$\mathbb{Z}_2$ lattice gauge theory coupled to fermionic matter

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Joint with: Sven Bachmann (UBC) and Leonardo Goller (SISSA)
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Overview

- Fermions on a lattice, mathematical challenges arising from condensed matter physics. Fermi surface, and its influence on many-body systems at zero temperature.

- Coupling matter to gauge fields. Simple example: $\mathbb{Z}_2$ gauge theory in $d = 2$ coupled to fermions. Interesting phase diagram (numerics). Different phases, associated to different shapes of the Fermi surface.

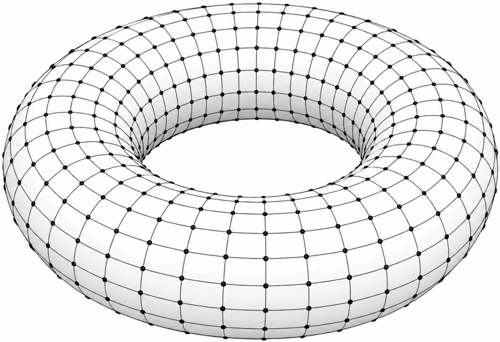
- **Results.** Construction of the ground states of the gauge theory. Emergence of the π-flux phase and of massless Dirac fermions. [Joint with L. Goller.]

  Opening a spectral gap: 4-dimensional ground state space, local indistinguishability. Nontrivial braiding of magnetic and charge excitations. [Joint with S. Bachmann and L. Goller.]

- Conclusions and open problems.

Fermions on a lattice

- Setting: fermions on a d-dimensional lattice, as a model for a crystalline solid. Lattice: $\Gamma_L = \mathbb{Z}^d / L\mathbb{Z}^d$, $d \geq 2$.



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- Single-particle Hilbert space: $\mathfrak{h} = \ell^2(\Gamma_L \times \{1, \ldots, M\}) \simeq \mathbb{C}^{ML^d}$. We will be interested in many particle systems, in a grand-canonical formulation. Fermionic Fock space:

$$\mathcal{F} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \mathfrak{h}^{\wedge n} , \qquad \mathfrak{h}^{\wedge n} = \text{space of } n\text{-particle fermionic wave fcns.}$$

- Useful tool: fermionic creation and annihilation operators,

$$a_{x,\sigma} : \mathcal{F}^{(n)} \to \mathcal{F}^{(n-1)} , \qquad a^*_{x,\sigma} : \mathcal{F}^{(n)} \to \mathcal{F}^{(n+1)} ,$$
$$\{a_{x,\sigma}, a^*_{y,\sigma'}\} = \delta_{\sigma,\sigma'} \delta_{x,y} , \qquad \{a_{x,\sigma}, a_{y,\sigma'}\} = \{a^*_{x,\sigma}, a^*_{y,\sigma'}\} = 0$$

for $x \in \Gamma_L$ and $\sigma \in \{1, \ldots, M\}$.

- $\mathcal{F}$ can be generated by repeated applications of $\{a^*_{x,\sigma}\}$ on $\Omega = (1, 0, \ldots, 0, \ldots)$ (vacuum) and taking linear combinations.

Hamiltonian and Gibbs state

- Typical grand-canonical Hamiltonian:

$$\mathcal{H} = \sum_{x,y} \sum_{\sigma,\sigma'} a^*_{x,\sigma} H_{\sigma\sigma'}(x,y) a_{y,\sigma'} + \lambda \sum_{x,y} \sum_{\sigma,\sigma'} n_{x,\sigma} v_{\sigma\sigma'}(x,y) n_{y,\sigma'}$$

  with: $n_{x,\sigma} = a^*_{x,\sigma} a_{x,\sigma}$; H and v finite-ranged, s.t. $\mathcal{H} = \mathcal{H}^*$.

- Let $\mathcal{O}_X$ be an operator acting non-trivially on fermions on $X \subset \Lambda_L$. Grand-canonical Gibbs state $\langle \cdot \rangle_{\beta,\mu,L}$:

$$\langle \mathcal{O}_X \rangle_{\beta,\mu,L} := \frac{\mathrm{Tr}\, \mathcal{O}_X e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\mathrm{Tr}\, e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}$$

  where $\beta = 1/(\text{temperature})$, $\mu =$ chemical potential, $\mathcal{N} = \sum_{x,\sigma} n_{x,\sigma}$.

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- General (and difficult) questions, as $L \to \infty$ and $\beta \to \infty$:
  - Large scale behavior of the correlation functions of the system? Scaling limits, universality classes? Long-range order?
  - How does the equilibrium state responds to perturbations?

Very hard already for λ small. Heavily dep. on spectral properties of H.

Fermi surface

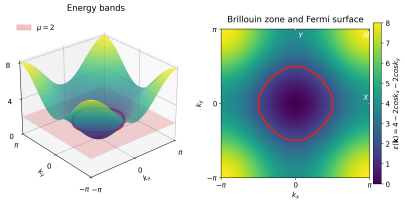
- Suppose that $H_{\sigma\sigma'}(x,y) \equiv H_{\sigma\sigma'}(x-y)$ (translation invariant). The Hamiltonian can be diagonalized in momentum space:

$$H \simeq \bigoplus_{k \in \mathbb{T}_L^d} \hat{H}(k) \,, \qquad \hat{H}(k) = \hat{H}(k)^* \in \mathbb{M}_{M\times M}(\mathbb{C}) \,.$$

- The Fermi surface of (H,μ) is defined as (for $L \to \infty$):

$$F_\mu(H) = \left\{ k \in \mathbb{T}^d \;\middle|\; \mu \in \sigma(\hat{H}(k)) \right\} .$$

Example. If $H = -\Delta$, $d = 2$, $M = 1$: $\hat{H}(k) = 4 - 2\cos k_x - 2\cos k_y$.



- Generically, $F_\mu(H)$ is a closed curve, of codimension 1. Metallic system.

Perturbing the Fermi surface

- $\lambda = 0.$ At $T = 0$, state of the system = projector over the ground state of $\mathcal{H}$. Slater determinant:

$$\psi_0 = \bigwedge_{(k,j):\, \varepsilon_j(k) \le \mu} f_{j,k} \ , \qquad H f_{j,k} = \varepsilon_j(k) f_{j,k} \ ,$$

with $f_{j,k}(x) = L^{-d/2} u_{j,k} e^{ik\cdot x}$ and $u_{j,k} \in \mathbb{C}^M$ (Bloch function).

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- $\lambda \neq 0$. The interaction has a dramatic effect. $U(1)$ gauge symmetry is expected to break down, giving rise to a superconducting transition:

$$\langle \hat{a}_{k,\uparrow} \hat{a}_{-k,\downarrow} \rangle_{T=0} \neq 0 \qquad \text{together with LRO for } O_x = a_{x,\uparrow} a_{x,\downarrow}.$$

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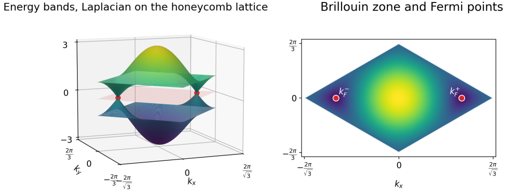
  Formation of Cooper pairs. Predicted by BCS in 1957, no rigorous proof. Big open problem. Rigorous renormalization group allows to prove:

  - No transition for $T \gtrsim e^{-1/|\lambda|}$ [Disertori-Rivasseau '00, Benfatto-Giuliani-Mastropietro '06]
  - No transition for all T for asymmetric enough $F_\mu(H)$ [Feldman-Knörrer-Trubowitz '04]

Semimetals

- Semimetals: $d \geq 2$ systems with pointlike Fermi surface. Most celebrated example: graphene [Geim-Novoselov, Nobel prize '11].



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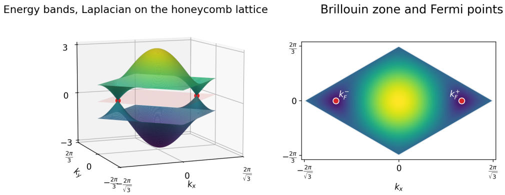

The interacting ground state can be constructed via rigorous RG. No BCS instability. Scaling limit: renormalized massless Dirac fermions. Universal transport properties. [Giuliani-Mastropietro-P. '11$^+$.]

- Conical intersections arise in several condensed matter systems. E.g.: topological insulators at phase transition, Weyl semimetals ($d = 3$). [Giuliani, Mastropietro, P. '17, '22.]

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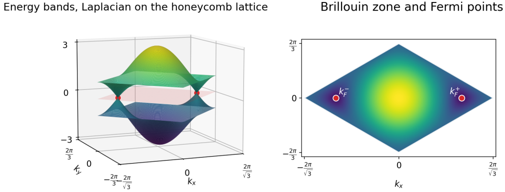

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- Conical intersections arise in several condensed matter systems. E.g.: topological insulators at phase transition, Weyl semimetals ($d = 3$). [Giuliani, Mastropietro, P. '17, '22.]

- **Today:** $\mathbb{Z}_2$ gauge theory coupled to fermionic matter in $d = 2$. Rich phase diagram. Numerics predict a metal-semimetal transition.

Coupling fermions to $\mathbb{Z}_2$ gauge fields

Coupling matter to gauge fields

- Let U_α a local phase transformation: $U_\alpha^* a_x U_\alpha = e^{i\alpha(x)} a_x$. In a $U(1)$ lattice gauge theory, one promotes:

$$H(x;y) \to H(x;y) e^{i\int_{x\to y} A\cdot d\ell} \quad \text{(Peierls' substitution)}$$

with A the vector potential. The full Hamiltonian is gauge invariant:

$$U_\alpha^* \mathcal{H}(A) U_\alpha = \mathcal{H}(A + \nabla\alpha) \ .$$

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- $\mathbb{Z}_2$ gauge theory [Gazit-Randeria-Vishwanath, Nature Phys. 2017]: Simplified model, $d = 2$, in which the Peierls' phase is replaced by $\sigma^z_{(x,y)}$ (third Pauli matrix). Hamiltonian:

$$\mathcal{H} = \mathcal{H}_\mathrm{g} + \mathcal{H}_\mathrm{f} \qquad \text{where:}$$

$$\mathcal{H}_\mathrm{g} = -\sum_{P\subset\Gamma_L} \prod_{b\in\partial P} \sigma^z_b\ , \qquad \mathcal{H}_\mathrm{f} = -t\sum_{x\sim y}\Big(a^*_x \sigma^z_{(x,y)} a_y + \mathrm{h.c.}\Big)$$

($P =$ plaquette.) It acts on the total Hilbert space: $\mathcal{F} \otimes \mathbb{C}^{2|E(\Gamma_L)|}$.

Gauge symmetry and physical Hilbert space

- The model is invariant under the following local transformation:

$$a_x^{\pm} \to \tau_x a_x^{\pm} \,, \quad \sigma^z_{(x,y)} \to \tau_x \sigma^z_{(x,y)} \tau_y \qquad \tau_x = \pm 1 \,.$$

In particular, $\mathcal{H} = Q_x \mathcal{H} Q_x$ with $Q_x = A_x(-1)^{n_x}$, where:

$$A_x = \sigma^x_{(x,x+e_1)} \sigma^x_{(x,x+e_2)} \sigma^x_{(x-e_1,x)} \sigma^x_{(x,x-e_2)} \qquad \text{(star operator)}$$

and $\sigma^x_{(x,y)}$ is the first Pauli matrix.

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  and $\sigma^x_{(x,y)}$ is the first Pauli matrix.

- A state $\psi \in \mathcal{F} \otimes \mathbb{C}^{2|E(\Gamma_L)|}$ satisfies the Gauss' law if:

$$\sigma^x_{(x,x+e_1)} \sigma^x_{(x,x+e_2)} \sigma^x_{(x-e_1,x)} \sigma^x_{(x,x-e_2)} \psi = (-1)^{n_x} \psi \quad \forall x \in \Gamma_L$$

  Interpreting σ^x_b as $e^{i\pi E_b}$, exponentiated $\mathbb{Z}_2$-version of $\nabla \cdot E_x = n_x$.

- We define the physical Hilbert space as:

$$\mathfrak{H}_{\text{phys}} := \left\{ \psi \in \mathcal{F} \otimes \mathbb{C}^{2|E(\Gamma_L)|} \;|\; Q_x \psi = \psi \quad \forall x \right\}$$

Gibbs state of the model

- The Gibbs state of the model is:

$$\langle \mathcal{O} \rangle_{\beta,\mu,L} = \frac{\mathrm{Tr} P_{\mathrm{G}} \mathcal{O} e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\mathrm{Tr} P_{\mathrm{G}} e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}$$

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- The plaquette term is minimized by the 0-flux phase. E.g.: $\sigma^z_{(x,y)} = 1$ for every bond (x, y). In this case, $\mathcal{H}_f$ is the lattice Laplacian. Spectrum:

$$\varepsilon(k) = t(4 - 2\cos k_x - 2\cos k_y)$$

- For a given choice of chemical potential, the ground state is obtained by populating all modes k such that $\varepsilon(k) \leq \mu$.

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- In our gauge theory, it is expected that for t small the 0-flux state is energy minimizing. Not easy to prove. What about large t?

Phase diagram

- Monte Carlo analysis: Gazit, Randeria, Vishwanath 2017. They considered a generalization of the model:

$$\mathcal{H}_h = \mathcal{H} + h \sum \sigma_b^x \ .$$

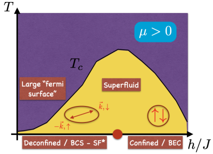


FIG. 1: Schematic phase diagram at fixed $\mu > 0$ as a function of h/J and the temperature T.

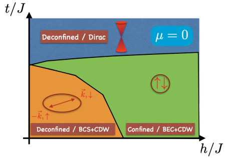


FIG. 2: Schematic $T = 0$ phase diagram at half filling with $\mu = 0$.

- At the half-filling point $\mu = 0$, the model has an unexpected extra phase: pointlike Fermi surface, relativistic low-energy spectrum.

- Effect of Hubbard interaction (antiferromagnetic transition): Gazit, Assaad, Sachdev, Vishwanath, Wang; PNAS 2018

- Large and fast growing literature on cold atoms realizations and studies via quantum computers.

Emergent Dirac fermions

- We will focus on the emergent semimetallic phase (Dirac fermions), expected to arise at large enough hopping t.

- The probe for such semimetallic phase is the magnetic susceptibility: $\chi(q) = \partial \hat{M}(q)/\partial \hat{B}(q)$ (see later for precise def.)

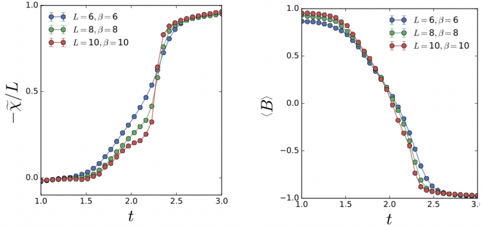


Figure: From Gazit, Randeria, Vishwanath; Nature Phys. 2017

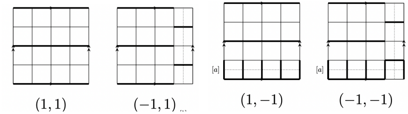
Left (susceptibility): the value 1 is the prediction for $2d$ massless Dirac fermions. Right (flux per plaquette): at large t, a π-flux phase emerges.

On the π-flux phase

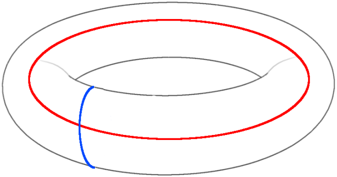
- Key insight: Lieb '94. Let $\mathcal{H}_f(\boldsymbol{\sigma})$ be the fermionic Hamiltonian, with frozen spins $\boldsymbol{\sigma} = \{\sigma_b^z\}$. By reflection positivity (later):

$$\mathrm{Tr}_{\mathcal{F}}\, e^{-\beta \mathcal{H}_f(\boldsymbol{\sigma})} \leq \mathrm{Tr}_{\mathcal{F}}\, e^{-\beta \mathcal{H}_f(\boldsymbol{\sigma}_\pi)}$$

The configuration $\boldsymbol{\sigma}_\pi$ describes a π-flux phase.



4 inequivalent π-phases, depending on fluxes on non-contractible loops.



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  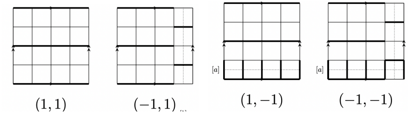

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- **But:** $\boldsymbol{\sigma}_\pi$ maximises $\mathcal{H}_{\mathrm{g}}(\boldsymbol{\sigma})$! Energetic competition between gauge fields and fermions. To analyze the gauge theory we need to:
  - quantify the energetic cost to "remove monopoles";
  - include the gauge constraint in the RP arguments;
  - study the π-flux phases, compute the susceptibility.

Dirac fermions in the π-flux phase

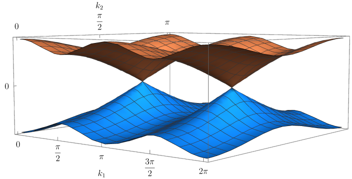


Figure: Energy bands associated with the π-flux phase.

- The object we shall consider is, for $p = (0, p_2)$:

$$\chi_{\beta,L}(p_2) := -\frac{1}{p_2^2} \frac{\partial^2}{\partial \hat{A}_1(p) \partial \hat{A}_1(-p)} \frac{1}{\beta L^2} \log \mathrm{Tr} P_{\mathrm{G}} e^{-\beta \mathcal{H}(A)} \Big|_{A=0}$$

  For massless $2d$ Dirac fermions with velocity v, it is as $|p_2| \to 0$:

$$\chi_{\mathrm{D}}(p_2) = -\frac{v}{8|p_2|}$$

- In the figure: the velocity is $v = 2t$ with t the hopping parameter.

Results

Restriction to the π-flux phase

- Given a physical observable O, one has:

$$\langle O\rangle_{\beta,L} = \frac{\mathrm{Tr} P_{\mathrm{G}} O e^{-\beta\mathcal{H}}}{\mathrm{Tr} P_{\mathrm{G}} e^{-\beta\mathcal{H}}} = \frac{\sum_{\boldsymbol{\sigma}} \mathrm{Tr}_{\mathcal{F}}(1+(-1)^{\mathcal{N}})O(\boldsymbol{\sigma})e^{-\beta\mathcal{H}(\boldsymbol{\sigma})}}{\sum_{\boldsymbol{\sigma}} \mathrm{Tr}(1+(-1)^{\mathcal{N}})e^{-\beta\mathcal{H}(\boldsymbol{\sigma})}}$$

with $O(\boldsymbol{\sigma}) = \langle\boldsymbol{\sigma}|O|\boldsymbol{\sigma}\rangle$. Gauge constraint $\equiv$ parity constraint.

- Remark: (Lieb-Loss '93.) The spectrum of $\mathcal{H}(\boldsymbol{\sigma})$ only depends on the fluxes across elementary plaquettes and non-contractible loops.

- Restriction to the π-flux phase. Let us define, with $(a,b)\in\mathbb{Z}_2^2$ = fluxes across non-contractible loops:

$$\langle O\rangle^{\pi}_{\beta,L} := \frac{\sum_{(a,b)\in\mathbb{Z}_2^2} \mathrm{Tr}_{\mathcal{F}}(1+(-1)^{\mathcal{N}})O(\pi,a,b)e^{-\beta\mathcal{H}_f(\pi,a,b)}}{\sum_{(a,b)\in\mathbb{Z}_2^2} \mathrm{Tr}_{\mathcal{F}}(1+(-1)^{\mathcal{N}})e^{-\beta\mathcal{H}_f(\pi,a,b)}}\,.$$

We would like to compare $\langle\cdot\rangle_{\beta,L}$ and $\langle\cdot\rangle^{\pi}_{\beta,L}$.

Stability of the π-flux phase

Theorem (Goller-P.; J. Stat. Phys. 2026)

Let L be even. For t large enough (indep of β, L) and $\beta \geq C \log L$:

$$\Big| \langle O \rangle_{\beta,L} - \langle \mathcal{O} \rangle^{\pi}_{\beta,L} \Big| \leq C \|O\| L^2 e^{-\beta(\Delta - 1)}$$

where $\Delta = \kappa t$ and $\kappa = 0.036\ldots$. Also, let:

$$E_{0;L}(\pi, a, b) = - \lim_{\beta \to \infty} \frac{1}{\beta} \log Z_{\beta,L}(\pi, a, b) \; .$$

The ground state is asymptotically 4-fold degenerate. That is, for $\varepsilon > 0$:

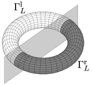
$$E_{0;L}(\pi, -1, -1) \leq E_{0;L}(\pi, a, b) \leq E_{0;L}(\pi, -1, -1) + \frac{C_\varepsilon}{L^{1-\varepsilon}} \; .$$

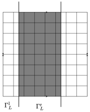
Furthermore, the ground state susceptibility is, for $|p_2| \ll 1$:

$$\lim_{L \to \infty} \lim_{\beta \to \infty} \chi_{\beta,L}(p_2) = - \frac{2t}{8|p_2|} + o\Big(\frac{1}{|p_2|}\Big) \; .$$

Remarks

- About reflection positivity. Cut the torus in two halves, as in the figure:





The reflection operator Θ is an antiunitary operator s.t.:

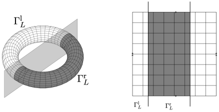
$$\Theta(a_{x,\sigma}) = a^*_{R(x),\sigma} \qquad R = \text{reflection through plane.}$$

RP of $\mathcal{H}$ is the observation that (up to a gauge transformation):

$$-\mathcal{H}(\boldsymbol{\sigma}) = -\mathcal{H}_{\text{left}}(\boldsymbol{\sigma}_{\text{left}}) - \mathcal{H}_{\text{right}}(\boldsymbol{\sigma}_{\text{right}}) + \sum_\alpha X_\alpha \Theta X_\alpha \quad \text{for suitable } X_\alpha\text{'s.}$$

Remarks

- About reflection positivity. Cut the torus in two halves, as in the figure:



The reflection operator Θ is an antiunitary operator s.t.:

$$\Theta(a_{x,\sigma}) = a^*_{R(x),\sigma} \qquad R = \text{reflection through plane.}$$

RP of $\mathcal{H}$ is the observation that (up to a gauge transformation):

$$-\mathcal{H}(\boldsymbol{\sigma}) = -\mathcal{H}_{\text{left}}(\boldsymbol{\sigma}_{\text{left}}) - \mathcal{H}_{\text{right}}(\boldsymbol{\sigma}_{\text{right}}) + \sum_\alpha X_\alpha \Theta X_\alpha \quad \text{for suitable } X_\alpha\text{'s.}$$

- RP $\Rightarrow$ the π-flux phase maximises the partition function [Lieb '93].
A more quantitative bound can be obtained via the chessboard estimate:

$$\mathrm{Tr}_{\mathcal{F}}(\pm 1)^{\mathcal{N}} e^{-\beta\mathcal{H}(\boldsymbol{\sigma})} \leq e^{-\beta k(\boldsymbol{\sigma})\Delta_\pm} \mathrm{Tr}_{\mathcal{F}}(\pm 1)^{\mathcal{N}} e^{-\beta\mathcal{H}(\boldsymbol{\sigma}_\pi)}$$

$k(\boldsymbol{\sigma})$ = # of zero fluxes, $\Delta_\pm > 0$ the energy of a periodic flux config.

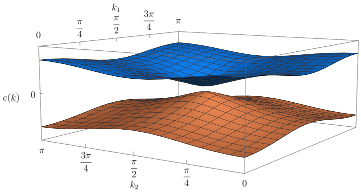
Opening a charge gap

- Our previous result shows the existence of a gap for the magnetic excitations. Still, the charge excitations are gapless.

- Staggered mass (with spin). Let $n_{x,\sigma} = a^*_{x,\sigma} a_{x,\sigma}$. Consider:

$$\mathcal{H}_{m,\lambda} = \mathcal{H} + m \sum_{x \in \Gamma_L} \sum_{\sigma = \uparrow\downarrow} (-1)^x n_{x,\sigma} + \lambda \sum_x \left(n_{x,\uparrow} - \frac{1}{2}\right)\left(n_{x,\downarrow} - \frac{1}{2}\right)$$

  The parameter m opens a gap in the spectrum of the π-flux phase. The Hamiltonian $\mathcal{H}_{m,\lambda}$ is still compatible with RP.



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- Degenerate gapped ground states might exhibit interesting braiding properties. Example: toric code [Kitaev, 1997] (next slide).

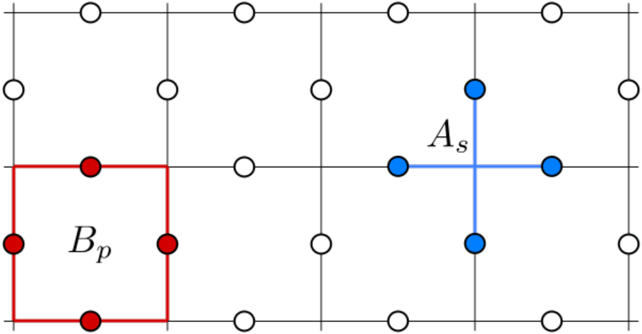
- The toric code is expected to describe the topological properties of a class of fermionic models (*e.g.* Kitaev's honeycomb model, 2006).
  Rigorous connection?

About the toric code

- The Hamiltonian of the toric code is:

$$\mathcal{H}_{\mathrm{TC}} = \sum_P B_P + \sum_v A_v$$

B_P = product of σ^z on a plaquette, while A_v = product of σ^x on a star.



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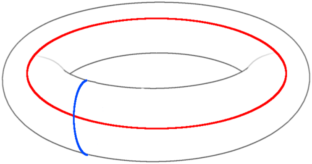
- The ground states of the model are given by:

$$\psi = c\Big[\prod_v (1 - A_v)\Big]\big|\boldsymbol{\sigma}_\pi\big\rangle \qquad \text{with } B_P\big|\boldsymbol{\sigma}_\pi\big\rangle = -\big|\boldsymbol{\sigma}_\pi\big\rangle \text{ for all } P$$

There are four degenerate ground states, parametrized by the eigenvalues of $W^z_{\mathcal{C}} = \prod_{b\in\mathcal{C}} \sigma^z_b$ with $\mathcal{C}$ the non-contractible loops of the torus.



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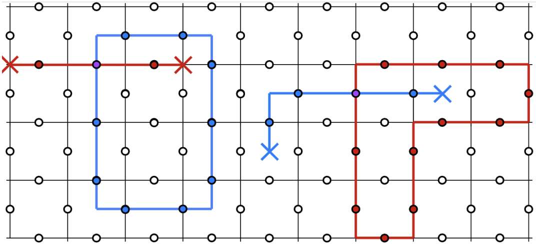
- The ground states cannot be distinguished via expectations of local observables. They can be connected applying $W^x_{\mathcal{C}^*} = \prod_{b\in\mathcal{C}^*} \sigma^x_b$ with $\mathcal{C}^*$ a non-contractible loop in the dual lattice since:

$$[W^x_{\mathcal{C}^*}, H_{\mathrm{TC}}] = 0\ , \qquad W^z_{\mathcal{C}} W^x_{\mathcal{C}^*} = -W^x_{\mathcal{C}^*} W^z_{\mathcal{C}} \quad \text{if } \mathcal{C}^* \perp \mathcal{C}.$$

$W^z_{\mathcal{C}}$, $W^x_{\mathcal{C}^*}$ are related to charge and magnetic excitations.
Nontrivial commutator $\equiv$ nontrivial braiding of the excitations.

Excitations



- A magnetic excitation at P corresponds to $B_P = +1$. They always come in pairs. They can be produced acting with strings of σ^x on ψ.
- A charge excitation at v corresponds to $A_v = +1$. They always come in pairs. They can be produced acting with strings of σ^z on ψ.

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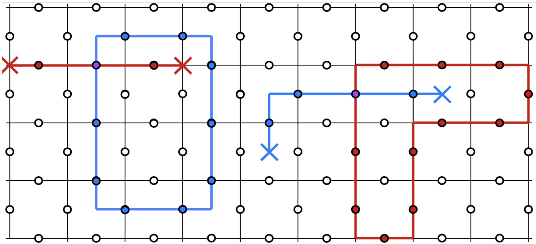

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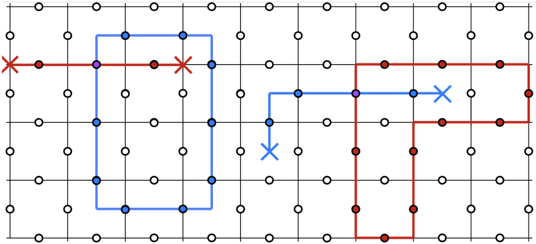

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- The Wilson loops $W^z_{\mathcal{C}}$, $W^x_{\mathcal{C}^*}$ describe the process of annihilation of excitations, after threading them around a non-contractible path.
- Their order matters: they have a nontrivial braiding ("abelian anyons"). Similar structure in a model coupled to dynamical fermions?

Gapped ground states in the $\mathbb{Z}_2$ gauge theory

Theorem (Bachmann-Goller-P.; Comm. Math. Phys. (2026))

For $m \neq 0$, $|\lambda|$ small enough, and t large enough, the following is true. The ground states of $\mathcal{H}_{m,\lambda}$ on $\mathfrak{H}_{phys}$ are, as $L \to \infty$:

$$\Omega_{a,b} = \frac{1}{\Xi} \cdot P_G \, \psi_{a,b}(\pi) \otimes \xi_{a,b}(\pi)$$

where:

- $\psi_{a,b}(\pi)$ = *ground state of the π-flux fermionic Hamiltonian, with (a,b)-boundary conditions;*
- $\xi_{a,b}(\pi)$ *is such that*

$$B_P \xi_{a,b}(\pi) = -\xi_{a,b}(\pi)\ , \quad \mathcal{W}^z_{\mathcal{C}_x} \xi_{a,b}(\pi) = a\xi_{a,b}(\pi) \quad \mathcal{W}^z_{\mathcal{C}_y} \xi_{a,b}(\pi) = b\xi_{a,b}(\pi)\ ;$$

- *The ground states are gapped. Let $P = \sum_{a,b} |\Omega_{a,b}\rangle\langle\Omega_{a,b}|$. Then:*

$$g := \inf spec(P^\perp \mathcal{H}_{m,\lambda} P^\perp) - \sup spec(P \mathcal{H}_{m,\lambda} P) > 0 \qquad \text{uniformly in } L.$$

Indistinguishability and braiding properties

Theorem (Bachmann-Goller-P.; Comm. Math. Phys. (2026))

- *For local O, the ground states are indistinguishable:*

$$\left|\langle\Omega_{a,b}, O\Omega_{a,b}\rangle - \lim_{\beta\to\infty}\langle O\rangle_{\beta,L}\right| \leq C_O e^{-cL} \ .$$

- *They are connected by suitable loop operators $W^x_{\mathcal{C}^*}$, that can be explicitly constructed. The operators $W^x_{\mathcal{C}^*_i}, W^z_{\mathcal{C}_j}$ satisfy the toric code algebra:*

$$[W^z_{\mathcal{C}_j}, W^z_{\mathcal{C}_i}] = [W^x_{\mathcal{C}^*_i}, W^z_{\mathcal{C}_i}] = 0 \ , \qquad \{W^x_{\mathcal{C}^*_1}, W^z_{\mathcal{C}_2}\} = 0 \ ,$$

and:

$$(W^x_{\mathcal{C}^*_1})^*(W^x_{\mathcal{C}^*_2})^* W^x_{\mathcal{C}^*_1} W^x_{\mathcal{C}^*_2}\Omega_{a,b} = e^{i\frac{\pi}{2}\sigma_H}\Omega_{a,b}$$

where σ_H is the Hall conductivity of the π-flux fermionic Hamiltonian, which vanishes.

Remarks

- The magnetic gap is established via RP and chessboard estimates. The charge gap and local indistinguishability are established by fermionic cluster expansion [Brydges-Battle-Federbush formula.]

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  Given a path $[0;1] \ni s \mapsto \gamma(s) \in \mathbb{T}^2$, it follows that:

$$P(s) = V(s)^* P(0) V(s) \; , \qquad P(s) \equiv P(\gamma(s)) \; , \quad V(s) \equiv V(\gamma(s))$$

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- The ground states of the gauge theory are swapped by the action of:
  $$W^x_{\mathcal{C}^*_x} := \Big[\prod_{b \in \mathcal{C}^*_x} \sigma^x_b \Big] V((\pi, 0)) \; , \qquad W^x_{\mathcal{C}^*_y} := \Big[\prod_{b \in \mathcal{C}^*_y} \sigma^x_b \Big] V((0, \pi)) \; .$$
  The commutator of these operators, evaluated on a ground state, can be related to the Hall conductivity of the fermionic sector.

Conclusions and open problems

- Lattice gauge theories are relevant for current condensed matter physics. Progress can be obtained via the combination of different mathematical methods: cluster expansion, reflection positivity, quasi-adiabatic flow.

- Ongoing work (with Leonardo Goller): Rigorous RG methods and RP can be combined to prove the stability of the (massless) π-flux phase against many-body interactions.

- Many open problems:
  - Different gauge groups? E.g. $\mathbb{Z}_N$? Asymptotics for $N \gg 1$?
  - BEC phase at large external field h?
  - Optimal flux away from half-filling? Flux-phase conjecture (Lieb): the optimal flux should be $2\pi \times$ filling factor.
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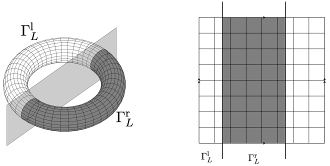
- **Thank you!**

Reflection positivity

- Rewrite the fermionic Hamiltonian with frozen spins as:

$$\mathcal{H}_f(\boldsymbol{\sigma}) = \mathcal{H}_l(\boldsymbol{\sigma}) + \mathcal{H}_r(\boldsymbol{\sigma}) + \mathcal{H}_{int}(\boldsymbol{\sigma})$$

where r, l refers to the right or the left of a cut:



- After a gauge transformation, the interface part can be written as:

$$\mathcal{H}_{int}(\boldsymbol{\sigma}) = -t \sum_{x:\, x \in \Gamma^{\mathrm{l}}_L,\, x+e_1 \in \Gamma^{\mathrm{r}}_L} (a^*_x a_{x+e_1} + a^*_{x+e_1} a_x)$$

so that $-\mathcal{H}_{int}$ has positive coefficients.

- The reflection operator Θ is an antiunitary operator s.t.:

$$\Theta(a_x) = a^*_{R(x)} \qquad R = \text{reflection through plane.}$$

Reflection positivity

- The Hamiltonian has the form, with X_α on the left half-torus:

$$-\mathcal{H}_f((\boldsymbol{\sigma}_l,\boldsymbol{\sigma}_r)) = -\mathcal{H}_l(\boldsymbol{\sigma}_l) - \mathcal{H}_r(\boldsymbol{\sigma}_r) + \sum_\alpha X_\alpha \Theta(X_\alpha)\ .$$

- Let $Z^\pm = \mathrm{Tr}\,(\pm 1)^N e^{-\beta\mathcal{H}_f}$. Following [Lieb '94] one has:

$$Z^\pm\big(\mathcal{H}_l(\boldsymbol{\sigma}_l),\mathcal{H}_r(\boldsymbol{\sigma}_r)\big)^2 \le Z^\pm\big(\mathcal{H}_l(\boldsymbol{\sigma}_l),\Theta(\mathcal{H}_l(\boldsymbol{\sigma}_l))\big) Z^\pm\big(\Theta(\mathcal{H}_r(\boldsymbol{\sigma}_r)),\mathcal{H}_r(\boldsymbol{\sigma}_r)\big)$$

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- Observe that the reflection flips the hoppings:

$$\Theta(a^+_{x,\sigma} a^-_{x+e_2,\sigma}) = a^-_{R(x),\sigma} a^+_{R(x+e_2),\sigma} = -a^+_{R(x+e_2),\sigma} a^-_{R(x),\sigma}$$

After cutting and reflecting, the new flux through the plaquettes across the cut is π. Iterating:

$$Z^\pm(\mathcal{H}(\boldsymbol{\sigma})) \le Z^\pm(\mathcal{H}(\boldsymbol{\sigma}_\pi))$$

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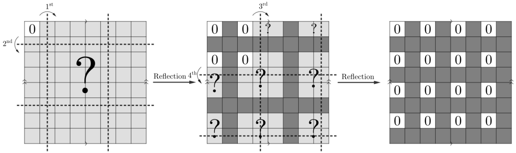
$$Z^\pm(\mathcal{H}(\boldsymbol{\sigma})) \leq Z^\pm(\mathcal{H}(\boldsymbol{\sigma}_\pi))$$

- Actually, we can quantify the cost of 0-fluxes using the chessboard estimate, another consequence of RP [Fröhlich, Israel, Lieb, Simon]:

$$Z^\pm(\{k \text{ 0-flux insertion}\}) \leq e^{-\beta k \Delta^\pm} Z^\pm(\mathcal{H}(\boldsymbol{\sigma}_\pi)) \quad \text{with } \Delta \propto t.$$

Removing monopoles



- Suppose $\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}(1)$ has a 0-flux P. By RP (similar analysis for Z^-):

$$\log Z^+(\boldsymbol{\sigma}(1)) \leq \frac{1}{2} \log Z^+(\boldsymbol{\sigma}(2)) + \frac{1}{2} \log Z^+(\boldsymbol{\sigma}_\pi) \quad \text{and iterating:}$$

$$\leq \frac{1}{2^{2m}} \log Z^+(\boldsymbol{\sigma}_*) + \left(1 - \frac{1}{2^{2m}}\right) \log Z^+(\boldsymbol{\sigma}_\pi)$$

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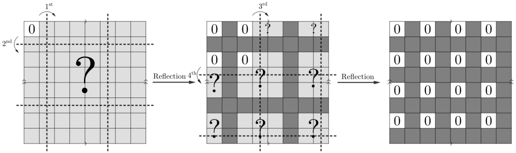

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$\boldsymbol{\sigma}_* =$ chessboard config. and $m = \lceil \log_2(L/2) \rceil$. That is, for $\beta, L \gg 1$:

$$\frac{1}{\beta} \log \frac{Z^+(\boldsymbol{\sigma})}{Z^+(\boldsymbol{\sigma}_\pi)} \leq \frac{1}{\beta L^2} \log \frac{Z^+(\boldsymbol{\sigma}_*)}{Z^+(\boldsymbol{\sigma}_\pi)} = -\Delta^+ \simeq -0.036 \cdot t \quad \text{for } U = 0$$

from an explicit Bloch-diagonalization of the chessboard Hamiltonian.

Braiding monopoles in presence of a staggered mass

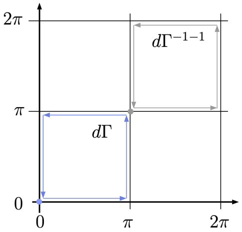
- Using that:

$$X_{\mathcal{C}_x^*} V((\phi_1, \phi_2)) X_{\mathcal{C}_x^*} = V((\phi_1, \phi_2 + \pi))$$
$$X_{\mathcal{C}_y^*} V((\phi_1, \phi_2)) X_{\mathcal{C}_y^*} = V((\phi_1 + \pi, \phi_2))$$

we can ultimately rewrite the braiding phase of the monopoles as:

$$\langle \Omega_{a,b}, (W^x_{\mathcal{C}_1^*})^* (W^x_{\mathcal{C}_2^*})^* W^x_{\mathcal{C}_1^*} W^x_{\mathcal{C}_2^*} \Omega_{a,b} \rangle = \langle \Omega_{a,b}, V(\gamma(1)) \Omega_{a,b} \rangle$$

with $\gamma(\cdot)$ a closed path:



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with $\gamma(\cdot)$ a closed path.

- $V(\gamma(1))\Omega_{a,b} = e^{i\omega}\Omega_{a,b}$ since the (a,b)-ground state is unique.
The phase is: $(\psi \equiv \psi(\phi)$, ground state with fluxes $\phi = (\phi_1, \phi_2))$

$$\omega = \int_{[0,1]} ds\, \dot{\gamma} \cdot \langle \psi, \nabla \psi \rangle = \int_{[0,\pi]^2} \mathrm{Im} \langle \partial_{\phi_1} \psi, \partial_{\phi_2} \psi \rangle = \frac{\pi}{2} \sigma_{\mathrm{H}}$$

by Stokes' theorem and by ϕ-independence of the integrand.