ON DOUBLE SHUFFLE RELATIONS BETWEEN MULTIPLE ZETA VALUES AND ASSOCIATORS

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MZVs: The numbers $\zeta(n_1,\dots,n_s) := \sum_{k_1>\dots>k_s} k_1^{-n_1}\cdots k_s^{-n_s}$ $n_1>1,\ n_2,\dots,n_s\geq 1$.

Sets of relations: (a) associator relations w. parameter $2\pi i$ (Drinfeld 89, Le-Murakami 95) based on study of the KZ differential system (b) double shuffle & regularization w. parameter $2\pi i$ based on alternative expression of MZVs as iterated integrals. for $\underline{k}\supset\mathbb{Q}$ ring and $\mu\in\underline{k}$:

$$\left\{\begin{array}{c}\text{Solutions of}\\ \text{associator}\\ \text{relations w.}\\ \text{parameter } \mu\end{array}\right\} \subset \left\{\begin{array}{c}\text{Solutions of the}\\ \text{double shuffle}\\ \text{equations w.}\\ \text{parameter } \mu\end{array}\right\} \subset \left\{(\zeta^{reg}(n_1,\dots,n_s))\in \underline{k}^{\{(n_1,\dots,n_s)\}}\right\}$$

$$\downarrow \qquad\qquad \downarrow$$

$$M_\mu(k) \qquad\qquad DMR_\mu(k)$$

- BE, HFurusho Selecta Math, 2023 (earlier proved in H. Furusho 11)

These sets are torsors under actions of subgroups $GRT_1(k)$ and $DMR_0(k)$ of group $(\exp(\mathrm{Lie}\langle e_0,e_1\rangle)^\wedge, \circledast)$ where

$$g(e_0,e_1)\circledast h(e_0,e_1) = h(ge_0g^{-1},e_1)g(e_0,e_1)$$

A almost-isomorphism $(\exp(\mathrm{Lie}\langle e_0,e_1\rangle)^\wedge,\circledast) \to TAut_2 = \{\alpha\in \mathrm{Aut}(\mathrm{Lie}(e_0,e_1)^\wedge) \mid \alpha(e_1)=e_1,\ \alpha(e_0)\sim e_0\}$

Inclusion of subgroups of $(\exp(\mathrm{Lie}\langle e_0,e_1\rangle)^\wedge,\circledast)$ underlying •: $GRT_1(k) \overset{\bullet}{\hookrightarrow} DMR_0(k)$

2 ingredients for •: ✳ Deligne-Terasoma 05 on a geometric viewpoint on double shuffle

✳ Dror's 1998 paper based on Drinfeld's 1990 paper on associators

Goal: explain inclusion • based on ✳✳ + sketch some applications of ✳

$DMR_\mu(k)$

$\{\phi\in k\langle\langle e_0,e_1\rangle\rangle \mid \phi \text{ sats (a)(b)(c)}\} \xrightarrow{\sim} DMR_\mu(k)$

$$\begin{array}{ccc} \underline{k}\langle\langle e_0,e_1\rangle\rangle & \to & \underline{k}^{\{(n_1,\dots,n_s)\mid n_1>1,\ n_2,\dots,n_s\geq 1\}} \\ \cup & & \cup \\ \Phi & \mapsto & \{\zeta_\Phi(e_0^{n_1-1}e_1\cdots e_0^{n_{k-1}}e_1) := (\Phi\mid e_0^{n_1}e_1\cdots e_0^{n_{k-1}}e_1)\} \end{array}$$

(a): $\Delta(\Phi)=\Phi\otimes\Phi$ where $\Delta: k\langle\langle e_0,e_1\rangle\rangle \to k\langle\langle e_0,e_1\rangle\rangle^{\otimes 2}$ st e_0, e_1 primitive

(b): $\mathrm{TT}(\Gamma_\Phi(-e_1)\Phi)\in \mathfrak{g}(W,\Delta_W)$

where $\Gamma_\Phi(-e_1) := \exp(\sum_{n\geq 1} -\frac{1}{n}(\Phi\mid e_0^{n-1}e_1)e_1^n)$

where $W = k + k\langle\langle e_0,e_1\rangle\rangle e_1 \subset k\langle\langle e_0,e_1\rangle\rangle$,

where $\pi: k\langle\langle e_0,e_1\rangle\rangle \to W$ is the map

$k\langle\langle e_0,e_1\rangle\rangle \twoheadrightarrow k\langle\langle e_0,e_1\rangle\rangle/k\langle\langle e_0,e_1\rangle\rangle e_0 \xrightarrow{\sim} W$ and $\Delta^W: W\to W\otimes W$ induced by $W = k\langle\langle y_1,y_2\cdots\rangle\rangle$ $(y_i := -e_0^{i-1}e_1)$ and $\Delta^W(y_n) = y_n\otimes 1 + 1\otimes y_n - \sum_{k=1}^{n-1} y_k\otimes y_{n-k}$

(c) $(\Phi\mid e_0e_1) = \mu^2/24$

③ **The group $DMR_0(k)$** $\mathcal{G} := \{ g \in \exp(\mathrm{Lie}(e_0,e_1)) \mid (g|e_0)=(g|e_1)=0 \} \subset (\exp(\mathrm{Lie}(e_0,e_1)), \otimes)$

$\mathcal{G} \times k[[t]]^\times$ acts on $\hat{W} \subset k\langle\langle e_0,e_1\rangle\rangle$ by $g(e_0) \bullet w(e_0,e_1) := w(geg^{-1}, e_1)$ $f(t) \bullet w(e_0,e_1) := f(e_1)\, w(e_0,e_1)\, f(e_1)^{-1}$

$\Rightarrow$ acts on $\mathrm{Hom}_{k\text{-alg}}(\hat{W}, \hat{W}^{\otimes 2}) \ni \hat{\Delta}^w$ — Thm (EFurusho 18) $DMR_0(k) = \mathrm{stab}_{\mathcal{G} \times k[[t]]^\times}(\hat{\Delta}^w)$

Gp morphism $\mathcal{G} \to k[[t]]^\times$, $g \mapsto \Gamma_g(t)^{-1}$ $\Rightarrow$ $GRT_1(k) \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{G} \times k[[t]]^\times \curvearrowright DMR_0(k)$

④ **Construction of Δ^w in terms of braids** (Following Deligne-Terasoma)

Spaces: $M_{0,n} := \{$curves of genus 0 w. n marked points$\}$ // $pr_i : M_{0,n} \to M_{0,n-1}$ $i=1,\dots,n$: forget i //

$pr_{ij} : M_{0,n} \to M_{0,n-1}^2 : pr_{ij}(x) = (pr_i(x), pr_j(x))$ // $\ell : M_{0,4} \supset M_{0,4}^{\mathbb{R}} \to M_{0,5}$ // $M_{0,4} \simeq \mathbb{P}^1 \setminus 3\,\mathrm{pts}$

Groups: $P_n := \pi_1 M_{0,n}$ // $pr_i : P_n \to P_{n-1}$ // $pr_{ij} : P_n \to P_{n-1}^2$ // $\ell : P_4 \to P_5$ // $P_4 \simeq F_2 = \langle X_0, X_1 \rangle$.

Tensor functors $F : \mathrm{Gps} \to \mathrm{HopfAlg}$: $F_B(\Gamma) = k\Gamma$ (gp algebra) // $F_{DR}(\Gamma) := \mathrm{gr}\, k\Gamma = \bigoplus_{i \geq 0} (k\Gamma)_+^i / (k\Gamma)_+^{i+1}$

(graded Hopf alg, $(k\Gamma)_+ =$ augmentation ideal)

Fact: $M := \lim_{\mathrm{ker}} (F(pr_3) \otimes F(pr_{12}) : F(P_5) \to F(P_4)^{\otimes 2})$ is a $F(P_4)$-$F(P_4)^{\otimes 2}$-bimodule $\Rightarrow$ alg morph $F(P_4) \xrightarrow{\rho_F} \mathrm{End}_{F(P_4)^{\otimes 2}\text{-mod}}(M)$

$F(P_4) \ni x_F := \begin{cases} X_1 - 1 \in kP_4 & \text{if } F = F_B \\ [X_1 - 1] \in \mathrm{gr}_1 kP_4 & \text{if } F = F_{DR} \end{cases}$

Fact: $\exists$ morphisms in $F(P_4)^{\otimes 2}$-mod $\mathrm{row}_F : M \to F(P_4)^{\otimes 2}$, $\mathrm{col}_F : F(P_4)^{\otimes 2} \to M$ st $\rho_F(x_F) = [M \xrightarrow{\mathrm{row}_F} F(P_4)^{\otimes 2} \xrightarrow{\mathrm{col}_F} M]$

Fact: $\exists$ algebra diagram

$$\begin{array}{ccccc} & & a x_F 1 & \longmapsto & \mathrm{row}_F \circ \rho_F(a) \circ \mathrm{col}_F \\ & & \cap & & \cap \\ F(P_4) & \hookrightarrow & F(P_4)_{x_F} & \to & \mathrm{End}_{F(P_4)^{\otimes 2}\text{-mod}}(F(P_4)^{\otimes 2}) \\ \Delta_F \downarrow & & & & \wr \\ (k + F(P_4)_{x_F})^{\otimes 2} & \hookrightarrow & & & F(P_4)^{\otimes 2} \end{array}$$

Fact: (1) $\Delta_{F_{DR}} = \Delta^w : k \oplus k\langle\langle e_0,e_1\rangle\rangle e_1 \to (k \oplus k\langle\langle e_0,e_1\rangle\rangle e_1)^{\otimes 2}$

(2) $\Delta_{F_B} : k + kF_2(X_1 - 1) \to (k + kF_2(X_1-1))^{\otimes 2}$ st $\mathrm{gr}_1(\Delta_{F_B}) = \Delta_{F_{DR}}$

⑤ **Sketch of proof of $GRT_1(k) \subset DMR_0(k)$** $\forall g \in GRT_1(k)$, g acts by automs of $F_{DR}(P_n)$, $F_{DR}(pr_i)$, $F_{DR}(\ell)$

$\Rightarrow (g, \Gamma_g(-t)) \in \mathrm{Stab}_{\mathcal{G} \times k[[t]]^\times}(\hat{\Delta}^w) = DMR_0(k)$ (detailed argument)

Sketch of proof of $M_\mu(k) \subset DMR_\mu(k)$: $\forall g \in M_\mu(k)$, g induces isos $F_B(P_n) \to F_{DR}(P_n)$ compatible w. $F_{B/DR}(pr_i)$, $F_{B/DR}(\ell)$ (detailed argument) $\Rightarrow (g, \Gamma_g(-t)) \in \mathrm{Stab}_{\mathcal{G} \times k[[t]]^\times}(\dots, \hat{\Delta}_{F_B}, \dots, \hat{\Delta}_{F_{DR}} = \hat{\Delta}^w) = DMR_\mu(k)$

where $i_\mu : k + kF_2(X_1-1) \to k\langle\langle e_0,e_1\rangle\rangle e_1$ is induced by $(kF_2)^\wedge \to k\langle\langle e_0,e_1\rangle\rangle$, $X_0 \mapsto e^{\mu e_0}$, $X_1 \mapsto e^{\mu e_1}$.

⑥ **Application of ④ (DT theory)** $(\mathcal{G}, \otimes) \xrightarrow{\sim} TAut_2 = \{ \alpha \in \mathrm{Aut}(\mathrm{Lie}\langle e_0, e_1 \rangle) \mid \alpha(e_1) = e_1,\ \alpha(e_0) \sim e_0 \}$

$DMR_0(k) \hookrightarrow IAut_2 = \{ \alpha$ such that $\alpha(e_\infty) \sim e_\infty \}$ where $e_\infty := -e_0 - e_1$

σ = involution induced by $e_0 \leftrightarrow e_\infty$, $e_1 \mapsto e_1$

Thm A (E + H. Furusho, 2025) (also Schneps 2025, Kawamura 2025) $DMR_0(k) \subset IAut_2$

Thm B (E + Furusho, 2025) $DMR_0(k)$ is stable under σ (involution of $IAut_2$)

⑦ **Cyclotomic analogues** Cyclotomic MZVs ($N > 1$): $\zeta\left(\begin{smallmatrix} n_1 & \cdots & n_s \\ \xi_1 & \cdots & \xi_s \end{smallmatrix}\right) = \sum_{k_1 > \cdots > k_s > 0} \xi_1^{k_1} \cdots \xi_s^{k_s} k_1^{-n_1} \cdots k_s^{-n_s}$

For $n_1 > 1, n_2, \dots, n_s \geq 1$, $\xi_1, \dots \xi_s \in \mu_N$. $M_\mu^{(N)}(k) \subset DMR_\mu^{(N)}(k)$ was proved by Furusho (13).

Partial results towards Deligne-Terasoma-type proof (w. K. Yaddaden), based on:

ⓐ replacing objects P_4, P_5 and morphisms pr_5, pr_{12}, ℓ in Gps by objects $P_4 \to \mu_N$ and $P_5 \to \mu_N^2$ and morphisms between them in Mor(Gps)

ⓑ replacing functors $F_B, F_{DR} : \mathrm{Gps} \to \mathrm{HopfAlgs}$ by functors $F_B, F_{DR} : \mathrm{Mor}(\mathrm{Gps}) \to \mathrm{HopfAlgs}$ related to relative completion