

Quantum Groups and Physics

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Hydrodynamic tails in chaotic spin chains with quantum group symmetry

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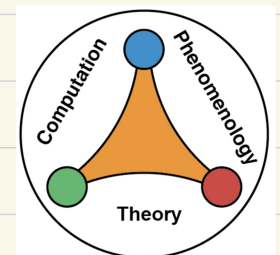
Chaos in Systems with Quantum Group Symmetry

Authors: Victor Gorbenko, Aleksandr Zhabin



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Outline

- General discussion of Quantum Groups and how they can act as symmetries
- Examples in field theory: non-local currents and topological defects
- Examples in spin chains: chaos and hydrodynamics
- Spontaneous Breaking and Goldstone Bosons

• Why QG?

- We have witnessed many times how the notion of symmetry was generalized. Most basic symmetries are either discrete, or continuous Lie groups and act on local objects
- When **Supersymmetry** was introduced, we understood that symmetries do not have to be groups. More recently generalizations to **non-invertible** and **higher-form** symmetries were proposed
- We would like to know what are the **most general useful symmetries**, but this question looks very hard.
- We take an existing mathematical construction, **deformation of Lie algebras**, called **Quantum Groups**, and study properties of systems with QG as a **global internal symmetries** should look like.
- We consider field theories and lattice models.

Disclaimer: QG is a very mathematically-heavy subject.
My goal here is, instead, to give an intuitive physical picture.

→ Imagine an experimentalist who discovers a physical system that has a $U(1)$ charge. Then she finds that many energy levels are degenerate, with degeneracies $2l+1$.
In particular, there are triplets with charges $+1, 0, -1$.
A natural conjecture is that the symmetry is $SU(2)$!

→ A prediction of $SU(2)$ would be that 3-point function of the triplets is proportional to ϵ_{ijk} , in particular, $\langle 000 \rangle$ should vanish.

→ Next, she measures 3-pt. functions and finds it does not!

Moreover, the ratio $\frac{\langle 000 \rangle}{\langle +0- \rangle} = \text{const}$



This is exactly what **Quantum Group** symmetry **$SU_q(2)$** would predict, and $\text{const} = \frac{q^2 - 1}{q}$, where $q-1$ is the deformation parameter.

- In this talk we will focus on $SU_q(2)$, or $U_q(\mathfrak{sl}(2))$ – a continuous deformation of $SU(2)$ by a complex parameter q .

What is $U_q(\mathfrak{sl}_2)$?

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- Remember $\mathfrak{sl}_2$ ($SU(2)$) commutation relations:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

- They can be deformed in the following way:

$$[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad q^H E q^{-H} = q^2 E, \quad q^H F q^{-H} = q^{-2} F. \quad [q \rightarrow 1 : \mathfrak{sl}_2]$$

- What is special about this deformation? For a physicist an important point is that representation theory is (almost) intact: $2\ell+1$ -dimensional representations with $U(1)$ charge $m = -\ell \dots \ell$
- What is different? The action on the tensor product of representations (coproduct). Consider a system which consists of two subsystems $V = V_1 \otimes V_2$. We know action on V_1 and V_2 :

$$\Delta(H) = H \otimes 1 + 1 \otimes H.$$

↑ usual

$$\Delta(E) = E \otimes 1 + q^{-H} \otimes E, \quad \Delta(F) = F \otimes q^H + 1 \otimes F$$

↑ unusual

- Historically, QG were discovered in the studies of integrable systems
Takhtajan, Faddeev '79

- They were developed as a mathematical subject by Drinfeld and Jimbo '85
And then returned to physics in the context of Chern-Simons theories Witten '90

Rational 2D conformal theories were shown to have a "hidden" QG symmetry (to be explained)
Moore, Seiberg '89, Alvarez-Gaume et al. '89, Felder et al. '89

- Bernard and Leclair '91 understood the role of QG in integrable scattering, which lead to identification of "hidden" QG symmetry in certain worldsheet sigma-models. e.g. Thompson 1904.11561

- Recently QG appeared in DSSYK model and its holographic duals. Berkooz et al. '19

- QG are used in Quantum computations Zahe et al '23

- Nevertheless, not much discussion exists on QGs as global symmetries in QFTs, without any relation to integrability. Known examples are limited to Chiral theories Gaberdiel '95; Mack, Schomerus '92 or BCFTs Pasquier, Saleur '90

Symmetries in physics

- Physicists think about symmetries in many different ways.

Hamiltonian: $\{Q, H\} = 0 \quad [\hat{Q}, \hat{H}] = 0 \quad Q \text{ is a sym.}$

Lagrangian: $S = \int dt dx \mathcal{L}[\varphi(x, t)]$

$$Q_\varepsilon \varphi(x, t) \approx \varphi + \delta_Q \varphi, \quad \delta S = 0$$

Noether theorem: $\exists j^\mu(x, t): \partial_\mu j^\mu = 0$

$$Q = \int dx j^0 \Leftrightarrow \frac{d}{dt} Q = 0$$

Topological:



symmetry ~ topological lines

Example: free complex field

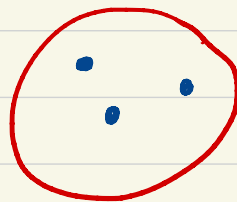
$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi^* \quad S = \int dt dx \mathcal{L} \quad \partial_\mu \partial^\mu \varphi = 0$$

$$j^\mu = \varphi \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi \quad \partial_\mu j^\mu = 0$$

$$Q = \int \varphi \dot{\varphi}^* - \varphi^* \dot{\varphi}$$

$$\varphi \rightarrow e^{i\alpha} \varphi \approx \varphi + i\alpha \varphi$$

$$\delta_Q \varphi(x,t) = i\alpha \varphi(x,t)$$

 $\rightarrow$ counts charges

It is natural that continuous symmetries
act **locally** on fields and form a **Lie group**
but is it always the case?

Consequences of symmetries

- $[\hat{Q}_i, \hat{H}]$, $Q_i \in G$ or $Q_i \in A \Rightarrow H \sim \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_2 & \\ & & & \ddots \end{pmatrix}$
 $\Rightarrow$ Hilbert space can be decomposed into
 irreps of G or $A \Rightarrow$ Energies within irrep are the same

- $\langle R | \hat{G} [O_1 O_2 \dots O_n] | R \rangle = 0 \Rightarrow \sum_{i=1}^n \langle R | O_1 \dots [\hat{G}, O_i] \dots O_n | R \rangle = 0$
 $\hat{G} | R \rangle = 0 \rightarrow$ vacuum is invariant
↑
 Ward identities

$$0 = \begin{array}{c} \circlearrowleft \\ \bullet \end{array} Q \begin{array}{c} \circlearrowleft \\ \bullet \end{array} O_2 \begin{array}{c} \circlearrowleft \\ \bullet \end{array} O_3 = \begin{array}{c} \circlearrowleft \\ \bullet \end{array} Q \begin{array}{c} \circlearrowleft \\ \bullet \end{array} O_2 \begin{array}{c} \circlearrowleft \\ \bullet \end{array} O_3 + \begin{array}{c} \circlearrowleft \\ \bullet \end{array} Q \begin{array}{c} \circlearrowleft \\ \bullet \end{array} O_2 \begin{array}{c} \circlearrowleft \\ \bullet \end{array} O_3 + \begin{array}{c} \circlearrowleft \\ \bullet \end{array} Q \begin{array}{c} \circlearrowleft \\ \bullet \end{array} O_2 \begin{array}{c} \circlearrowleft \\ \bullet \end{array} O_3$$

(+ for A , $\times$ for G)

Spontaneous Sym. Breaking

If vacuum is not invariant: $\hat{G}|\Omega\rangle \neq 0$

Symmetry is spontaneously broken $\rightarrow$ There is a "vacuum manifold" $\{|\Omega\rangle : \hat{G}|\Omega\rangle = 0\}$

Generically (always in relativistic theory) there are **Goldstone bosons** - massless particles

$$\hat{G} = \int j^0(x) dx \quad \langle \Omega | \hat{G}^\dagger H \hat{G} | \Omega \rangle = 0$$

$$\hat{G}_k \equiv \int e^{ikx} j(x) dx : \quad E(k) \equiv \langle \Omega | \hat{G}_k^\dagger H \hat{G}_k | \Omega \rangle$$

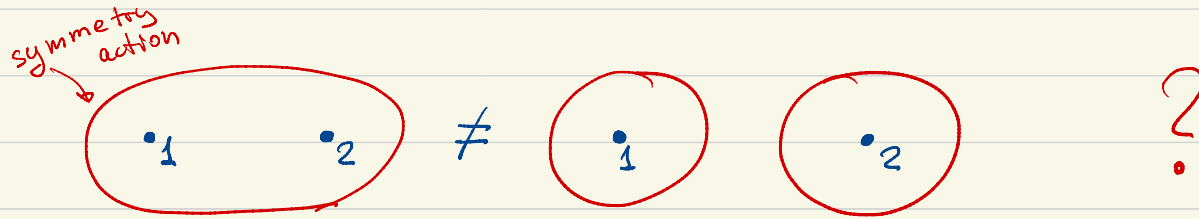
$$E(k) \xrightarrow{k \rightarrow 0} 0$$

QG Symmetry: General Construction

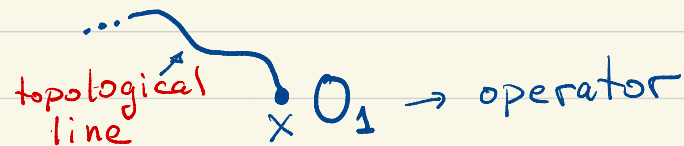
- How do we implement QG symmetry in a QFT? At first its action appears very **non-local** due to the coproduct

$$\Delta(E) = E \otimes 1 + q^{-H} \otimes E, \quad \Delta(F) = F \otimes q^H + 1 \otimes F$$

In particular, it depends on the order of subsystems.



- Let's consider 2D (**Euclidean**) FP. We suggest that a natural object on which a QG can act is a **defect-ending operator**:



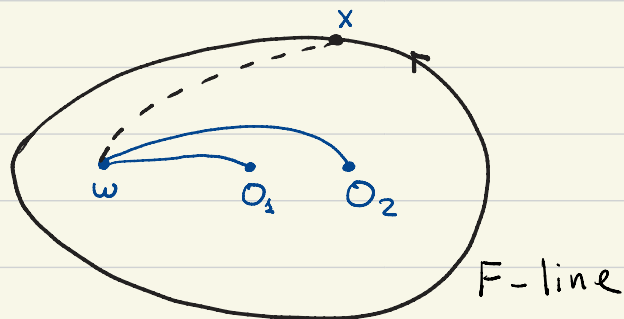
- Operators now can be ordered using their lines:



- Point **w** is arbitrary.

- Now let's define QG algebra generators. Usual symmetry generators are represented by topological lines. For QG we need a generalization:

$F(O_1 O_2)$:



Additional dashed line from w to x is q^H line. Point x is integrated over the F-line

- For simplicity, we can assume existence of a current j_F , such that $F = \int_{\Sigma} dx j_F(x)$ then j_F is itself a defect-ending operator
Bernard, Leclair '91

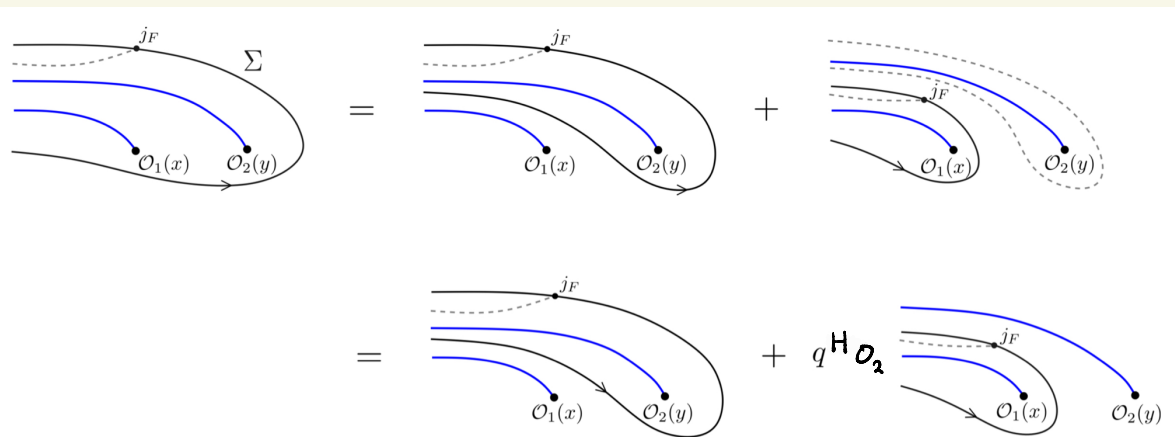


Figure 6: The action of quantum group generator F on two defect-ending operators, given by the $U_q(sl_2)$ coproduct, $\Delta(F) = 1 \otimes F + F \otimes q^H$. The black contour represents the line Σ along which the current operator j_F is integrated. The dashed line denotes the additional q^H -line attached to j_F , while the blue lines are the defect lines attached to O_1 and O_2 .

- Let us check that the coproduct works:

- Applying the F generator to an arbitrary set of operators with total H -charge $+1$ we obtain the **Ward identities** that relate the correlation functions

$$\sum_{i=1}^n q^{2(m_{i+1}+m_{i+2}+\dots+m_n)} f_{m_i}^{\ell_i} \langle \mathcal{O}_{1,\ell_1,m_1}(x_1) \dots \mathcal{O}_{i,\ell_i,m_i-1}(x_i) \dots \mathcal{O}_{n,\ell_n,m_n}(x_n) \rangle = 0$$

$$m_i = -\ell_i \dots \ell_i \\ (\sum_i m_i = 1)$$

$$f_m^\ell = q^{m-1} \sqrt{[\ell+m]_q [\ell-m+1]_q}$$

- Dependence of correlation functions on the **rep. index** (m_i) is **fully fixed by symmetries**, they are proportional to the **invariant tensors** of the group (as for usual symmetries)

$$\langle \mathcal{O}_{\ell_1}^{m_1} \mathcal{O}_{\ell_2}^{m_2} \mathcal{O}_{\ell_3}^{m_3} \rangle \sim \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \quad (3j\text{-symbol})$$

XXZ_q 2D CFT

- To illustrate this general picture let's consider a concrete example. We will consider a CFT, which is UV-complete by a spin-chain model, and in which QG symmetry is realized the way we described.

- It has $U_q(\mathfrak{sl}_2)$ symmetry with $q = e^{i\varphi} = e^{i\pi \frac{\mu}{\mu+1}}$

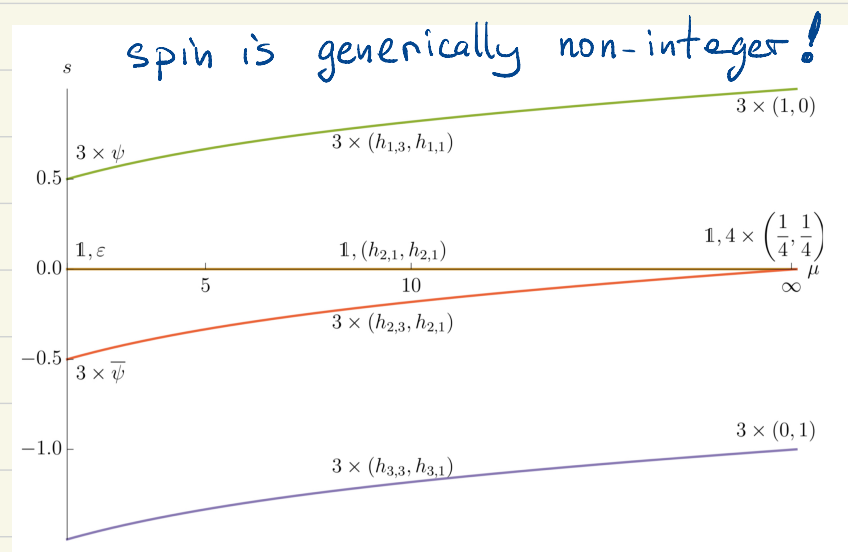
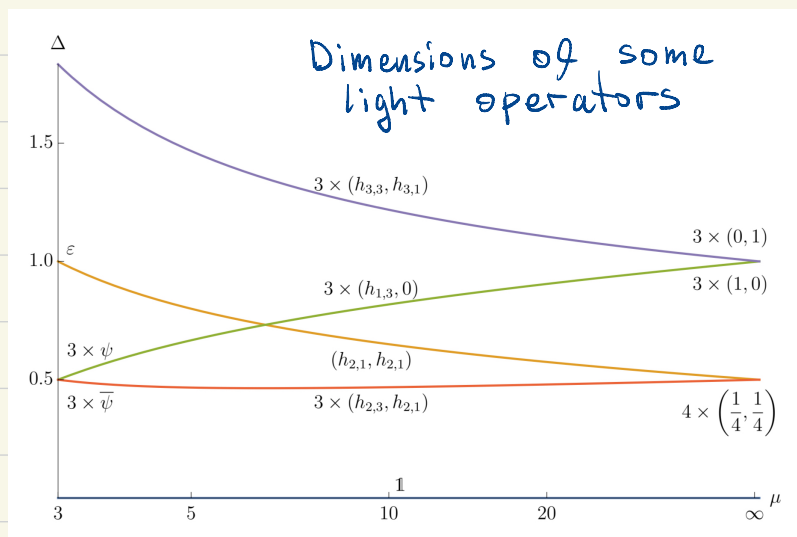
- Torus partition function reads:

$$Z(\tau, \bar{\tau}) = \sum_{\ell=0}^{\infty} (2\ell+1) \sum_{r=1}^{\infty} \chi_{r,2\ell+1}(\tau) \chi_{r,1}(\bar{\tau})$$

$$h_{r,s} = \frac{[r(\mu+1) - s\mu]^2 - 1}{4\mu(\mu+1)}$$

Virasoro characters

Multiplicities = dimensions of irreps



- For $\mu = \infty$ we get $SU(2)_1$ WZW model

Quantum Groups in Minimal Models

- Spectrum of operators in diagonal (generalized) M.M. consists of operators $(h_{r,s}, \bar{h}_{r,s})$ each with multiplicity one.
- To exhibit a QG structure we first consider an auxiliary extended model that has **four** groups:

$O_{r,s;r',s'}$ lives in $r \times s \times r' \times s'$ -dimensional representation.

But to construct a M.M. we need to "**confine**" all **QGs**, similar to the projection we used in our model for two of the groups.

$$\phi_{r,s}(z, \bar{z}) \equiv \sum_{m^- = -\ell^-}^{\ell^-} \sum_{m^+ = -\ell^+}^{\ell^+} \tilde{V}_{(\ell^+, \ell^-), (m^+, m^-)}(z) \bar{\tilde{V}}_{(\ell^+, \ell^-), (m^+, m^-)}(\bar{z})$$

So none of the QG acts on the Hilbert space of M.M., but some consequences of it remain ("**hidden**" QG symmetry)

For example, crossing symmetry of XXZ_q CFT explains the appearance of 6j symbols in the crossing kernel for conformal blocks:

$$\mathcal{F}_{r',s'}^{(t)} = \sum_{r,s} \frac{C_{(r_1,s_1)(r_2,s_2)(r,s)} C_{(r,s)(r_3,s_3)(r_4,s_4)}}{C_{(r_2,s_2)(r_3,s_3)(r',s')} C_{(r_1,s_1)(r',s')(r_4,s_4)}} \left\{ \frac{r_1-1}{2} \quad \frac{r_2-1}{2} \quad \frac{r-1}{2} \right\}_{q_+} \left\{ \frac{s_1-1}{2} \quad \frac{s_2-1}{2} \quad \frac{s-1}{2} \right\}_{q_-} \mathcal{F}_{r,s}^{(s)} \quad (4.42)$$

XXZ_q DPE coefficients

Conformal blocks are the same in (Generalized) MM, so the crossing kernel still applies, even though there is no symmetry

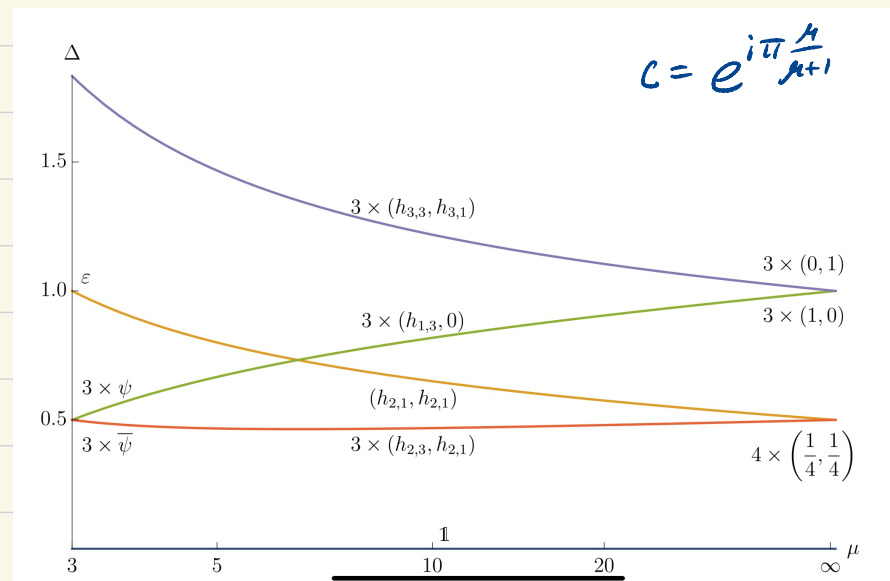
Ising model as a subsector of XXZ_q

- For $q = e^{i\pi \frac{3}{4}}$ c is equal to $\frac{1}{2}$ and operators of the (fermionic) Ising CFT belong to the spectrum

- Fermions are particular components of $SU_q(2)$ triplets:

$$\psi \equiv \lim_{\mu \rightarrow 3} \frac{W_{1,3}^1 + W_{1,3}^{-1}}{2^{\frac{1}{4}}}$$

$$\bar{\psi} \equiv \lim_{\mu \rightarrow 3} \frac{W_{2,3}^1 + W_{2,3}^{-1}}{2^{\frac{1}{4}}}$$



Spin chains related to XXZ_q CFT.

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- We would like to address the question whether our non-local operators have a microscopic definition. To do this we go to the spin chain model and look for operators that flow to W 's in the IR
- Consider open XXZ spin chain : $\uparrow \uparrow \downarrow \uparrow \downarrow \uparrow$
Each spin : $\alpha |\uparrow\rangle + \beta |\downarrow\rangle$

$$H_{PS} = \sum_{i=1}^N \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q + q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right) - \frac{q - q^{-1}}{2} (\sigma_1^z - \sigma_N^z)$$

and the generators

Pasquier, Saleur

$$E = \sum_{i=1}^N q^{-\sum_{j=1}^{i-1} H_j} E_i$$

$$F = \sum_{i=1}^N F_i q^{\sum_{j=i+1}^N H_j}$$

$$q^H = \prod_{i=1}^N q^{H_i}$$

where $H_i = \sigma_i^z$, $E_i = \sigma_i^+$, $F_i = \sigma_i^-$.

Non-integrability = Chaos

- Important point of the talk is that a quantum group can be finite-dimensional symmetry and hence does not have to imply integrability, however, examples considered were in fact integrable.
- Let us now present an example which is manifestly not integrable. We add to the spin-chain Hamiltonian above a next-to-nearest interaction:

$$V(\lambda) = \lambda \sum_{i=1}^{L-2} \left(R_i R_{i+1} + R_{i+1} R_i - \left[q + \frac{q - q^{-1}}{2} \right] (R_i + R_{i+1}) + \left[q^2 - 1 + \frac{q^2 + q^{-2}}{4} \right] \mathbb{1} \right)$$

where R_i 's are elements of Hecke algebra, which all commute with $SU_q(2)$

$$R_i = \frac{1}{2} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q + q^{-1}}{2} (\sigma_i^z \sigma_{i+1}^z + 1) - \frac{q - q^{-1}}{2} (\sigma_i^z - \sigma_{i+1}^z - 2) \right]$$

- For large enough λ or L the system becomes chaotic.
- Chaotic systems cannot be solved, but they can be analyzed in a statistical way

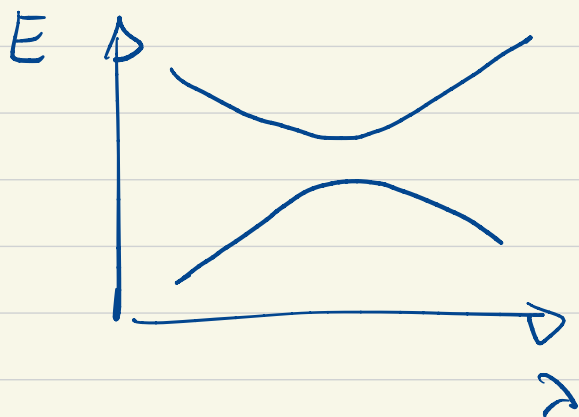
- Exact form of chaotic distribution depends on discrete space-time symmetries and unitarity. The main feature is **Eigenvalue repulsion**, absent for integrable systems, which typically **exhibit level-crossing**.

Integrable Hamiltonian: Poisson statistics

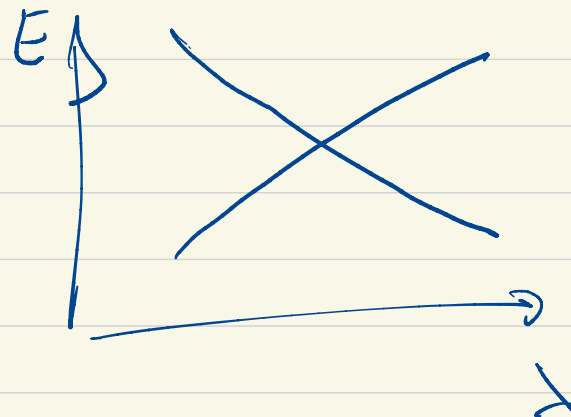
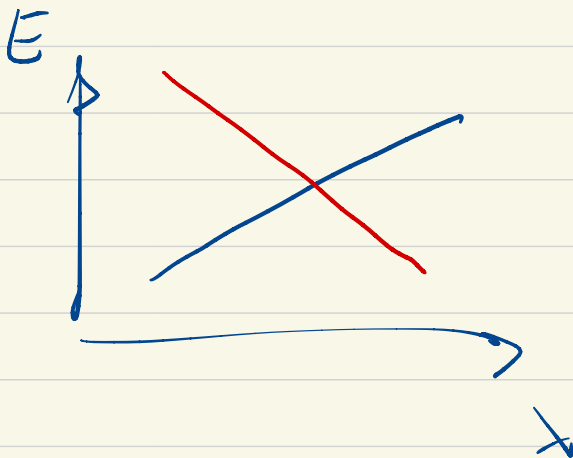
Chaotic Hamiltonian: Universal distribution a la Wigner

As long as we consider a sector of Hilbert space with **fixed $SU_q(2)$ quantum numbers (l, m)** .

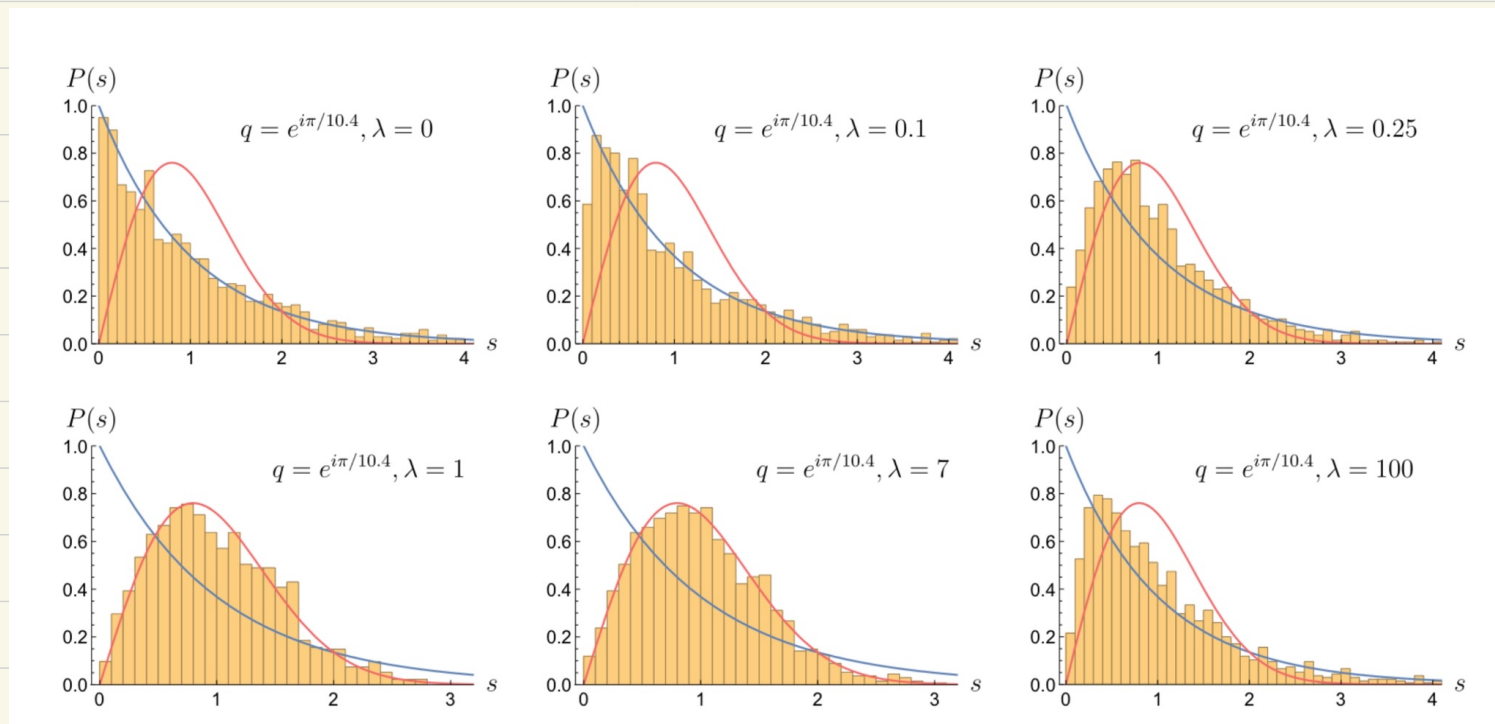
Chaotic



Symmetry



- We consider distribution of level spacing $s = \{E_{i+1} - E_i\}$ of states with fixed $SU_q(2)$ charges



— Poisson

— Wigner

$L=16$, $\sim 10^3$ states

- For large phase of q most eigenvalues become complex:

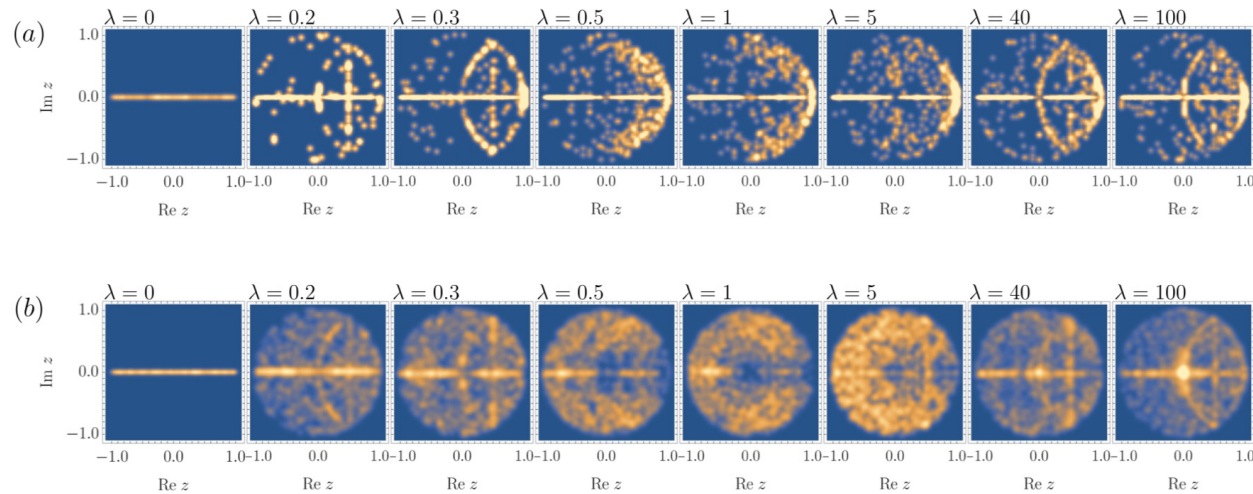


Figure 8: Complex level spacing ratio distributions $P(z)$ for different values of λ . Parameter q is: (a) $q = e^{i\pi/6.7}$; (b) $q = e^{i\pi/2.7}$. Spin chain length is $L = 16$. Eigenvalues are from q -parity even sector with $\{\ell = 1, S^z = 1\}$.

- Their statistics is also chaotic
- PT-symmetry gets "spontaneously broken"

Quantum Groups Hydrodynamics

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- In addition to detailed spectral properties chaotic systems exhibit universal properties governed by hydrodynamics ~ sound waves
- Hydrodynamics transport is related to conserved charges (symmetries)
- In modern language hydro modes are Goldstones:

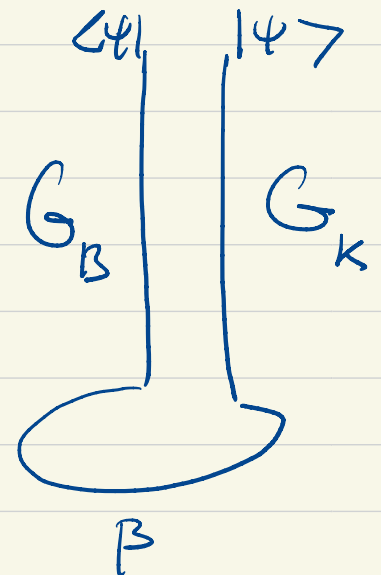
Crossley, Glorioso, Liu '17
Akyuz, Goon, Peacock '24

$$G_B \times G_K \rightarrow G_d$$

$$|\psi\rangle \rightarrow \hat{G}_K |\psi\rangle \Rightarrow$$

$$\langle\psi| \rightarrow \langle\psi| \hat{G}_B$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{a}$$



- Should QG symmetry lead to hydro modes?

- Goldstones are produced by currents
and we do not have local currents.

$$\hat{G}_k \equiv \int e^{ikx} j(x) dx : \quad E(k) \equiv \langle \Omega | \hat{G}_k^\dagger H \hat{G}_k | \Omega \rangle$$

$$E(k) \xrightarrow{k \rightarrow 0} 0$$

- It appears that people assumed the answer
is No!?

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- We now consider our spin chain at high (infinite) temperature

$$H = \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + V_i.$$

$$\text{Tr} e^{-\beta H} \sigma_i^z(0) \sigma_i^z(t) \longrightarrow t^{-\Delta} \quad \text{U(1) hydro mode}$$

$$\text{Tr} e^{-\beta H} \sigma_i^x(0) \sigma_i^x(t) \begin{cases} \nearrow t^{-\Delta'} & \sim (\text{gapless}) \\ \searrow e^{-t} & \sim (\text{mass gap}) \end{cases}$$

charged operators cannot couple to neutral U(1) or energy hydro modes

- At the moment we do not have theoretical understanding, but we can do numerical simulations

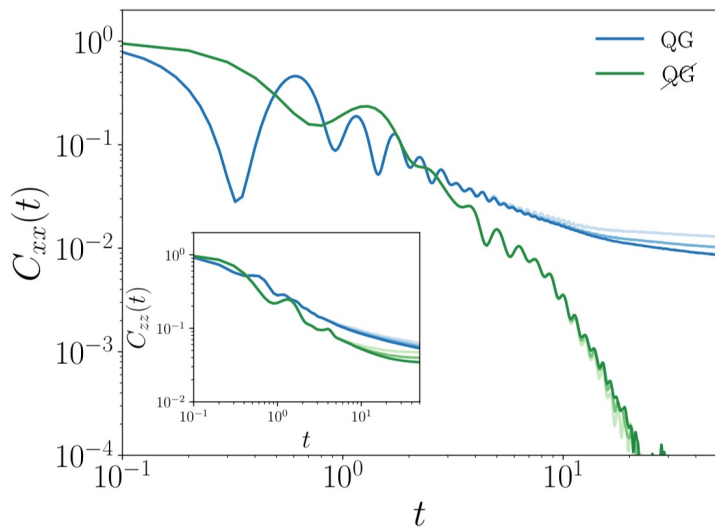


Figure 1. Spin autocorrelation function $C_{ab}(t) = \langle \sigma_i^a(t) \sigma_i^b \rangle$ in chaotic XXZ-like models with and without quantum group (QG) symmetry, for system sizes $L = 22, 26, 30$. Both models feature diffusive decay of the longitudinal spin σ_z , the conserved density for a conventional $U(1)$ symmetry (inset). In contrast, the decay of transverse spin is markedly different in both models: it is faster than polynomial in the model without QG symmetry, as expected for a non-conserved operator, but features slow polynomial decay in the QG symmetric model.

$$\sigma^z \sigma^z(t) \sim t^{-1/2} \text{ diffuses}$$

$$\sigma^x \sigma^x(t) \sim t^{-2/3}$$

demonstrates "super-diffusion"

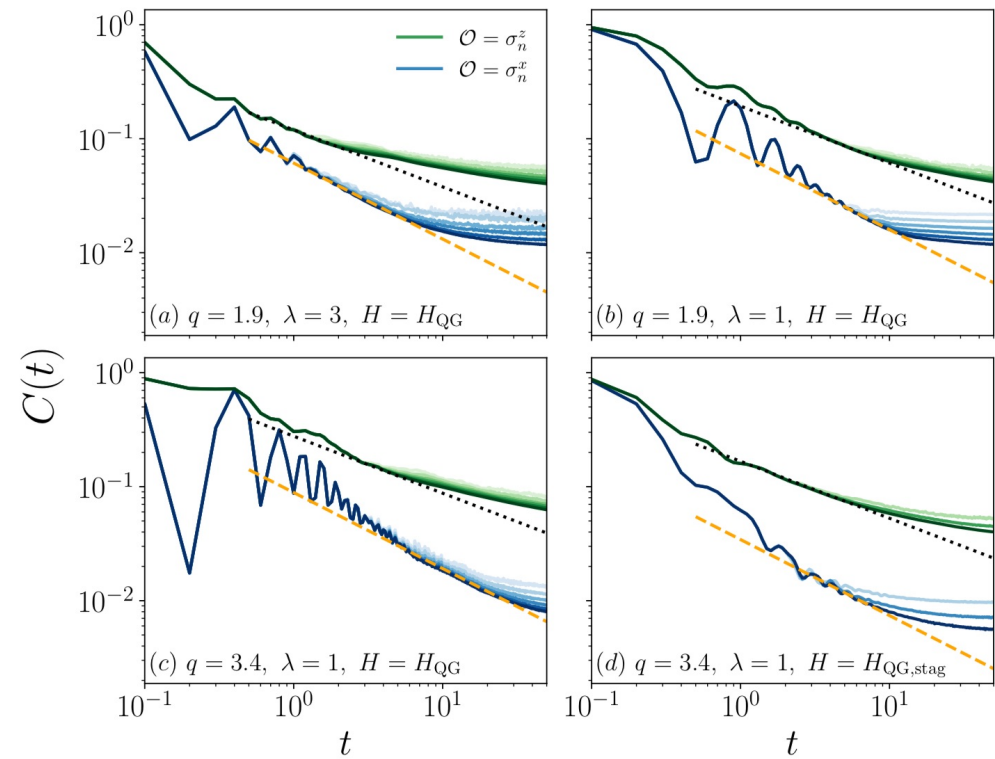


Figure 4. Infinite temperature autocorrelation function $C(t) = \text{Tr}(\mathcal{O}(t)\mathcal{O})$ for transverse and longitudinal spin $\mathcal{O} = \sigma^x, \sigma^z$ in QG symmetric Hamiltonian models (8) and (9) with various parameters. The light to dark colors correspond to increasing system size: $L = 20, 22, 24, 26, 28, 30$ in (a,b,c) and $L = 22, 26, 30$ in (d). Dotted ($1/t^{1/2}$) and dashed ($1/t^{2/3}$) lines are guides to the eye.

Conclusions

- Quantum groups appear to have much more widespread physical consequences than one could have thought...

Could we be missing some more non-local symmetries?