Membranes and Maps

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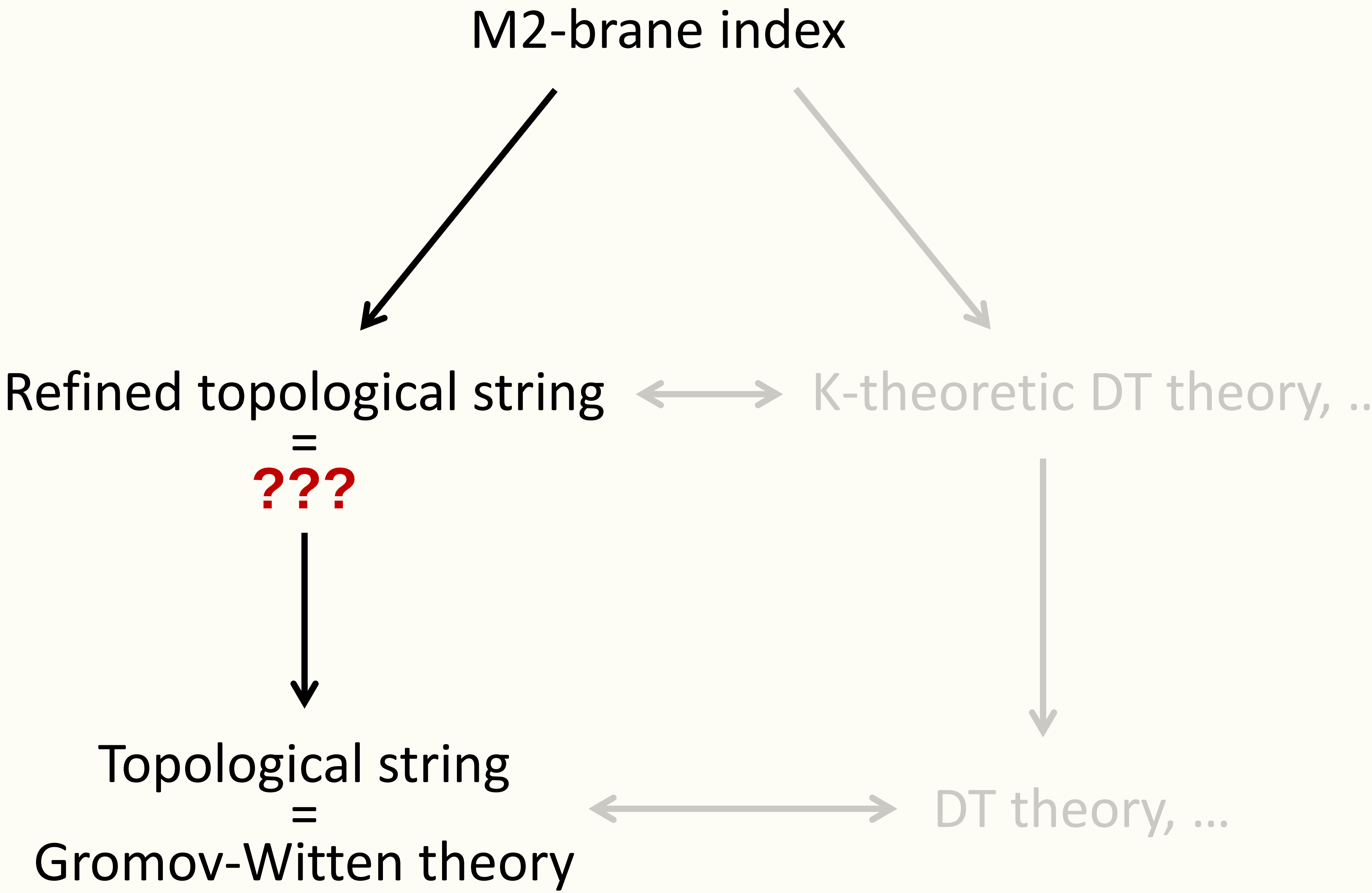
Algebra and Quantum Geometry of BPS Quivers
Les Diablerets

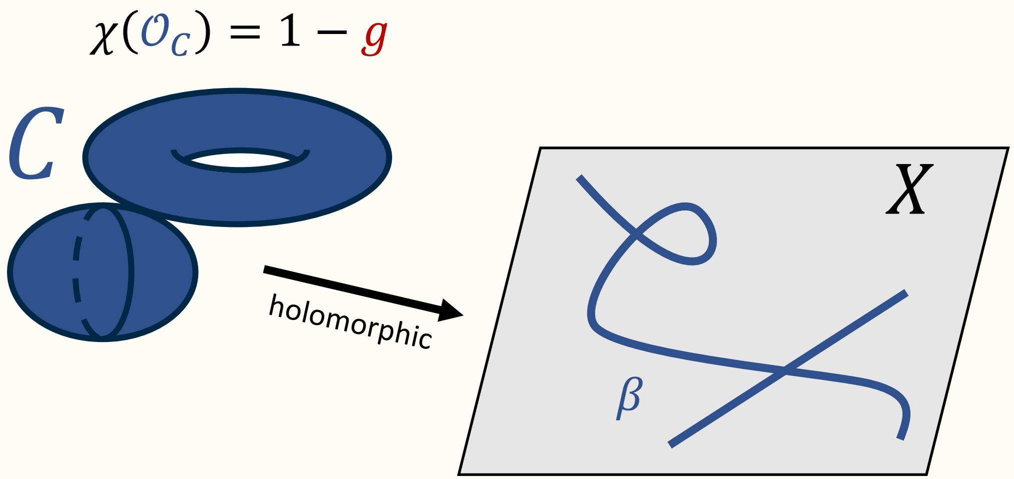
ETH zürich

joint with A. Brini
& ongoing with A. Giacchetto


M2-brane index
Refined topological string
=
???
K-theoretic DT theory, ...
Topological string
=
Gromov-Witten theory
DT theory, ...

curve class

$$\overline{\mathcal{M}}\,(X,\beta) = \left\{ \quad \right\} \Big/ \sim$$



assumed CY3!

$\Rightarrow \mathrm{virdim} = 0$

A-twisted topological string on X = Gromov-Witten theory of X

$$F_\beta^X := \sum_{g\geq 0} g_s^{2g-2} \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{\mathrm{vir}}} 1 \quad \in \mathbb{Q}((g_s))$$

M2 brane index

of $X \times \mathbb{C}^2 \curvearrowleft T$

$\rtimes S^1$

$\downarrow$

$$\overset{\epsilon_4}{} \quad \overset{\epsilon_5}{}$$

$$X \times \mathbb{C} \times \mathbb{C}$$

holomorphic 5-form **fixed!**

$$\circlearrowright$$

$$T \cong (\mathbb{C}^\times)^n$$

Refined TS = Equivariant GW

$$\mathcal{F}_\beta^{X\times\mathbb{C}^2}(\epsilon_4, \epsilon_5, \dots) := \underbrace{\sum_{g\geq 0} \int_{[\overline{\mathcal{M}}(X\times\mathbb{C}^2, \beta)]_T^{\mathrm{vir}}} 1}_{}$$

$\downarrow \epsilon_4 = -\epsilon_5 = \mathrm{i} g_s$

[Mumford '83]

TS = GW

$$F_\beta^X(g_s)$$

$$\in \mathrm{H}^T_{2(2-2g)}(\mathrm{pt})^{\mathrm{loc}}$$

$$\cong \mathbb{Q}[\epsilon_4, \epsilon_5, \dots]^{\mathrm{loc}}_{2g-2}$$

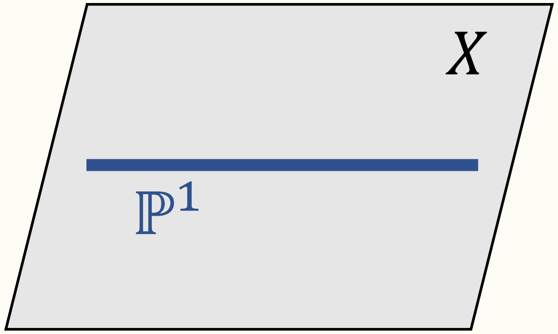
Example 1: Resolved Conifold

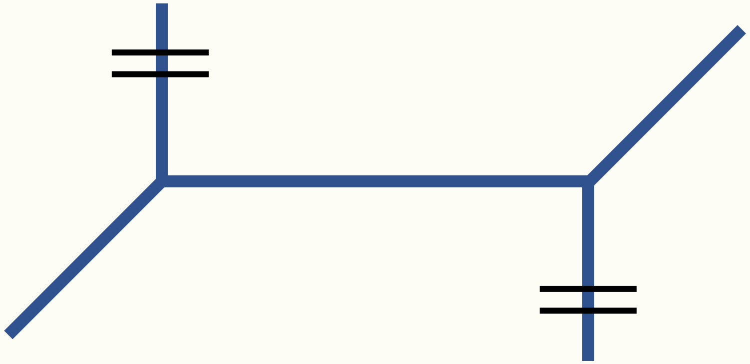
$$X = \mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

$$\mathcal{F}^{X\times\mathbb{C}^2}_{d[\mathbb{P}^1]}(\epsilon_4, \epsilon_5)$$

$$\overset{*}{=} \frac{1}{d \cdot 2\sinh\frac{d\epsilon_4}{2} \cdot 2\sinh\frac{d\epsilon_5}{2}}$$ [Brini-S]

$$= \text{Refined Topological Vertex}$$ [Iqbal-Kozcaz-Vafa]





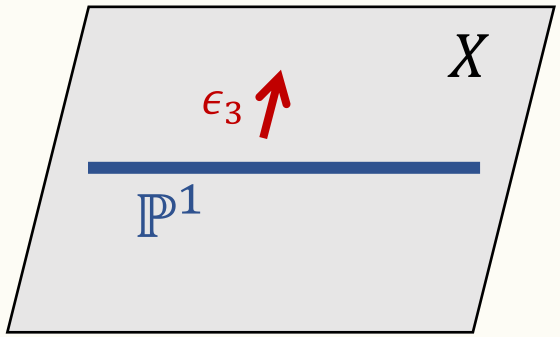
* proven in several limits and tested for $g \leq 5$ and $d \leq 3$ on a computer

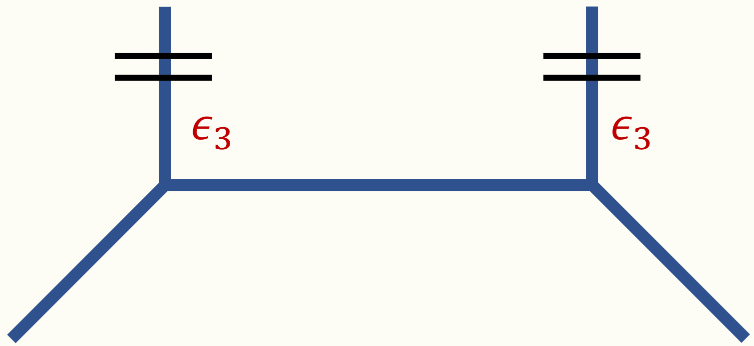
Example 2: Resolution of A_1 singularity

$$X = \mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-2) \times \mathbb{C}^{\epsilon_3}$$

$$\mathcal{F}_{d[\mathbb{P}^1]}^{X\times\mathbb{C}^2}(\epsilon_3,\epsilon_4,\epsilon_5) \overset{*}{=} \frac{-2\sinh\frac{d(\epsilon_3+\epsilon_4+\epsilon_5)}{2}}{d\cdot 2\sinh\frac{d\epsilon_3}{2}\cdot 2\sinh\frac{d\epsilon_4}{2}\cdot 2\sinh\frac{d\epsilon_5}{2}}$$

$$\xrightarrow[\epsilon_3\to\infty]{} -\frac{e^{\frac{d(\epsilon_4+\epsilon_5)}{2}}}{d\cdot 2\sinh\frac{d\epsilon_4}{2}\cdot 2\sinh\frac{d\epsilon_5}{2}} = \text{Refined Topological Vertex}$$





* proven in several limits and tested for $g \leq 5$ and $d \leq 3$ on a computer

Conjecture: For toric CY3 the refined topological vertex free energy is recovered in the the limit

$$\underbrace{\epsilon_1, \epsilon_2, \epsilon_3}_{X} \to \pm\infty \quad \text{and} \quad \underbrace{\epsilon_4 + \epsilon_5}_{\mathbb{C}^2} \text{ fixed}$$

- Experimentally no $\epsilon_1, \epsilon_2, \epsilon_3$-dependence for resolved conifold, local del Pezzo,…
- choice of limit $\leftrightarrow$ preferred direction
- analogous statement known in DT theory
  [Nekrasov-Okounkov, Arbesfeld]
- limit requires analytic lift!

M2 brane index of $X \times \mathbb{C}^2 \circlearrowleft T$

$\downarrow$

Refined TS = Equiv. GW

$\mathcal{F}_\beta^{X\times\mathbb{C}^2}(\epsilon_4, \epsilon_5, \dots)$

$\big\downarrow {\scriptstyle \epsilon_4 = -\epsilon_5 = i g_s}$

TS = GW

$F_\beta^X(g_s)$

Conjecture (weak): $\mathcal{F}_d^Z$ lifts to a rational function in $e^{\epsilon_i/2}$ for all T-actions fixing the holomorphic five-form of *any CY5* Z.

Conjecture (strong): This lift admits a modular interpretation as the M2-brane index of $Z \circlearrowleft T$.

- M2-brane moduli space only known in special cases [Nekrasov-Okounkov]
- implies refined GW/DT correspondence, …
- ∃ modification for constant map contributions

Example 2: Resolution of A_1 singularity

$$Z = \mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\epsilon_3} \oplus \mathcal{O}_{\mathbb{P}^1}^{\epsilon_4} \oplus \mathcal{O}_{\mathbb{P}^1}^{\epsilon_5}$$

$$\mathcal{F}^{Z}_{[\mathbb{P}^1]}(\epsilon_3, \epsilon_4, \epsilon_5) \stackrel{*}{=} \mathrm{ch}_T \frac{\overbrace{(q_3 q_4 q_5)^{-1/2}}^{\mathrm{H}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))} - (q_3 q_4 q_5)^{1/2}}{\prod_{i=3}^{5} \Big(\underbrace{q_i^{1/2}}_{\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\epsilon_i})} - q_i^{-1/2} \Big)}$$

$$= \mathrm{ch}_T\, \hat{\mathrm{a}}\big(\mathrm{H}^0(\mathbb{P}^1, N_{\mathbb{P}^1} Z) - \mathrm{H}^1(\mathbb{P}^1, N_{\mathbb{P}^1} Z) \big)$$

$\mathrm{ch}_T\, q_i = \mathrm{e}^{\epsilon_i}$

$$\hat{\mathrm{a}}\Big(\sum_i x_i - \sum_j y_j \Big) := \frac{\prod_j \big(y_j^{1/2} - y_j^{-1/2} \big)}{\prod_i \big(x_i^{1/2} - x_i^{-1/2} \big)}$$

Theorem: For any local curve

$$Z = \mathrm{Tot}\,\mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_5 \quad \longrightarrow \quad C$$

we have

$$\mathcal{F}^{Z}_{[C]} = \hat{\mathrm{a}}\big(\mathrm{H}^0(C, N_C Z) - \mathrm{H}^1(C, N_C Z)\big)$$

iff the formula holds in Example 1 & 2. (Tested for $g \leq 6$)

Corollary: Strong conjecture holds for local curves and

$$\mathrm{M2}_{[C]}(Z)^{T} = \mathrm{pt} = \{C \hookrightarrow Z\}$$

Theorem: Weak conjecture holds for local curves in *any degree* if $\epsilon_i + \epsilon_j = 0$ for $i \neq j \in \{2,3,4,5\}$.

M2-brane index on CY5 $Z \curvearrowleft T$

$$\hat{a}\big(\mathrm{M2}_\beta(Z)\big)$$

$$\Big\downarrow \ \mathrm{ch}_T$$

Refined TS = Equivariant GW on Z

$$\mathcal{F}^Z_\beta(\epsilon_4, \epsilon_5, \ldots)$$

$$Z = X \times \mathbb{C}^2 \ \Big\downarrow \ \epsilon_4 = -\epsilon_5 = \mathrm{i} g_s$$

TS = GW on X

$$F^X_\beta(g_s)$$