HOMEWORK FOR LECTURE 2

*Problems marked * will be discussed during the Exercise session on Tuesday.*

1. Recall the coproduct Δ on the extended shuffle algebras $\mathcal{V}^{\geq}$ and $\mathcal{V}^{\leq}$ defined via

$$\Delta(\varphi_i^{\pm}(z)) = \varphi_i^{\pm}(z) \otimes \varphi_i^{\pm}(z),$$

$$\Delta(R^+) \;=\; \sum_{\mathbf{l}\in\mathbb{N}^I\colon l_i\leq k_i} \frac{\left[\prod_{i\in I}^{a>l_i} \varphi_i^+(z_{ia})\right] * R^+(z_{i,a\leq l_i} \otimes z_{i,a>l_i})}{\prod_{i,i'\in I}\prod_{a\leq l_i}^{a'>l_{i'}} \zeta_{i'i}(z_{i'a'}/z_{ia})} \qquad \forall\, R^+ \in \mathcal{V}^+_{\mathbf{k}},$$

$$\Delta(R^-) \;=\; \sum_{\mathbf{l}\in\mathbb{N}^I\colon l_i\leq k_i} \frac{R^-(z_{i,a\leq l_i} \otimes z_{i,a>l_i}) * \left[\prod_{i\in I}^{a\leq l_i} \varphi_i^-(z_{ia})\right]}{\prod_{i,i'\in I}\prod_{a\leq l_i}^{a'>l_{i'}} \zeta_{ii'}(z_{ia}/z_{i'a'})} \qquad \forall\, R^- \in \mathcal{V}^-_{-\mathbf{k}}.$$

Show that it is indeed a coproduct by verifying:

(a) Δ is coassociative;

(b*) Δ preserves the multiplication.

2*. (a) For a simple Lie algebra $\mathfrak{g}$ recall the pairings

$$\langle\cdot,\cdot\rangle\colon\; \mathcal{V}^+ \otimes \widetilde{U}_q^-(L\mathfrak{g}) \longrightarrow \mathbb{C}(q) \qquad \text{and} \qquad \langle\cdot,\cdot\rangle\colon\; \widetilde{U}_q^+(L\mathfrak{g}) \otimes \mathcal{V}^- \longrightarrow \mathbb{C}(q)$$

given by

$$\left\langle R^+, f_{i_1,-d_1}\dots f_{i_k,-d_k}\right\rangle = \prod_{a=1}^{k}(q_{i_a}^{-1}-q_{i_a})^{-1}\int_{|z_1|\ll\cdots\ll|z_k|} \frac{R^+(z_1,\dots,z_k)z_1^{-d_1}\dots z_k^{-d_k}}{\prod_{1\leq a<b\leq k}\zeta_{i_a i_b}(z_a/z_b)},$$

$$\left\langle e_{i_1,d_1}\dots e_{i_k,d_k}, R^-\right\rangle = \prod_{a=1}^{k}(q_{i_a}^{-1}-q_{i_a})^{-1}\int_{|z_1|\gg\cdots\gg|z_k|} \frac{R^-(z_1,\dots,z_k)z_1^{d_1}\dots z_k^{d_k}}{\prod_{1\leq a<b\leq k}\zeta_{i_b i_a}(z_b/z_a)},$$

for any $k\in\mathbb{N}$, $i_1,\dots,i_k\in I$, $d_1,\dots,d_k\in\mathbb{Z}$, $R^{\pm}\in\mathcal{V}^{\pm}$ (pairings between elements of non-opposite degrees are set to be 0). Verify that these formulas are well-defined, i.e. compatible with the defining relations of $\widetilde{U}_q^{\pm}(L\mathfrak{g})$.

(b) Express the following power series as a sum of two products of δ-functions:

$$\mathrm{Sym}_{z_1,z_2}\frac{z_2-z_1}{z_2-q^2z_1}\left(\frac{1}{w-z_1q^{-1}}\frac{1}{w-z_2q^{-1}}+\frac{q+q^{-1}}{w-z_1q^{-1}}\frac{1}{z_2-wq^{-1}}+\frac{1}{z_1-wq^{-1}}\frac{1}{z_2-wq^{-1}}\right),$$

whereas $\frac{1}{x-y}=\sum_{k\geq 0}x^{-k-1}y^k$ (i.e. the power series expansion of that function in $|y|\ll|x|$).

(c) Use (b) to deduce for simply-laced $\mathfrak{g}$ that the pairings in (a) indeed descend to the pairings

$$\langle\cdot,\cdot\rangle\colon\; \mathcal{S}^+ \otimes U_q^-(L\mathfrak{g}) \longrightarrow \mathbb{C}(q) \qquad \text{and} \qquad \langle\cdot,\cdot\rangle\colon\; U_q^+(L\mathfrak{g}) \otimes \mathcal{S}^- \longrightarrow \mathbb{C}(q).$$

3. Let Δ be the root system of a simple $\mathfrak{g}$ with a polarization $\Delta = \Delta^+ \cup \Delta^-$. Recall that an order $<$ on Δ^+ is *convex* if for any $\alpha < \beta \in \Delta^+$ with $\alpha + \beta \in \Delta^+$, we have $\alpha < \alpha + \beta < \beta$.

(a) Show that for any $\{\gamma_a\}_{a=1}^k \subset \Delta^+$ with $k > 1$ such that $\gamma = \gamma_1 + \ldots + \gamma_k \in \Delta^+$ we have:

$$\min\{\gamma_1, \ldots, \gamma_k\} < \gamma < \max\{\gamma_1, \ldots, \gamma_k\}.$$

Hint: You may want to show first (arguing by induction on k) that one can choose a permutation $\sigma \in S(k-1)$ *such that* $\gamma_k + \gamma_{\sigma(1)} + \ldots + \gamma_{\sigma(a)} \in \Delta^+$ *for any* $1 \leq a \leq k-1$.

(b) Show that if $\alpha < \beta$ is a *minimal pair*, i.e. $\nexists\, \alpha < \alpha' < \beta' < \beta$ with $\alpha' + \beta' = \alpha + \beta$, then

$$\nexists\, \alpha < \gamma_1 < \ldots < \gamma_k < \beta \qquad \text{satisfying} \qquad \gamma_1 + \ldots + \gamma_k = \alpha + \beta \quad \text{and} \quad k > 1.$$

(c) Use Melançon's lemma from Lecture 2 to show that if $\ell_1, \ell_2, \ell_1\ell_2$ are standard Lyndon words then $\deg(\ell_1) < \deg(\ell_2)$ form a minimal pair w.r.t. lex.order, i.e. $\nexists\, \ell_1 < \ell_1' < \ell_2' < \ell_2$ with standard Lyndon words ℓ_1', ℓ_2' satisfying $\deg(\ell_1) + \deg(\ell_2) = \deg(\ell_1') + \deg(\ell_2')$.