From divergent series to geometry

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Introduction

Sometimes, concepts introduced in physics turn out to have a second life in a more geometric realm. A good example is **supersymmetry**, which provides a physical model of cohomology and leads to topological field theories, BPS invariants, etc.

In this talk I will argue that the ideas developed in physics to understand the information hidden in divergent series lead to novel perspectives in geometry, and even to precise conjectures. This is an example of what is called sometimes "physical mathematics."

Divergent series

Perturbative series have always played an important role in physics, but in quantum field theory (QFT) they become an essential tool. Dyson realized as far back as 1952 that perturbative series in QED must be divergent.

$$E(g) \sim \sum_{n\geq 0} a_n g^n$$

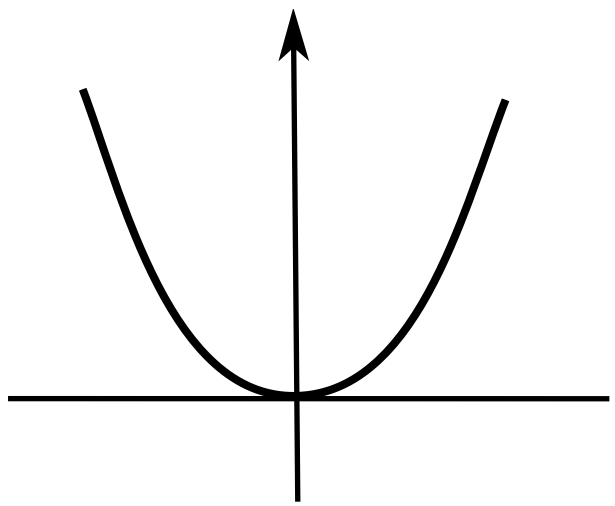
$$a_n \sim n!$$

Dyson's argument is qualitative and heuristic. He notes that in QED, the physics when the coupling constant is negative is very different from the usual physics, so we cannot expect it to be described by the same convergent series.

This argument became quantitative with the work of Bender and Wu in the early 1970s, in the context of quantum mechanics.

Bender-Wu and the quartic oscillator

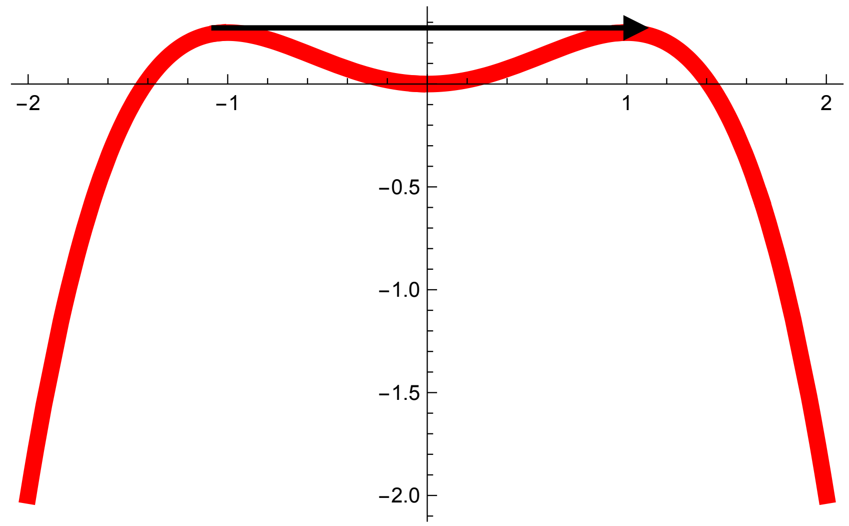
Bender and Wu studied the perturbative series for the ground state energy of the quartic oscillator, which is also factorially divergent



$$\mathsf{H} = \frac{\mathsf{p}^2}{2} + \frac{\mathsf{x}^2}{2} + g\mathsf{x}^4$$

$$E_0(g) \sim \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \cdots$$

Dyson's argument suggests that the theory with **negative** coupling could "explain" the divergence of the series.



$$g = -\lambda < 0$$

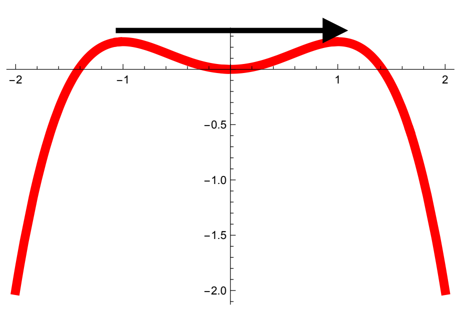
When the coupling is negative, the theory is very different: we have resonances, not bound states.

From the path integral point of view, there is a non-trivial saddle point, given by an **instanton** tunneling between the two critical points. This leads to a new sector of the theory.

Bender and Wu showed that the tunneling amplitude explains quantitatively the behavior of the divergent series:

$$\boxed{a_n \sim (-1)^n A^{-n} n! \longleftrightarrow \mathrm{Im}\, E_0(\lambda) \sim \mathrm{e}^{-A/\lambda}}$$

$\boxed{\text{large order behavior}} \longleftrightarrow \boxed{\text{instantons}}$



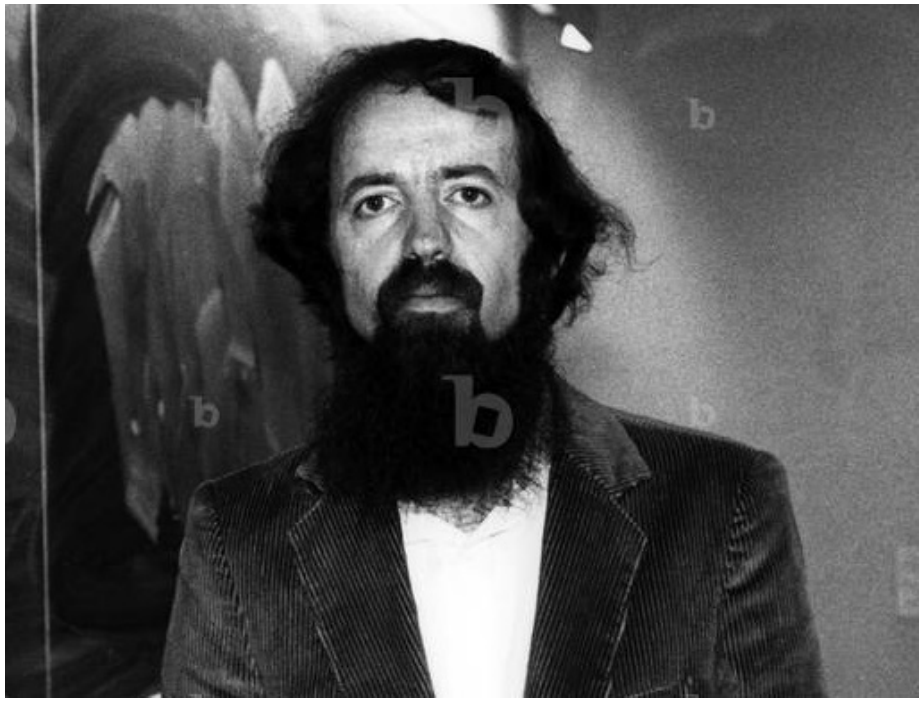
$$A = \frac{1}{3}$$

action of the instanton

To generalize the relation found by Bender and Wu it is convenient to introduce the concept of **resurgent structure,** which is a more powerful mathematical gadget

Resurgent structures

The mathematics of resurgent structures was introduced by Jean Ecalle in the early 1980s.



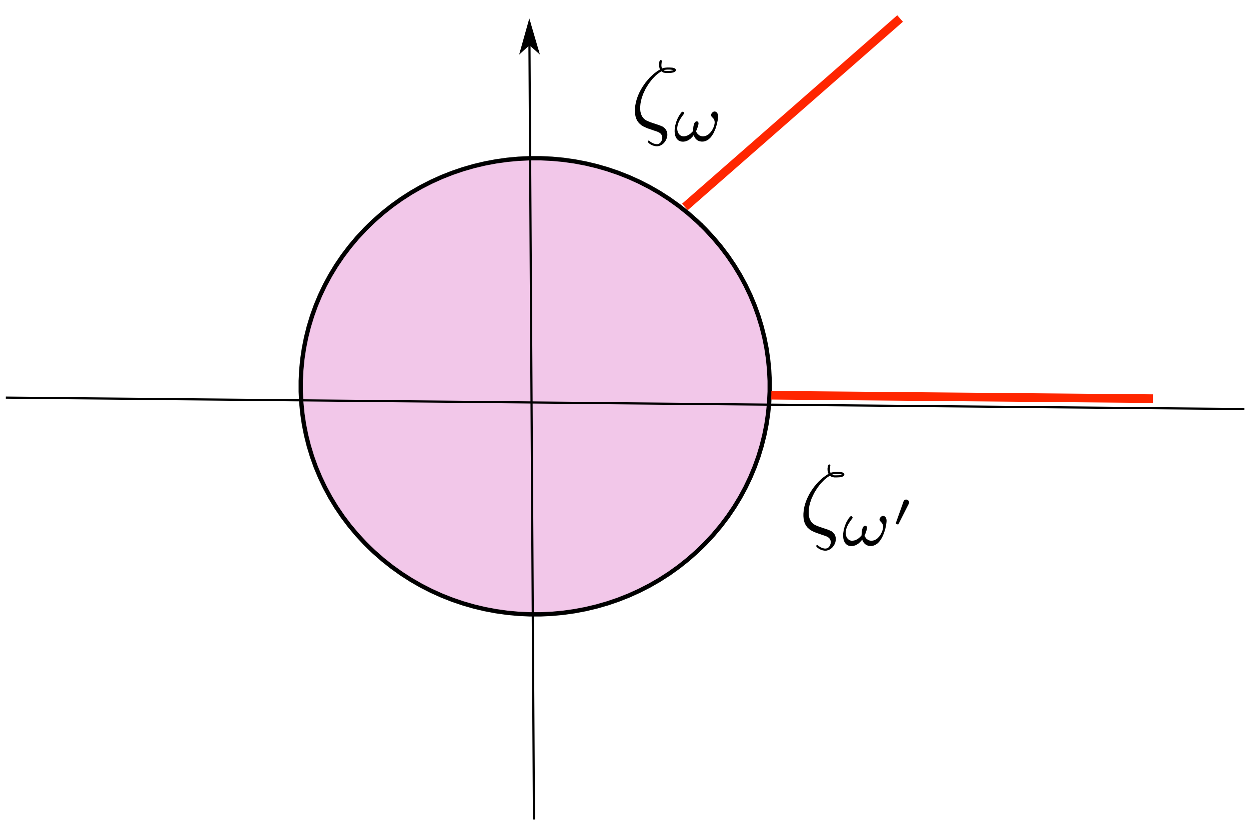
Ecalle was inspired by developments in physics, but he was particularly interested in the divergent series that appear in dynamical systems, when one solves non-linear ODEs. He constructed explicit relations between perturbative and "instanton" sectors for ODEs.

To define the resurgent structure of a factorially divergent series, we start with its **Borel transform**

$$\varphi(z) = \sum_{n \geq 0} a_n z^n \xrightarrow{\boxed{\text{Borel transform}}} \widehat{\varphi}(\zeta) = \sum_{n \geq 0} \frac{a_n}{n!} \zeta^n$$

$$a_n \sim n!$$

The Borel transform is **analytic** at the origin, but when analytically continued to the complex plane we'll find **singularities** (poles, branch cuts).

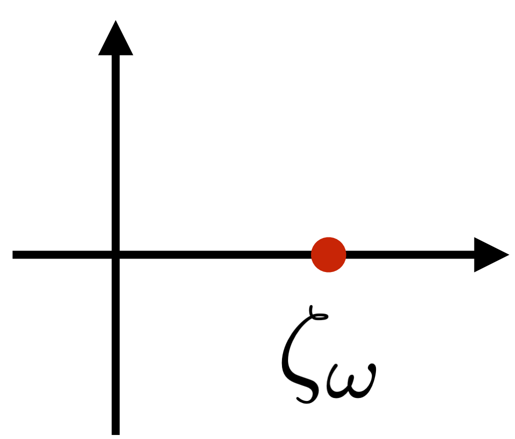


The expansion of the Borel transform around each singularity leads to **new information!**

Borel singularity ζ_ω → new power series $\varphi_\omega(z) = \sum_{n\geq 0} a_{n,\omega} z^n$

Borel singularity ζ_ω → Stokes constant S_ω

E.g. for a log singularity:



$$\widehat{\varphi}(\zeta_\omega + \xi) = \frac{\mathsf{S}_\omega}{2\pi \mathrm{i}} \left(\sum_{n\geq 0} \frac{a_{n,\omega}}{n!} \xi^n \right) \log(\xi) + \cdots$$

The resulting collection of power series and Stokes constants, for all singularities, is called the **resurgent structure** of the perturbative series.

Under some mild assumptions, finding this structure is a well-defined problem with a unique answer.

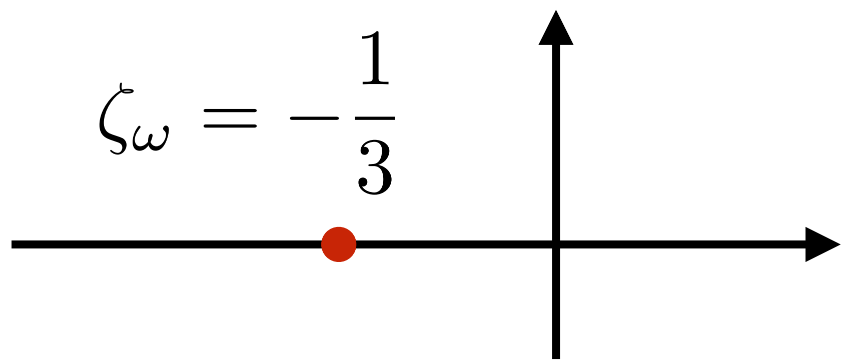
A window into NP physics

One of the reasons we are interested in the resurgent structure of perturbation theory is that it gives us a window into non-perturbative physics.

Indeed, with the information encoded in each Borel singularity one can construct a **non-perturbative object** or **trans-series**

$$\Phi_\omega(z) = \mathsf{S}_\omega \, \mathrm{e}^{-\zeta_\omega/z} \varphi_\omega(z)$$

In Bender-Wu's quartic oscillator, the leading Borel singularity for the perturbative series is on the negative real axis. The associated trans-series gives the all-orders asymptotics of the non-perturbative part of the resonant energy.



$$\mathrm{Im}\, E_0(\lambda) \sim \Phi_{-\frac{1}{3}}(-\lambda) = \frac{1}{\sqrt{\lambda}} \mathrm{e}^{-A/\lambda} \left(1 + \mathcal{O}(\lambda)\right)$$

Of course, this can be also obtained directly by a calculation in the instanton sector of the path integral. Resurgence relates the two sectors.

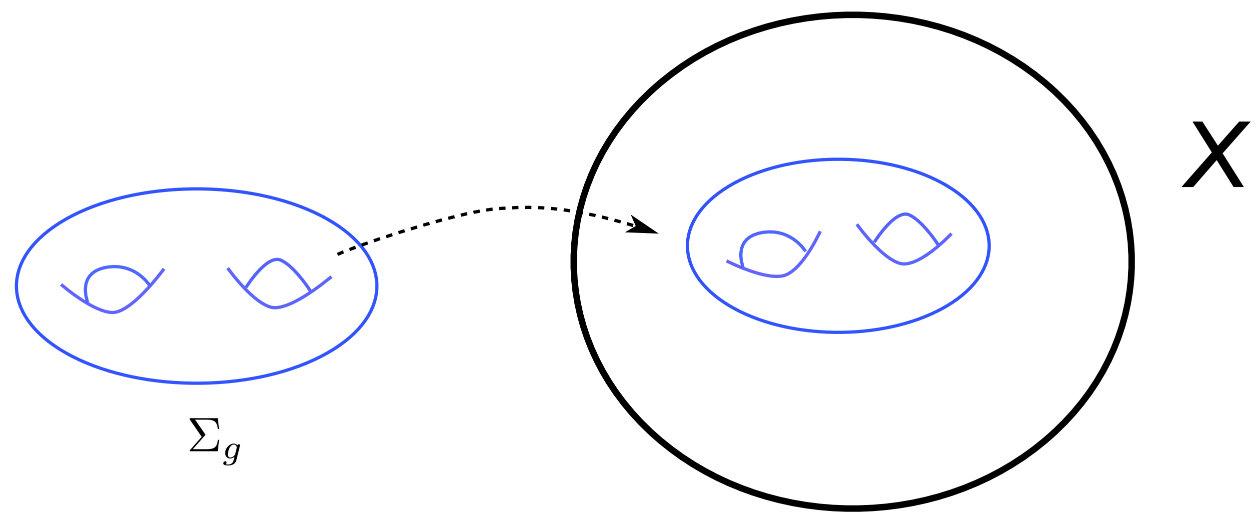
From resurgence to geometry

Can we use the connection between the factorial growth of perturbation theory and instanton sectors in order to obtain new results in geometry?

To do this, we need a physical theory which realizes geometric information. The natural candidates are **topological** theories of fields and strings. In this talk I will focus on **topological theories** of strings.

Perturbative (topological) strings

In string theory, perturbative series are given by genus expansions, since each order in perturbation theory corresponds to a Riemann surface of genus g.



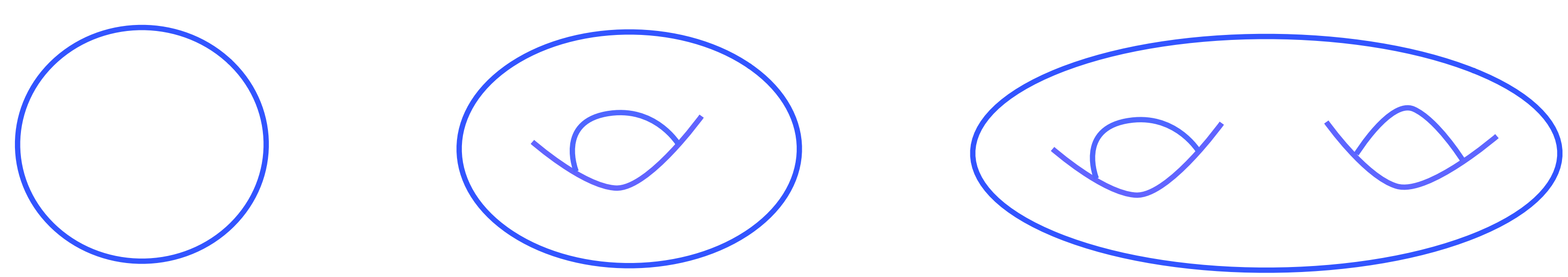
The corresponding coefficient in the series is a complicated function of the geometric moduli *t* of the ambient manifold *X*, called "genus g free energy" $F_g(t)$

In the topological case, one usually takes the ambient manifold to be a Calabi-Yau threefold (i.e. Kahler and Ricci flat) although this can be relaxed.

The genus g function can be obtained by counting holomorphic curves in that space and is the subject of **Gromov-Witten theory**. This is therefore the common arena we were looking for!

Physics suggests that we add the contributions of all genera into a formal power series

$$\sum_{g\geq 0} F_g(t) g_s^{2g-2} = g_s^{-2} F_0(t) + F_1(t) + g_s^2 F_2(t) \quad \cdots$$



Divergent series in string theory

What is the nature of the genus expansion in (topological) string theory? Explicit results, heuristic arguments and partial theorems indicate that

$$F_g(t) \sim (2g)!$$

so we have a factorial divergence similar to the one in quantum mechanics!

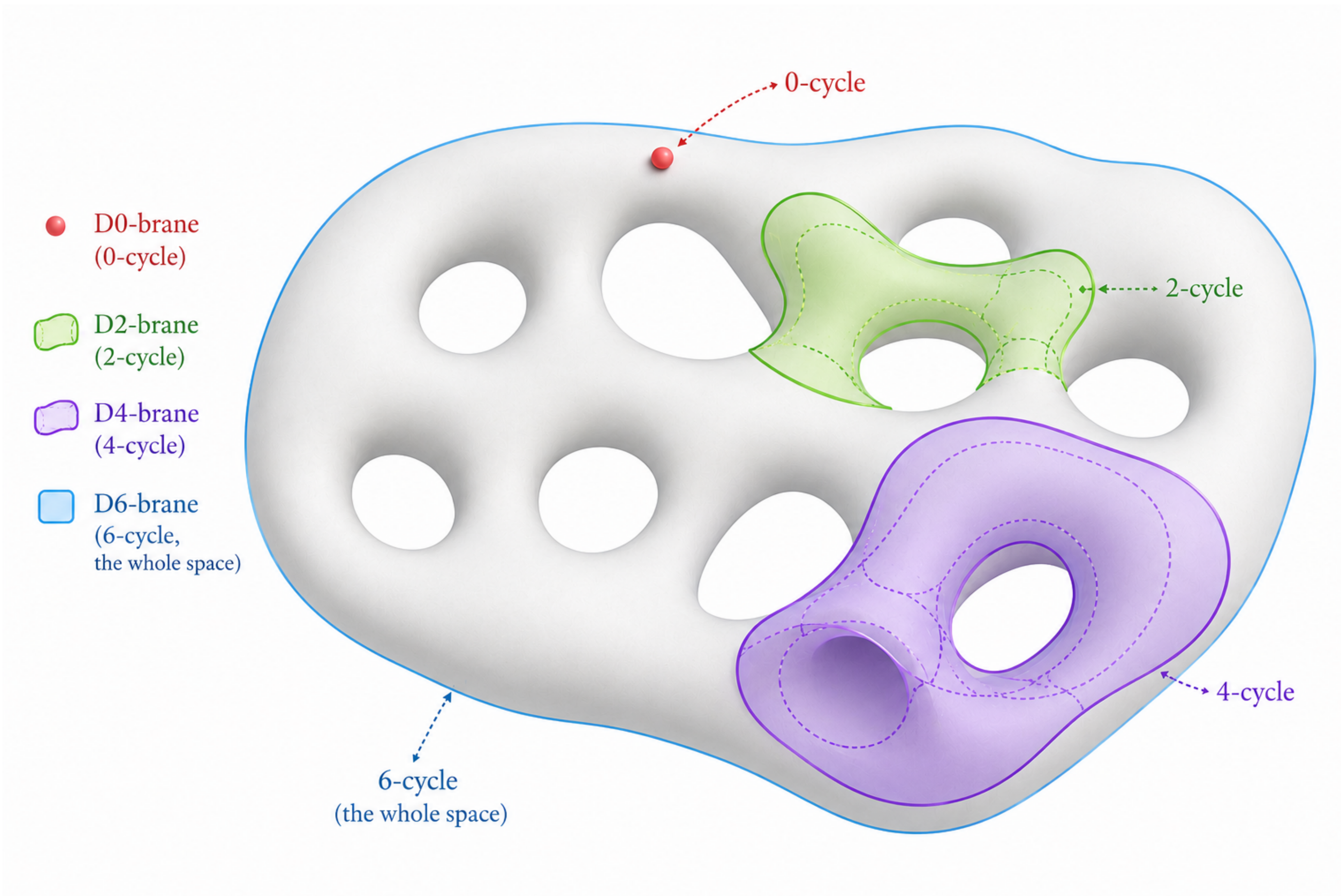
We can therefore ask: **what is the resurgent structure of this series? Where are the Borel singularities?**

Physics not only suggests the question, but also the answer! Based on the work of Bender and Wu, we should look at the **instantons of string theory**!

This is hard, since there is no string theory path integral, but in 1994 Polchinski proposed that these "instantons" are higher-dimensional objects, called **D-branes.**

Geometrically, D-branes are supported on odd or even dimensional homology classes γ of the spacetime manifold.

Strings see only curves, D-branes see higher-dimensional cycles.



It turns out that mathematicians have also developed a theory for counting D-brane configurations. Given a homology class and a value of the moduli *t*, the counting is given by the **Donaldson-Thomas invariant**

$$\Omega(\gamma; t)$$

If the physics discovered by Bender-Wu is still applicable here, we could expect that the perturbative series of the topological string "knows" about these instantons through its factorial growth, or resurgent structure.

One can in fact formulate a very precise conjecture

A conjecture

The sequence of Gromov-Witten genus *g* functions has good resurgent properties (it is a **resurgent function**, in Ecalle's terminology).

Its Borel singularities are labelled by even/odd homology classes of the ambient space, and they are located at the so-called **quantum volumes** of the cycles

$$Z(\gamma; t)$$

The corresponding Stokes constants coincide with the Donaldson-Thomas invariants.

$$\mathsf{S}(\gamma; t) = \Omega(\gamma; t)$$

This conjecture turns out to "explain" many properties of the Donaldson-Thomas invariants:

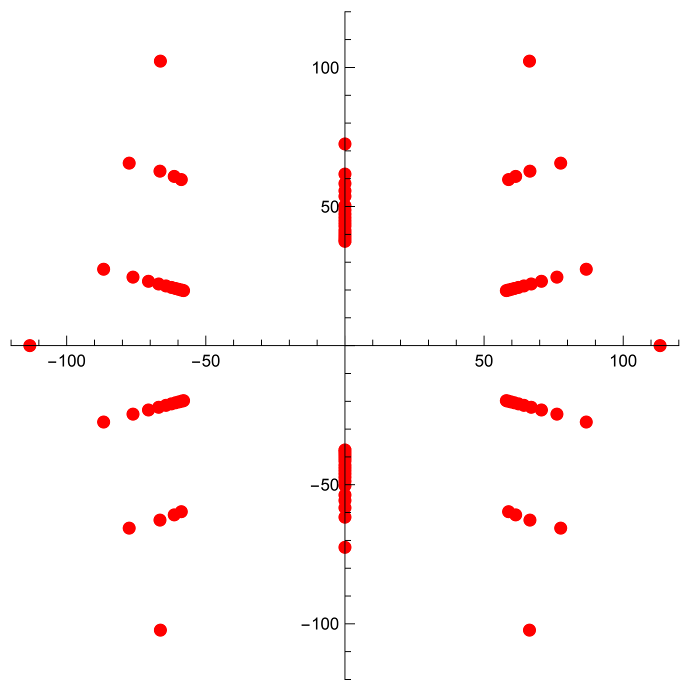
although the coefficients of the perturbative series depend smoothly on the geometric moduli of the CY manifold, the resurgent structure changes discontinuously at special walls.

This explains the so-called wall-crossing behavior of Donaldson-Thomas invariants!

An example

Recent numerical calculations confirm so far this conjecture

A beautiful example is $X = \mathcal{O}(-3) \to \mathbb{P}^2$



By counting holomorphic curves up to high genus (g=300!) one can find Borel singularities corresponding to D-branes made out of $\mathbb{P}^2$ and points on it (D4-D0 branes)

One checks numerically that the Stokes constants give the Donaldson-Thomas invariants, which in this case are Euler characteristics of Hilbert schemes of points on $\mathbb{P}^2$!

[Douaud-KashaniPoor]

Conclusions

The theory of resurgent structures, developed both by physicists and mathematicians, indicates that the factorial divergence of perturbative series is often a smoking gun for instanton sectors.

Surprisingly, when applied to topological string theories, this leads to a precise conjecture relating two of the most important invariants in algebraic geometry (Gromov-Witten and Donaldson-Thomas). There is compelling evidence for this relation, and I hope that mathematicians (or perhaps AI) will be able to find a proof!

Thank you for your attention!