a*) Assume now that the pairing (1) is non-degenerate, and let $R\in D(A,B)$ be the canonical tensor of $D(A,B)$, that is $R=\sum_{k\in S} a_k\otimes b_k$ where $\{a_k\}_{k\in S}$ and $\{b_k\}_{k\in S}$ are dual bases of A and B with respect to $\langle\cdot,\cdot\rangle$. Verify the following standard properties of R:

(i) $(\Delta\otimes\mathrm{Id})(R)=R_{13}R_{23}$ and $(\mathrm{Id}\otimes\Delta)(R)=R_{13}R_{12}$ (*cabling identities*),

(ii) $\Delta^{\mathrm{op}}(x)R=R\Delta(x)$ for any $x\in D(A,B)$,

(iii) $R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}$ (*Yang-Baxter equation*).

(b) For any representations (V_k,ρ_{V_k}) over $D(A,B)$, interpret above (i)–(iii) as the properties of the operators $\widehat{R}_{V_1,V_2}=\tau(\rho_{V_1}\otimes\rho_{V_2})(R)\colon V_1\otimes V_2\to V_2\otimes V_1$, where τ is the permutation map. Draw the corresponding commutative diagrams.