# Sphere packings, cyclotomic fields, and subconvexity bounds

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# Lattice packings

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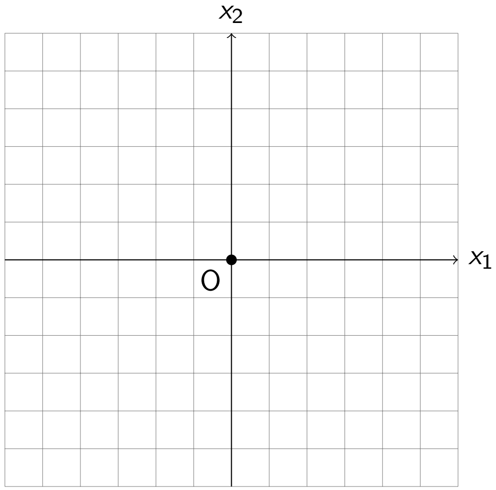

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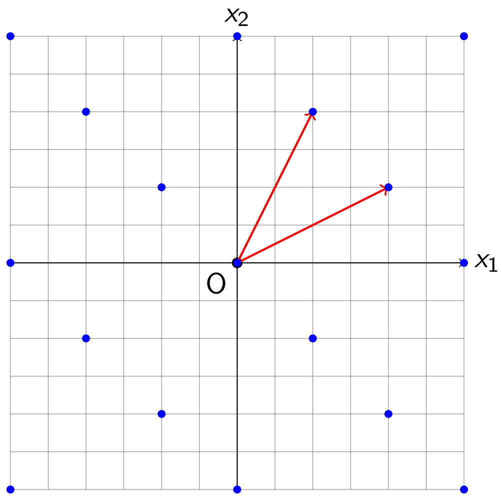

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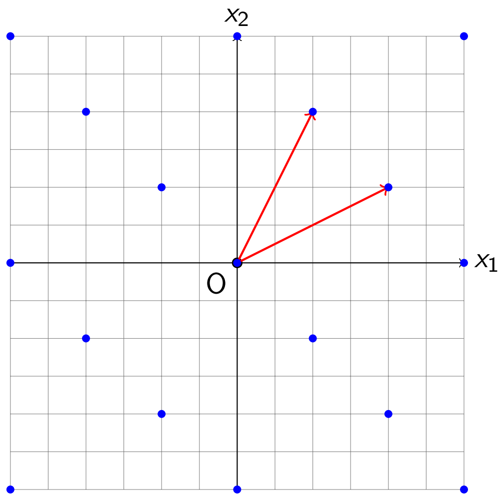

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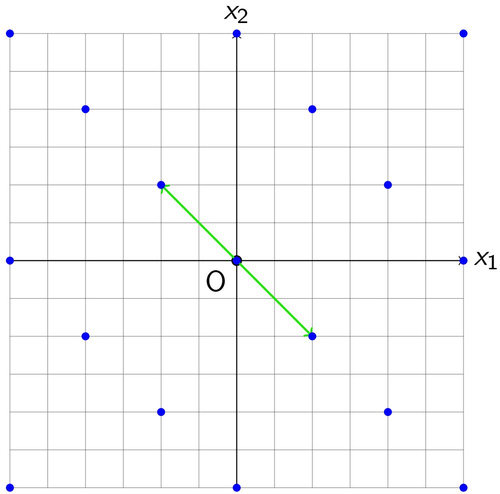

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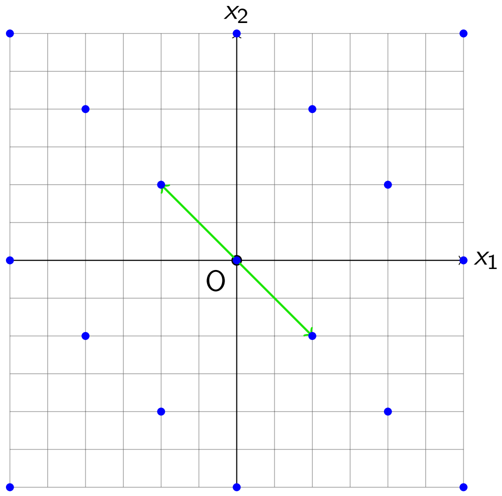

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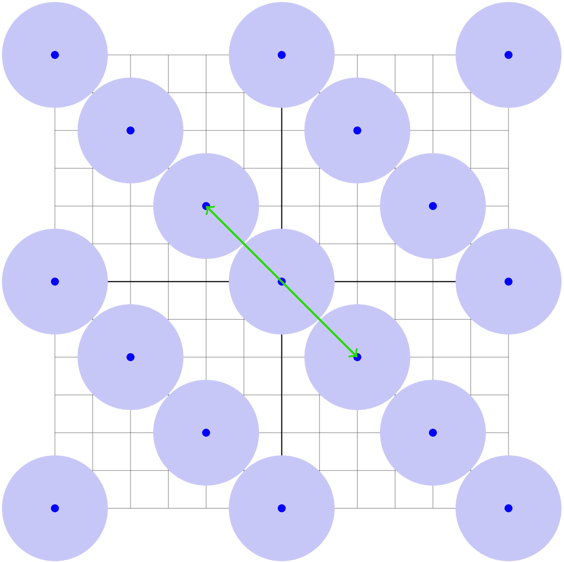

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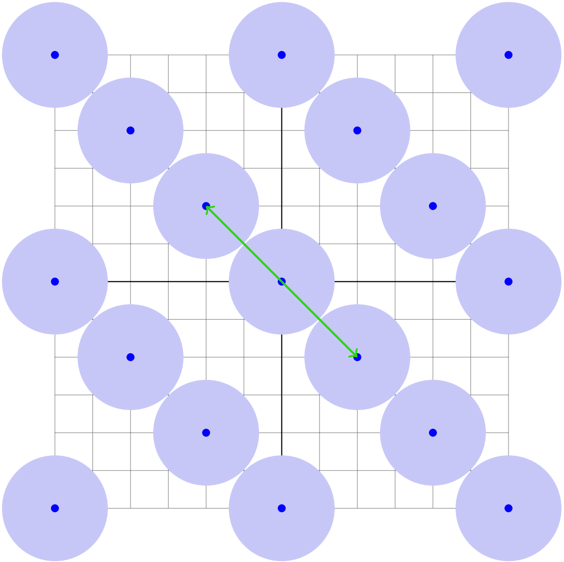
We denote by $\Delta(\Lambda)$ the density of this packing.

For each dimension $d$ we would like to know $\Delta_d^L := \sup_{\text{lattice } \Lambda \subset \mathbb{R}^d} \Delta(\Lambda)$.



# Lattice packings

The densest lattice packing is known in dimensions:

$$\mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 5, 6, 7, \mathbf{8}, \mathbf{24}.$$

# Asymptotic bounds for the sphere packing constant

Theorem (Minkowski 1905, Hlawka 1920s)

$$\Delta_d^L \geq 2^{-d+1}.$$

Theorem (Rogers 1947)

$$\Delta_d^L \geq c\,2^{-d+1}\,d(1+o(1)), \qquad d \to \infty.$$

Theorem (Venkatesh 2013)

$$\Delta_d^L \geq 2^{-d}\,d\,\log\log(d)$$

for a sequence of dimensions $d$ ($d = 2\phi(n)$ and $n = p_1 \cdot p_2 \cdots p_k$ has many prime factors.)

# Asymptotic bounds for the sphere packing constant

Theorem (M.Campos, M. Jensen, M. Michelin, and J. Sahasrabudhe 2023)

$$\Delta_d \geq 2^{-d-1}\, d\, \log(d)\,(1+o(1)), \qquad d \to \infty.$$

Theorem (B. Klartag 2025)

$$\Delta_d^L \geq c\, 2^{-d}\, d^2.$$

Theorem (E. B. Abuya, N. Gargava, Y. Zhao 2026)

There exists a universal constant $c > 0$ and a finite sequence of dimensions $d$ in which

$$\Delta_d^L \geq c\, 2^{-d}\, d^2\, \log\log(d).$$

# Random lattices

The space of random unimodular lattices in $\mathbb{R}^d$:

$$\mathcal{L}_d := \{\Lambda \subset \mathbb{R}^d \mid \mathrm{vol}(\mathbb{R}^d/\Lambda) = 1\} \simeq \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}).$$

$$\begin{aligned} \mu_d := & \text{ Haar measure normalized so that} \\ & \mu_d(\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})) = 1. \end{aligned}$$

Consider a random variable on $\mathcal{L}_d$:

$$N_r(\Lambda) := \#(\Lambda \cap B(0,r)) - 1, \; r > 0.$$

What is the expected value of $N_r$?

# Random lattices

## Theorem(Siegel)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a compactly supported bounded integrable function. Then

$$\int_{\mathcal{L}_d} \left( \sum_{\ell \in \Lambda \setminus \{0\}} f(\ell) \right) d\mu_d(\Lambda) = \int_{\mathbb{R}^d} f(x)\, dx.$$

## Corollary

$$\mathbb{E}(N_r) = \int_{\mathcal{L}_d} \left( \sum_{\ell \in \Lambda \setminus \{0\}} \mathbb{I}(\ell \in B(0,r)) \right) d\mu_d(\Lambda) = \mathrm{vol}(B(0,r)).$$

## Corollary(Minkowski-Hlawka)

$$\Delta_d^L \geq 2^{-d+1}.$$

# Geometry of numbers

Let $K$ be a number field and let $\mathcal{O}_K$ denote its ring of integers.
Suppose that $K$ has $r_1$ real embeddings and $2r_2$ complex embeddings.
Then $r_1 + 2r_2 = [K : \mathbb{Q}]$.
Minkowski embedding:

$$K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$$

$$K \to K_{\mathbb{R}}, \qquad x \mapsto (\sigma(x))_{\sigma},$$

where $\sigma$ runs over $r_1$ real and $r_2$ complex (one for each pair of complex conjugates) embeddings of $K$.

# Geometry of numbers. Lattices constructed from number fields

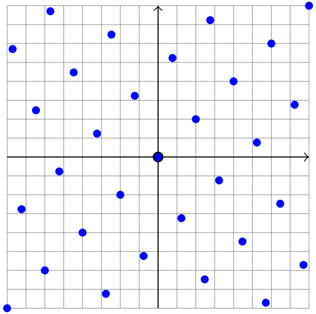
Let $K$ be a number field and let $\mathcal{O}_K$ denote its ring of integers. Let $K_\mathbb{R} = K \otimes_\mathbb{Q} \mathbb{R}$ denote the $[K:\mathbb{Q}]$-dimensional Euclidean space associated to $K$, the Minkowski space.

The scalar product is defined by $\langle x, y\rangle = \Delta_K^{\frac{-2}{[K:\mathbb{Q}]}} \mathrm{tr}(x\overline{y})$.
Here $\Delta_K$ is the absolute value of the discriminant of the number field $K$. The normalization in the above equation ensures that $\mathcal{O}_K \subset K_\mathbb{R}$ is a lattice with unit covolume.

Examples:

$K = \mathbb{Q}, \mathcal{O}_K = \mathbb{Z}$

$K = \mathbb{Q}(\sqrt{5}), \mathcal{O}_K = \mathbb{Z} + \frac{1+\sqrt{5}}{2}\mathbb{Z}$



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Examples:

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The lattice obtained by embedding of $\mathcal{O}_K$ is invariant under the action of $\mathcal{U}_K$, the units in $\mathcal{O}_K$.

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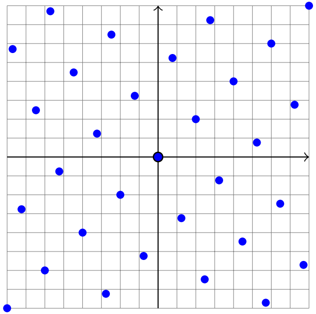


# Lattices with the symmetry of a number field

We consider the space of rank $t$ free unimodular $\mathcal{O}_K$ -lattices

$$\mathcal{L}_{K,t} := \mathrm{SL}_t(K_{\mathbb{R}})/\mathrm{SL}_t(\mathcal{O}_K).$$

$\mathrm{SL}_t(K_{\mathbb{R}}) \simeq \mathrm{SL}_t(\mathbb{R})^{r_1} \times \mathrm{SL}_t(\mathbb{C})^{r_2}.$

When $K$ does not have class number one and $\mathcal{O}_K$ is not a principal ideal domain, one may consider more general $\mathcal{O}_K$-modules.

## Random lattices with the symmetry of a number field

Obviously, $\mathcal{L}_{K,t}$ can be seen as a subset of $\mathcal{L}_{[K:\mathbb{Q}]t}$.

Each lattice $\Lambda \subset \mathcal{L}_{K,t}$ is invariant under the action of $\mu_K$, the finite units in $\mathcal{O}_K$. Therefore $\#(\Lambda \cap B(0,r)) - 1$ is divisible by $\omega_K := \#\mu_K$.

### Theorem (Siegel-Weil)

For $t \geq 2$

$$\mathbb{E}(\#(\Lambda \cap B(0,r))) \;=\; \mathrm{vol}(B(0,r)) + 1.$$

### Corollary (Venkatesh)

$$\Delta_d^L \geq 2^{-d}\, d \,\log\log(d).$$

What about $t = 1$?

# Arakelow class group

The Arakelov class group $\widetilde{\mathrm{Cl}}(K)$ is defined as the set of ideal lattices $(g, \mathcal{I})$ where $\mathcal{I} \subseteq K$ is a finitely generated $\mathcal{O}_K$-module and $g \in K_{\mathbb{R}}^{\times}$. We identify two ideal lattices as per the rule

$$(g_1, \mathcal{I}_1) = (g_2, \mathcal{I}_2) \Leftrightarrow g_1\mathcal{I}_1 = c \cdot g_2\mathcal{I}_2 \text{ for some } c > 0.$$

Here the equality on the right is the equality of sublattices in $K_{\mathbb{R}}$.
The following sequence of topological abelian groups is exact.

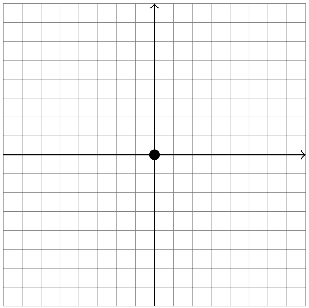
$$0 \to K_{\mathbb{R}}^{(1)}/\mathcal{O}_K^{\times} \to \widetilde{\mathrm{Cl}}(K) \to \mathrm{Cl}(K) \to 0$$

Here $K_{\mathbb{R}}^{(1)}$ is the group

$$K_{\mathbb{R}}^{(1)} = \{a \in K_{\mathbb{R}} \mid |\mathrm{N}(a)| = 1\}.$$

The map $\mathrm{N} : K_{\mathbb{R}} \to \mathbb{R}^{\times}$ is the norm map. Dirichlet's unit theorem tells us that $K_{\mathbb{R}}^{(1)}/\mathcal{O}_K^{\times}$ is a compact torus.

# Lattices constructed from number fields



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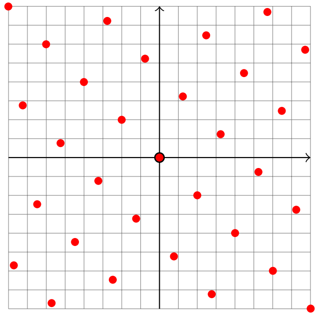

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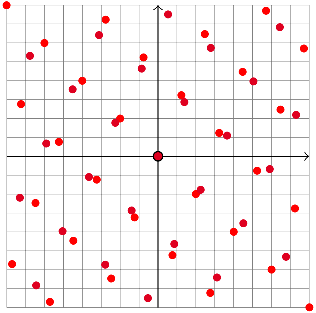

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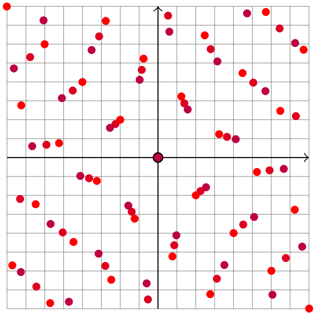

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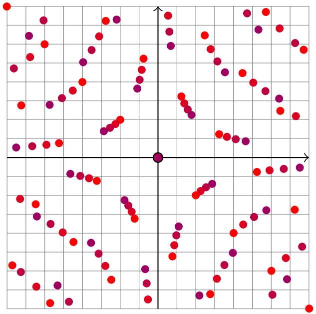

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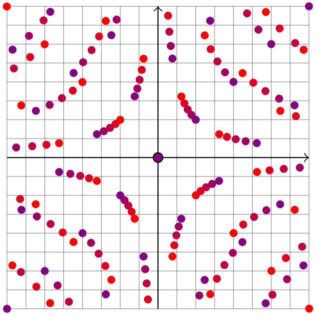

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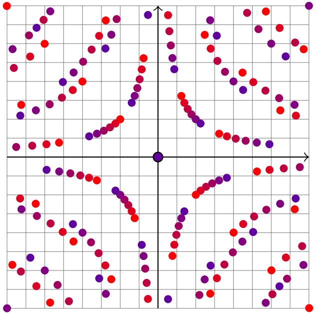

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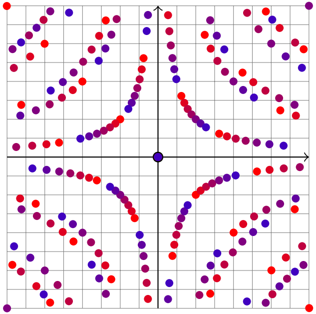

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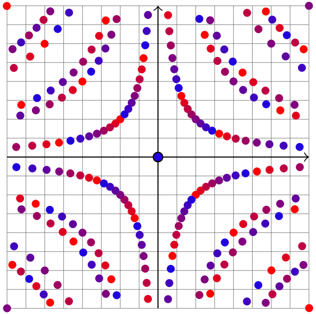

# Random ideal lattices

### Theorem(Gargava, V. 2024)

Let $K$ be a cyclotomic number field. For each $K$, choose $B \subseteq K \otimes \mathbb{R}$ to be a ball of volume $V$ with respect to the trace form. Then, as $\deg K \to \infty$, we have for some $\eta > 0$

$$\mathbb{E}(\# B \cap \Lambda) = 1 + V + \varepsilon(V, K) \text{ for a Haar-random unit covolume } \Lambda \in \widetilde{\mathrm{Cl}}(K),$$

where $\varepsilon(V, K)$ is an error term, that behaves like

$$|\varepsilon(V, K)| \ll \sqrt{V} \Delta_K^{-\eta} \text{ as } \deg K \to \infty.$$

The discriminant $\Delta_K$ of a cyclotomic field $K$ with $\deg K = d$ is at least $e^{cd \log d}$ for some constant $c > 0$. Therefore, there exist constants $c_1, c_2 > 0$ chosen uniformly for all cyclotomic number fields $K$ such that

$$|\varepsilon(V, K)| \leq V^{1/2} \exp\left(-c_1\, d \log(d) + c_2\, d\right).$$

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## Corollary

Improvement on the result by Venkatesh by a factor of $1/2$.

# Idea of the proof: Hecke´s integration formula

**Theorem**

Let $f_r : \mathbb{R}_{>0} \to \mathbb{R}$ be a compactly supported smooth function and $f : K_{\mathbb{R}} \to \mathbb{R}$ be $f(x) = f_r(\|x\|)$. Then

$$\int_{\Lambda \in \widetilde{\mathrm{Cl}}(K)} \left( \sum_{v \in \Lambda \setminus \{0\}} f(v) \right) d\Lambda = \int_{x \in K_{\mathbb{R}}} f(x)\, dx + \varepsilon(K, f),$$

where

$$\varepsilon(K, f) = \frac{(2\pi)^{r_2}}{\mathrm{vol}(\widetilde{\mathrm{Cl}}(K))\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \Delta_K^{\frac{s}{2}}\, \zeta_K(s)\, \mathcal{M} f_r(sd) \left( \frac{\Gamma(\frac{s}{2})^{r_1} \Gamma(s)^{r_2}}{2^{s r_2} \Gamma(\frac{sd}{2})} \right) ds.$$

# Idea of the proof: Hecke's integration formula + Subconvexity bounds for Dirichlet zeta function

## Corollary

Let $K$ be a totally imaginary number field, that is $r_1 = 0$. Let $B_R \subseteq K_{\mathbb{R}}$ be an origin-centered ball of radius $R$. Then we have that if $r_2 \geq 4$, then

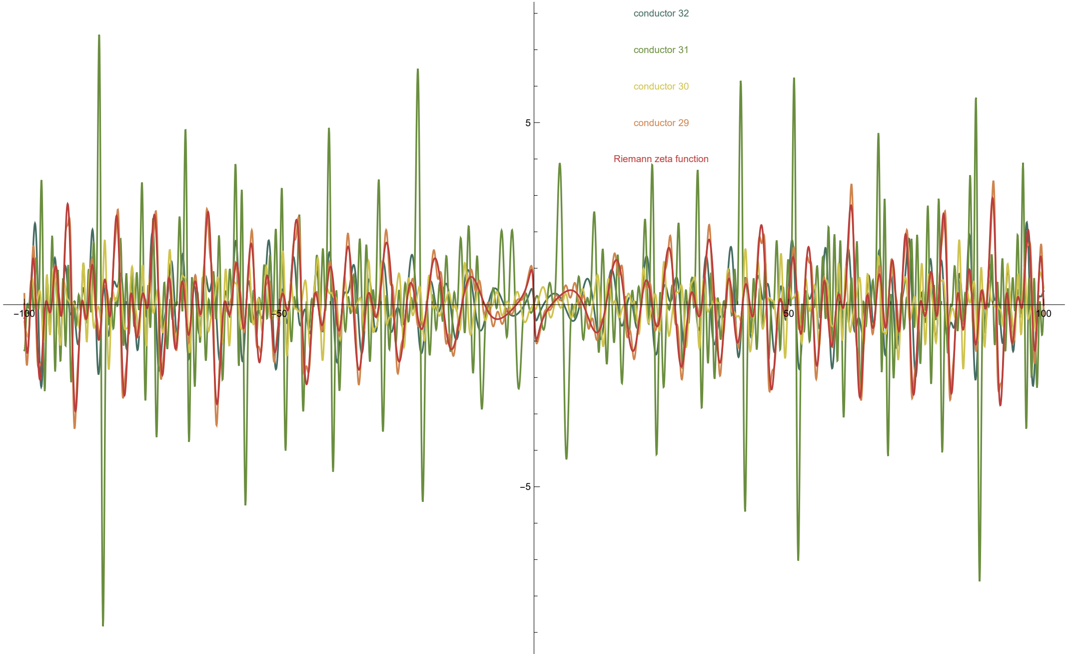
$$\int_{\Lambda \in \widetilde{\mathrm{Cl}}(K)} \# (B_R \cap \Lambda) \, d\Lambda = 1 + \mathrm{vol}(B_R) + \varepsilon(R, K).$$

where

$$\varepsilon(R, K) = \frac{(2\pi)^{r_2}}{\mathrm{vol}(\widetilde{\mathrm{Cl}}(K))\pi i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \Delta_K^{\frac{s}{2}} \zeta_K(s) \frac{R^{2sr_2}}{2sr_2} \left( \frac{\Gamma(s)^{r_2}}{2^{sr_2} \Gamma(sr_2)} \right) ds.$$

Here the $\frac{1}{2}$ may be replaced by any other real in the interval $(\frac{1}{2}, 1)$.

# Idea of the proof: Hecke‘s integration formula + **Subconvexity bounds**



# Idea of the proof: Hecke's integration formula + **Subconvexity bounds**

## Theorem (Petrov-Young, 2023)

For any Dirichlet character $\chi$ with conductor $q = q(\chi)$, for $t \in \mathbb{R}$ and any $\epsilon > 0$

$$|L(\tfrac{1}{2} + it, \chi)| \ll_{\varepsilon} q^{\frac{1}{6}+\varepsilon}(|t| + 1)^{\frac{1}{6}+\varepsilon}. \tag{1}$$

## Corollary

There exist positive numbers $\eta_1, \eta_2$ such that for cyclotomic number fields $K$ and for all $t \in \mathbb{R}$

$$\zeta_K(\tfrac{1}{2} + it) \ll_{\eta_1,\eta_2} \Delta_K^{\frac{1}{4}-\eta_1}(|t| + 1)^{\deg K \cdot (\frac{1}{4}-\eta_2)}.$$

Here $\Delta_K$ is the discriminant and $\deg K$ is the degree of a number field $K$. The positive numbers $\eta_1, \eta_2$ can be choosen arbitrarily close to $\frac{1}{12}$.

## Conclusion for Part 1

Random ideal lattices for cyclotomic fields are almost as good as random lattices.

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### Part 2

Let us look closer at modular lattices over number fields.

# Moments of the number of lattice points in a bounded set

## Theorem (Rogers, 1956)

Let $\Lambda \subseteq \mathbb{R}^t$ be a random unit covolume lattice and let $S$ be a centrally symmetric Borel set of volume $V$. Consider the random variable

$$\rho(\Lambda) = \#\left(S \cap (\Lambda \setminus \{0\})\right). \tag{2}$$

Then, provided $t \geq \lceil \frac{1}{4}n^2 + 3 \rceil$, it follows that the $n$-th moment of the number of non-zero lattice points in $S$ satisfies

$$2^n \cdot m_n(\tfrac{V}{2}) \leq \mathbb{E}[\rho(\Lambda)^n] \leq 2^n \cdot m_n(\tfrac{V}{2}) + E_{n,t} \cdot (V+1)^n,$$

where

$$m_n(\lambda) = e^{-\lambda} \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} r^n = \mathbb{E}_{X \sim \mathcal{P}(\lambda)}(X^n),$$

is the $n$th moment of a Poisson distribution with parameter $\lambda$ and where $E_{n,t}$ is an error term decaying exponentially as $t$ increases:

$$E_{n,t} \leq 2 \cdot 3^{\lceil \frac{n^2}{4} \rceil} \cdot (\tfrac{\sqrt{3}}{2})^t + 21 \cdot 5^{\lceil \frac{n^2}{4} \rceil} \cdot (\tfrac{1}{2})^t.$$

# Poisson distribution on Wikipedia

Chewing gum on a sidewalk. The number of chewing gums on a single tile is approximately Poisson distributed.

# Random lattices with the symmetry of a number field

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What is the distribution of

$$\frac{\#(\Lambda \cap B(0,r) \setminus \{0\})}{\#\mu_K}, \Lambda \in \mathcal{L}_{K,t} \ ?$$

Is it close to the Poisson distribution?

# Lattices constructed from number fields

**Theorem (Gargava, Serban, Viazovska 2023)**

Let $K$ be any number field of fixed degree $d$. There is an explicit constant $t_0(K, n) = O(n^2 \log\log n/\sqrt{d})$ for large $n$, as well as constants $C_K, \varepsilon_K > 0$, such that the $n$-th moment $\mathbb{E}[\rho(\Lambda)^n]$ of the number of nonzero $\mathcal{O}_K$-lattice points in an origin-centered sphere of volume $V$ over the probability space of $\mathcal{O}_K$-lattices of rank $t$ and unit covolume satisfies

$$\omega_K^n \cdot m_n(\tfrac{V}{\omega_K}) \leq \mathbb{E}[\rho(\Lambda)^n] \leq \omega_K^n \cdot m_n(\tfrac{V}{\omega_K}) + C_K \cdot e^{-\varepsilon_K(t-t_0+1)} \cdot (V+1)^n.$$

provided $t \geq t_0(K, n)$.

# Cyclotomic fields

### Theorem (Gargava, Serban, Viazovska 2024)

Consider a sequence of cyclotomic number fields given by $K_i = \mathbb{Q}(\zeta_{k_i})$ of degree $d_i = \varphi(k_i)$ and let $t_0 = 27$. There then exist uniform constants $C, \varepsilon > 0$ such that for any $t \geq t_0$ and for any degree $d_i > (\frac{252}{t})^2$ the second moment $\mathbb{E}[\rho(\Lambda)^2]$ of the number of nonzero $\mathcal{O}_K$-lattice points in an origin-centered sphere of volume $V$ over the space of $\mathcal{O}_K$-lattices of rank $t$ and unit covolume satisfies:

$$V^2 + V \cdot \omega_{K_i} \leq \mathbb{E}[\rho(\Lambda)^2] \leq V^2 + V \cdot \omega_{K_i} \cdot (1 + C \cdot e^{-\varepsilon \cdot d_i (t - t_0 + 1)}).$$

# Mahler measures and the Bogomolov property

For an algebraic number $\alpha \in K^\times$, recall that the Mahler measure (or unnormalised exponential Weil height) is given by the product over the set of places $M_K$ of $K$:

$$H_M(\alpha) := \prod_{v \in M_K} \max\{1, |\alpha|_v\}.$$

We also define, keeping only the infinite places, the closely related

$$H_W(\alpha) = \prod_{\sigma : K \to \mathbb{C}} \max\{1, |\sigma(\alpha)|\}$$

which will be more directly relevant for estimates in the Euclidean space associated to $K$. The two coincide for algebraic integers and in general differ by a denominator. We also recall that the **absolute** Mahler measure (or exponential Weil height) of an algebraic integer $\alpha$ is given by $H_M(\alpha)^{1/\deg(\alpha)}$ and we shall denote by

$$h(\alpha) = \log(H_W(\alpha)^{1/\deg(\alpha)})$$

the **Weil height** of an algebraic integer. More generally for non-integers, we shall write $h_M(\alpha)$ for the Weil height $\log(H_M(\alpha)^{1/\deg(\alpha)})$ and keep $h(\alpha) = \log(H_W(\alpha)^{1/\deg(\alpha)})$.

Note that the absolute Mahler measure and Weil heights are independent of the subfield over which one is considering an algebraic integer. That is, if $\beta \in K$ we have $\deg \beta = \#\{\sigma : \mathbb{Q}(\beta) \to \mathbb{C}\}$ and

$$\frac{\log\left(\prod_{\sigma:K\to\mathbb{C}} \max\{1, |\sigma(\beta)|\}\right)}{[K:\mathbb{Q}]} = \frac{\log\left(\prod_{\sigma:\mathbb{Q}(\beta)\to\mathbb{C}} \max\{1, |\sigma(\beta)|\}\right)}{[\mathbb{Q}(\beta):\mathbb{Q}]}.$$

Lehmer's famous problem asks for a uniform lower bound for $h_M(\alpha)\deg(\alpha)$. We shall consider algebraic numbers related to the stronger property:

### Definition

A subset $S \subset \overline{\mathbb{Q}}$ is said to satisfy the **Bogomolov property** if there exists a constant $C > 0$ such that

$$h_M(\alpha) \geq C$$

provided $\alpha \in S$ has infinite multiplicative order.

## Hypothesis

As $K$ varies among the number fields considered, there exist uniform constants $c_0 \geq c_1 > 0$ such that the absolute (logarithmic) Weil heights satisfy

$$h(\alpha) > c_0 \text{ for } \alpha \in \mathcal{O}_K \setminus \mu_K$$

and

$$h_M(\alpha) > c_1 \text{ for } \alpha \in K^\times \setminus \mu_K.$$

We now recall some important examples from the literature when the Bogomolov property is satisfied:

## Theorem (Schinzel 1973)

Assume that an algebraic number $\alpha$ of infinite multiplicative order is contained in a totally real field. Then, denoting by $\varphi = \frac{1+\sqrt{5}}{2}$ the golden ratio, we have

$$h_M(\alpha) \geq \frac{1}{2}\log\varphi \approx 0.2406\ldots.$$

Moreover, the same is true for $\alpha$ in a CM field provided one (and equivalently, all) of its Archimedean embeddings satisfy $|\alpha| \neq 1$.

### Theorem (Amoroso–Dvornicich 2000)

Assume that an algebraic number $\alpha$ of infinite multiplicative order is contained in an abelian extension of $\mathbb{Q}$. Then we have

$$h_M(\alpha) \geq \frac{\log 5}{12} \approx 0.1341\ldots.$$

We may therefore record as a special case:

Any tower of cyclotomic fields satisfies Hypothesis with constants $c_0 = \frac{1}{2}\log\varphi \approx 0.2406$ and $c_1 = \frac{\log 5}{12} \approx 0.1341$.

### Theorem (Gargava, Serban, V. 2023)

Let $\{K_i\}_{i\in\mathbb{N}}$ denote a sequence of number fields such that the absolute Weil height for infinite order units in $\mathcal{O}_{K_i}^{\times}$ has a uniform lower bound $c_0 > 0$. Then, there are explicit constants $t_0(n) = O(n^2 \log\log n)$ and $d_0(n) = O(n^2)$ for large $n$ as well as $C, \varepsilon > 0$, all uniform in $i$, such that for any $t \geq t_0$ and for any $d = [K_i : \mathbb{Q}] \geq d_0$ the $n$-th moment $\mathbb{E}[\rho(\Lambda)^n]$ of the number of nonzero $\mathcal{O}_K$-lattice points in an origin-centered sphere of volume $V$ over the space of $\mathcal{O}_K$-lattices of rank $t$ and unit covolume satisfies:

$$\omega_i^n \cdot m_n(\tfrac{1}{\omega_i} V) \leq \mathbb{E}[\rho(\Lambda)^n] \leq \omega_i^n \cdot m_n(\tfrac{1}{\omega_i} V) \\ + C \cdot e^{-\varepsilon d(t-t_0+1)} \cdot (V+1)^n \cdot Z(K_i, t, n).$$

Here $\omega_i = \omega_{K_i}$ are the number of roots of unity in $K_i$, $Z(K_i, t, n)$ denotes a finite product of Dedekind zeta values $\zeta_{K_i}$ at certain real values $> 1$ and $m_n$ is as before.

# No limiting Poisson distribution

## Proposition (Gargava, Serban, Viazovska 2023)

Let $f_n$ be a sequence of irreducible polynomials of degree $n$ such that their Mahler measures are uniformly bounded from above. Let $K_n := \mathbb{Q}(f_n)$ be the resulting sequence of number fields. Then for any fixed number of copies $t > 2$ there exists $C_t > 0$ such that over any number field $K_n$ the second moment satisfies

$$\mathbb{E}[\rho(\Lambda)^2] \geq \; C_t + \omega_{K_n}^2 \, m_2(\frac{V}{\omega_{K_n}})$$

There are many sequences satisfying the assumptions of the proposition. For example, one has limiting results for Mahler measures such as :

$$\lim_{n\to\infty} H_W(\alpha_n) = 1.3815\ldots, \text{ where } \alpha_n^n - \alpha_n + 1 = 0,$$

so where $\alpha_n$ is a root of $f_n(x) = x^n - x + 1$.

# An application of second moment estimates

## Theorem (Jianwei Li, Phong Q Nguyen 2020)

Let $\Lambda \subseteq \mathbb{R}^d$ be a Haar-random lattice chosen from the space $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. Then, as $d \to \infty$, with probability $1 - o(1)$ we have

$$1 - \frac{\log \log d}{d} \leq \frac{\lambda_1(\Lambda)}{\gamma(d)} \leq 1 + \frac{\log \log d}{d},$$

where $\gamma(d)$ is radius of a ball of unit volume in $d$ dimensions.

### Remark

For large $d$, $\gamma(d) \simeq \sqrt{\frac{d}{2\pi e}}$ by Stirling's approximation.

## Theorem (Gargava, Serban, V., Viglino 2024)

Let $t \geq 11$ be fixed. Let $K = \mathbb{Q}(\mu_k)$ be a cyclotomic number field and let $d = t \cdot \deg K = t\varphi(k)$. Consider a Haar-random unit covolume module lattice $\Lambda \subseteq K^t \otimes \mathbb{R}$ in the moduli space of rank-$t$ module lattices over $K$. Then, as $k \to \infty$, we have with probability $1 - o(1)$ that

$$1 - \frac{\log \log k}{d} \leq k^{-\frac{1}{d}} \frac{\lambda_1(\Lambda)}{\gamma(d)} \leq 1 + \frac{\log \log k}{d}.$$