Topological correlators of massive $\mathcal{N} = 2$ SQCD and 5d $\mathcal{N} = 1$ YM on S^1

Jan Manschot

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Coláiste na Tríonóide, Baile Átha Cliath
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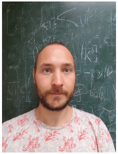
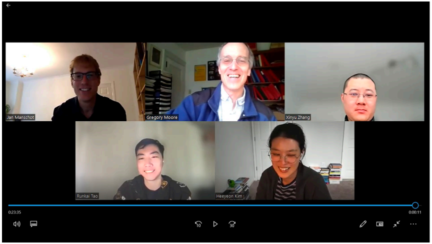
This talk is mostly based on



"Topological Twists of Massive SQCD" arXiv:2206.08943 + 2312.11616 with Johannes Aspman and Elias Furrer,



and "Path Integral Derivations of K-Theoretic Donaldson Invariants" to appear with Heeyeon Kim, Greg Moore, Runkai Tao, and Xinyu Zhang.

Correlation functions

Correlation functions are at heart of quantum field theory:

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle = \int [\mathcal{D}\mathcal{X}]\,\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\,e^{-\mathcal{S}(\mathcal{X})}$$

Large effort to include all perturbative and non-perturbative effects, and to increase n.

Motivation to study theories where such effects can be included.

Topologically twisted Yang-Mills theories

Path integrals and correlation functions can be evaluated exactly for topologically twisted $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Yang-Mills theories in many cases. These observables feature many crucial non-perturbative phenomena in Yang-Mills theory.

Such results provide at the same time deep connections to the geometry of four-manifolds and instanton moduli spaces, as well as to analytic number theory.

The correlation functions involve topological invariants of four-manifolds, such as Donaldson-Witten invariants, Seiberg-Witten invariants, Vafa-Witten invariants, *Segre numbers* and *K-theoretic Donaldson invariants*.

On a compact four-manifold X, the path integral is a functional integral over all fields of the topologically twisted theory. For manifolds with $b_2^+(X) = 1$, there is a contribution from the Coulomb branch of the theory.

$\Rightarrow$ analysis of Coulomb branches and effective couplings

Pure $\mathcal{N} = 2$, $SU(2)$ Yang-Mills

The gauge group is spontaneously broken by a vev of the vector multiplet scalar ϕ,

$$\phi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$$

with (classical) gauge invariant order parameter

$$u = \left\langle \mathrm{Tr}\phi^2 \right\rangle = 2a^2$$

The perturbative part of the effective coupling reads

$$\tau = \frac{\theta}{4\pi} + \frac{4\pi i}{g^2} \sim \frac{4i}{\pi} \log(a/\Lambda) + \dots$$

Pure $\mathcal{N} = 2$, $SU(2)$ Yang-Mills

SW curve provides full solution:

$$y^2 = x^3 - u\,x^2 + \frac{1}{4}\Lambda_0^4 x, \qquad a = \int_A \lambda_{SW}$$

Seiberg, Witten (1994)

$\Rightarrow$ Expresses u in terms of Jacobi theta series:

$$\begin{aligned}\frac{u(\tau)}{\Lambda_0^2} &= -\frac{1}{2}\frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{\vartheta_2(\tau)^2\vartheta_3(\tau)^2} \\ &= -\frac{1}{8}(q^{-1/4} + 20q^{1/4} - 62q^{3/4} + 216q^{5/4} + \mathcal{O}(q^{7/4})),\end{aligned}$$

with $q = e^{2\pi i\tau}$ and ϑ_j Jacobi theta series.

Matone (1996), Nahm (1996),...

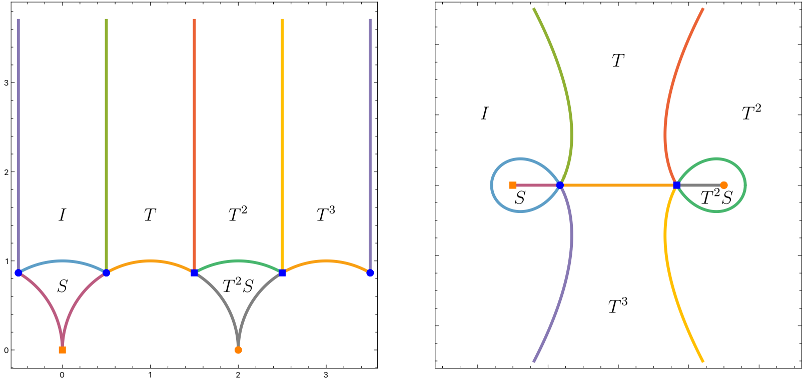
Theory has two strong coupling singularities for $u = \pm\Lambda_0^2$, where a monopole and dyon become massless.

Monodromies around the singularities generate the group $\Gamma^0(4)$, which leave u invariant

$$u\left(\frac{a\tau + b}{c\tau + d}\right) = u(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(4)$$

$$\Gamma^0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : b = 0 \mod 4 \right\}$$

Fundamental domain for pure $\mathcal{N} = 2$



Left: fundamental domain for $\mathbb{H}/\Gamma^0(4)$
Right: u-plane, and its partitioning given by the images of $\mathcal{F}_I$ in $\mathbb{H}/\Gamma^0(4)$

Aspman, Furrer, JM (2021)

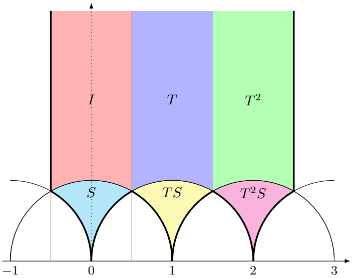
Include one hypermultiplet in fundamental representation of $SU(2)$

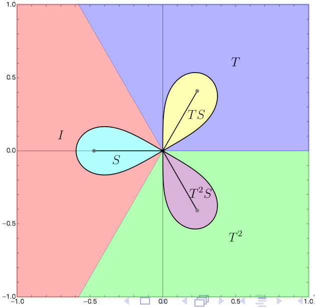
Global symmetry group: $(Spin(4) \times SU(2)_R \times U(1)^{(f)})/\mathbb{Z}_2$

Effective coupling: The perturbative contributions read

$$\tau \sim \frac{4i}{\pi}\log(a/\Lambda) - \frac{i}{2\pi}\log((a+m)/\Lambda) - \frac{i}{2\pi}\log((a-m)/\Lambda) + \dots$$

Massless theory has three strong coupling singularities.





The theory with $N_f = 1$: u

u for this theory reads:

$$\frac{u(\tau)}{\Lambda_1^2} = -\frac{3}{2^{\frac{7}{3}}}\frac{\sqrt{E_4(\tau)}}{\sqrt[3]{E_4(\tau)^{\frac{3}{2}} - E_6(\tau)}}$$
$$= -\frac{1}{16}(q^{-1/3} + 104q^{2/3} - 7396q^{5/3} + \mathcal{O}(q^{8/3}))$$

Nahm (1996) $\Rightarrow$ thus u is expressed in terms of square roots of modular forms, which are in general *not* modular forms.
One can verify that it is left invariant under the "path" transformations.

Sextic polynomial for u

To determine u, we bring the SW curve to Weierstrass form

$$y^2 = 4x^3 - g_2\,x - g_3$$

This gives rise to the polynomial,

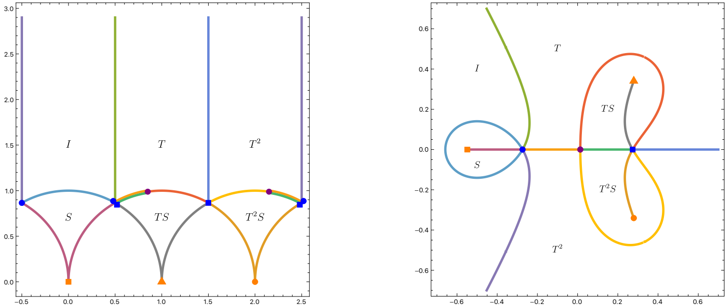
$$\begin{aligned} P_{N_f}(X) &= \left(g_2(X,\boldsymbol{m},\Lambda)^3 - 27g_3(X,\boldsymbol{m},\Lambda)^2\right)j - 12^3 g_2(X,\boldsymbol{m},\Lambda)^3 \\ &= a_6\,X^6 + a_5\,X^5 + \ldots + a_1\,X + a_0, \end{aligned}$$

where the coefficients $a_i = a_i(\boldsymbol{m},\Lambda,j)$ are polynomial functions of $\boldsymbol{m}$, Λ, and the j function, $a_i(\boldsymbol{m},\Lambda,j) \in \mathbb{C}[\boldsymbol{m},\Lambda,j]$.

The polynomials can thus be viewed as polynomials over the field $\mathbb{C}[\boldsymbol{m},\Lambda,j]$. In general no explicit solution X available. X can be considered as a single-valued function on a 6-fold branched cover of $\mathbb{H}/SL_2(\mathbb{Z})$. The branch points can occur if two or more (j-dependent) roots of P_{N_f} coincide for a specific value of j_{bp} of j.

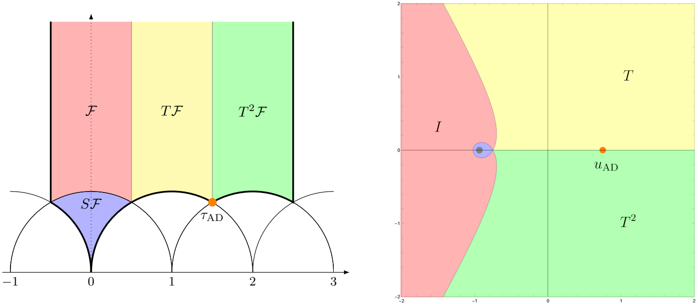
$N_f = 1$ continued: Mass $m > 0$

Increasing the mass $m > 0$, the branch points move to the interior such that we get branch cuts in the fundamental domain:



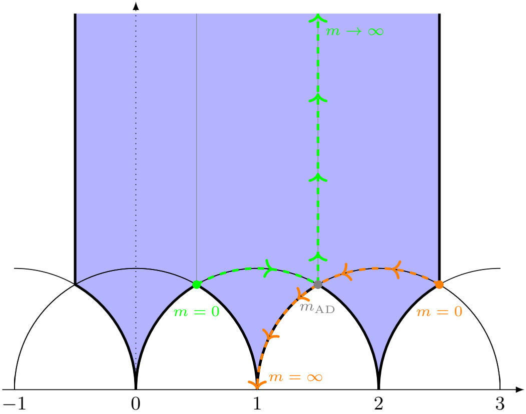
Argyres-Douglas mass m_{AD}

For the Argyres-Douglas mass $m_{\text{AD}} = \frac{3}{4}\Lambda_1$, two mutually non-local singularities collide in the u-plane, and the theory becomes superconformal. Within $\mathcal{F}_1$, the two branch points collide for this mass at $e^{\pi i/3} + 1$, and annihilate each other. The two strong coupling regions are disconnected from the other four. The resulting domain is a domain for $\Gamma^0(3)$.



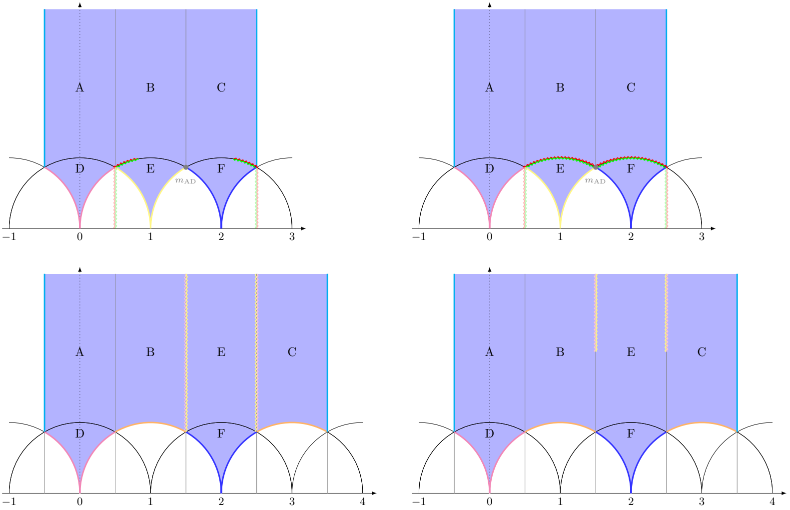
Mass $m > m_{AD}$

Beyond m_{AD}, the branch points return. They follow the following path:



Decoupling of hypermultiplet: cutting and gluing

In this way, we can see that the hypermultiplet smoothly decouples, returning to the one for $N_f = 0$



5d theory on $\mathbb{R}^4 \times S^1$

Another important theory is the 5-dimensional $\mathcal{N} = 1$ gauge theory theory compactified on a circle of radius R. The theory in 4d includes a full KK tower of states.

Work in progress together with Kim, Moore, Tao and Zhang

Bosonic fields: gauge field A_m, $m = 0, \ldots, 4$, real scalar σ

Global symmetries: $(Spin(4) \times SU(2)_R)/\mathbb{Z}_2 \times U(1)^{(I)}$

The current for the $U(1)^{(I)}$ symmetry is $j = *\frac{1}{8\pi^2}\mathrm{Tr}F \wedge F$ and the charged particles are instanton particles.

Seiberg (1996); Morrison, Seiberg, Intrilligator (1996),. . .

1-form center symmetry: shift of A_m by a $\mathbb{Z}_N$-valued flat connection for $SU(N)$

5d action

SUSY action:

$$S_{YM} = \frac{1}{g_{5d}^2} \int dx^5 \, \mathrm{Tr}\left[\frac{1}{2} F_{mn} F^{mn} + (D_m \sigma)^2 + \ldots\right]$$

Chern-Simons term:

$$S_{\mathrm{CS}} = -\frac{i\kappa}{24\pi^2} \int_{X \times S^1} \mathrm{Tr}[A \wedge F \wedge F] + \ldots$$

Tachikawa (2004)

Weakly gauge $U(1)^{(I)}$ by introducing "frozen" vector multiplet (A_m^I, σ^I). Mixed CS-term

$$S_{\text{mixed CS}} = \int_{X \times S^1} iF^I \wedge \mathrm{Tr}[AdA + \frac{2}{3} A^3] - \frac{1}{8\pi^2} \sigma^I \mathrm{Tr}[F \wedge *F] + \ldots$$

Baulieu, Losev, Nekrasov (1997)

Gives rise to standard kinetic terms in 4d with $\sigma^I = -\frac{8\pi^2}{g_{5d}^2}$

We include a background flux $\boldsymbol{n} = [F^I/2\pi] \in H^2(X, \mathbb{Z})$ for the topological global $U(1)^{(I)}$ symmetry.

4d KK theory on $\mathbb{R}^4 \times S^1$

Parameters: 4d scale Λ, and S^1 radius R

Dimensionless parameter: $\mathcal{R} = \Lambda R = e^{-8\pi^2 R/g_{5d}^2 + i\theta}$ with θ the holonomy of A^I

Electric BPS particles for gauge group $SU(2)$:

- W-bosons: $Z_a = 2a = \frac{2}{R}\int_{S^1}(\sigma + iA_5)dx^5$
- instanton particle: $Z_I = \frac{1}{R}\log(\mathcal{R}^4)$
- unit momentum around S^1: $Z_K = \frac{2\pi i}{R}$

Prepotential

$$\mathcal{F}(a, R, \Lambda) = \frac{2}{R^2}\left(\mathrm{Li}_3(e^{-Ra} - \zeta(3))\right) + a^2\left(\log(\mathcal{R}) - \frac{\pi i}{2}\right) + O(\mathcal{R})$$

Order parameter

The Coulomb branch order parameter is the vev of the Wilson line operator

$$U = \left\langle \mathrm{Tr}\,\mathrm{P}\exp\Big(\int_{S_1}(\sigma + iA_5)dx_5\Big)\right\rangle = e^{Ra} + e^{-Ra} + O(\mathcal{R})$$

Action of 1-form symmetry T:

$$T : \begin{cases} a \mapsto a + \frac{\pi i}{R} \\ U \mapsto -U \end{cases}$$

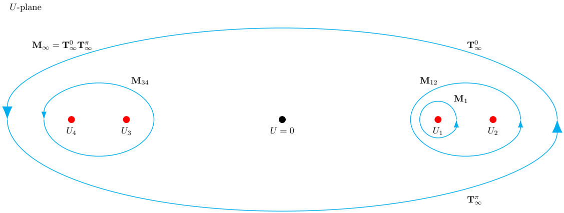
5d $SU(2)$ theory on $\mathbb{R}^4 \times S^1$

SW curve for $\kappa = 0$:

$$\Sigma: \quad Y^2 = P(X)^2 - 4X^2\mathcal{R}^4, \qquad P(X) = X^2 + UX + 1,$$

Nekrasov (1996); Ganor, Morrison, Seiberg (1996); Göttsche, Yoshioka, Nakajima (2006),. . .

Four singularities: $U = \pm 2(\mathcal{R}^2 \pm 1)$



Using the theory of elliptic curves, one can demonstrate

$$U^2 = -8\mathcal{R}^2\mathrm{u} + 4\mathcal{R}^4 + 4,$$

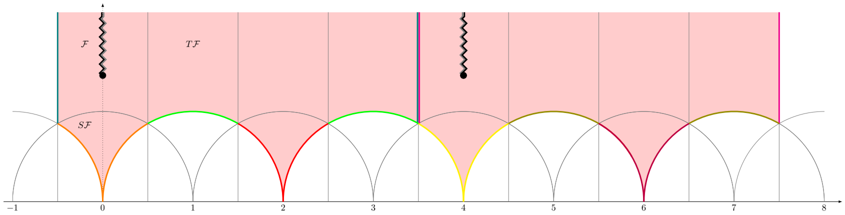
with

$$\mathrm{u}(\tau) = \frac{\vartheta_2(\tau)^4 + \vartheta_3(\tau)^4}{2\vartheta_2(\tau)^2\,\vartheta_3(\tau)^2}, \qquad \mathcal{R} = R\Lambda$$

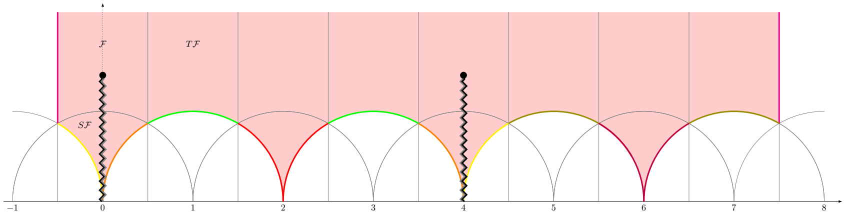
with τ the complex structure of Σ

5d $SU(2)$ theory on $\mathbb{R}^4 \times S^1$

$U(\tau)$ is a function on the double cover of the pure $SU(2)$ domain ($\mathbb{H}/\Gamma^0(4)$). It includes a branch point and cuts:



We can rearrange the domain to avoid cuts at infinity.



In the limit $\mathcal{R} \to 1$ the branch points disappear and the U is a modular form for (a congruence of) $\Gamma^0(8)$.

See also Closset, Magureanu (2021)

$H^2(X, \mathbb{Z})$ together with the intersection form

$$B(\boldsymbol{k}_1, \boldsymbol{k}_2) = \int_X \boldsymbol{k}_1 \wedge \boldsymbol{k}_2, \qquad \boldsymbol{k}_{1,2} \in H^2(X, \mathbb{Z})$$

gives rise to an integral, uni-modular lattice L (the image of $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{R})$)

The lattice has signature (b_2^+, b_2^-). For non-vanishing correlators, $b_2^+ + b_1$ =odd. We restrict to $b_1 = 0$.

For $b_2^+ = 1$, let J be the normalized generator of the unique self-dual direction in $H^2(X, \mathbb{R})$. It provides the projection of $\boldsymbol{k} \in L$ to $(L \otimes \mathbb{R})^+$,

$$\boldsymbol{k}_+ = B(\boldsymbol{k}, J)\, J$$

Topological twisting

Assume X is spin, such that the chiral $SU(2)$ spin bundles are well-defined.

Donaldson-Witten twist: Replace $SU(2)_+$ representation by that of the diagonally embedded subgroup in $SU(2)_+ \times SU(2)_R$
$\Rightarrow \phi$ and A_μ remain a vector and scalar, but hypermultiplet scalars $(Q, \tilde{Q}^\dagger)$ become space-time spinors $(M_{\dot\alpha}, \bar{M}_{\dot\alpha})$

Topological twisting

Spinors $M_{\dot\alpha}$ are problematic for the generalization to non-spin X. We cure this by coupling the hypermultiplet to the Spin^c line bundle $\mathcal{L}$, such that

$$W^+ = S^+ \otimes \mathcal{L}^{1/2}$$

is a well-defined Spin^c bundle

$$c_1(\mathcal{L}) = \bar{w}_2(X) + \bar{w}_2(E) + 2L$$

Aspman, Furrer, JM (2022)

For $\mathfrak{s}$ canonically determined by an ACS

$$W^+ \simeq \Lambda^0 \oplus \Lambda^{0,2}, \qquad W^- \simeq \Lambda^{0,1}$$

For Spin^c structures for fundamental matter: Hyun, Park, Park (1995), Labastida, Marino (1997)

For Spin^c structures for adjoint matter: JM, Moore (2021)

More generally Moore, Saxena, Singh (2024)

Relation to Donaldson invariants

Examples of observables in the $\mathcal{Q}$-cohomology of DW-theory are:

- Point observable:

$$\mathcal{O}^{(0)}(p) = u(p),$$

- Surface observable:

$$\mathcal{O}^{(2)}(\boldsymbol{x}) = I_-(\boldsymbol{x}) = \frac{1}{4\pi^2}\int_{\boldsymbol{x}} \mathrm{Tr}\Big[\tfrac{1}{8}\psi \wedge \psi - \tfrac{1}{\sqrt{2}}\phi F\Big], \quad \boldsymbol{x} \in H_2(M,\mathbb{Z})$$

These observables correspond to differential forms on the moduli space, a 4-form ω_u and a 2-form $\omega_I(\boldsymbol{x})$.

Provide physical understanding for the Donaldson invariants of a compact smooth four-manifold M:

$$\Big\langle e^{\mathcal{O}^{(0)}(p)+\mathcal{O}^{(2)}(\boldsymbol{x})}\Big\rangle = \sum_{\ell,n\geq 0} D_{\ell,n}\, p^\ell \boldsymbol{x}^n,$$

where the $D_{\ell,n}$ are Donaldson invariants,

$$D_{\ell,n} p^\ell \boldsymbol{x}^n = \int_{\mathcal{M}_k} \omega_u(p)^\ell \wedge \omega_I(\boldsymbol{x})^n$$

Evaluation of correlation functions

- For compact four-manifolds, the path integral includes integral over u:

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_{u-\text{plane}} + \langle \mathcal{O} \rangle_{\text{SW}}$$

  where $\langle \mathcal{O} \rangle_{\text{SW}}$ has δ-function support on the cusps $u = \pm 1$

- $\langle \mathcal{O} \rangle_{u-\text{plane}} =: \Phi^J_{\boldsymbol{\mu}}[\mathcal{O}]$ is non-vanishing only for $b_2^+ \leq 1$. Such four-manifolds provide a testing ground for the analysis of Coulomb branches.
- For $b_2^+ = 1$, the path integral reduces to an integral over zero modes A_μ, $\phi_0 = a$, η_0, ψ_0, χ_0.

Witten (1995); Moore, Witten (1997); Losev, Nekrasov, Shatashvili (1997)

Partition function

Assume X has in addition $b_1 = 0 \Rightarrow$ no ψ zero modes

Partition function for theory on X:

$$\Phi^J_{\boldsymbol{\mu}} = \sum_{fluxes} \int da\, d\bar{a}\, d\eta_0\, d\chi_0\; A(u)^{\chi(X)} B(u)^{\sigma(X)}\, e^{-\int_X \mathcal{L}_0}$$
$$= \int_{\mathcal{F}_T} d\tau \wedge d\bar{\tau}\, \nu(\tau)\, \Psi^J_{\boldsymbol{\mu}}(\tau,\bar{\tau})$$

with $(q = e^{2\pi i \tau})$

- $\nu(\tau) = \frac{da}{d\tau} A(u)^{\chi(M)} B(u)^{\sigma(M)} = q^{-\frac{3}{8}} + \dots$
- Sum over fluxes:

$$\Psi^J_{\boldsymbol{\mu}}(\tau,\bar{\tau}) = \frac{1}{\sqrt{y}} \sum_{\boldsymbol{k}\in L+\boldsymbol{\mu}} B(\boldsymbol{k},J)\, q^{-\boldsymbol{k}_-^2/2}\, \bar{q}^{\boldsymbol{k}_+^2/2}$$

where $L = H^2(X,\mathbb{Z})$/torsion with bilinear form $B(\cdot,\cdot)$, J is the period point of M, $\boldsymbol{\mu} \in L/2$ is the 't Hooft flux

Efficient evaluation using mock modular forms for all X with $b_2^+ = 1$

Korpas, JM (2017); Korpas, JM, Moore, Nidaiev (2019); JM, Moore (2021)

Construction of a suitable anti-derivative:

$$\frac{\partial \widehat{F}(\tau,\bar{\tau})}{\partial \tau} = \Psi^J_{\boldsymbol{\mu}}(\tau,\bar{\tau}),$$

Then

$$\Phi^J_{\boldsymbol{\mu}}[\mathcal{O}] = \left[\mathcal{O}\,\nu(\tau)\,F(\tau)\right]_{q^0} + \text{contributions from other cusps},$$

with $F(\tau) = \sum_n c(n)\,q^n$ the holomorphic part of $\widehat{F}(\tau,\bar{\tau})$

SW contributions

General form of partition function:

$$Z^J_{\boldsymbol{\mu}} = \Phi^J_{\boldsymbol{\mu}} + \sum_{j=1}^{2+N_f} Z^J_{SW,j,\boldsymbol{\mu}}$$

with

$$Z^J_{SW,j,\boldsymbol{\mu}} = \mathcal{A}_j^\chi \mathcal{B}_j^\sigma \sum_c SW(c;J)\, \mathcal{E}_j^{c^2}\, \mathcal{F}_{\boldsymbol{\mu}}(c)$$

The terms on the rhs undergo wall-crossing upon varying J. Wall-crossing from the singularity u_j of $\Phi^J_{\boldsymbol{\mu}}$ is absorbed by the wall-crossing of $Z^J_{SW,j,\boldsymbol{\mu}}$:

$$\left[\Phi^{J^+}_{\boldsymbol{\mu}} - \Phi^{J^-}_{\boldsymbol{\mu}}\right]_j = Z^{J^-}_{SW,j,\boldsymbol{\mu}} - Z^{J^+}_{SW,j,\boldsymbol{\mu}}$$

This makes it possible to derive $Z^J_{SW,j,\boldsymbol{\mu}}$ in terms SW$(c;J)$ with c the IR Spin^c structure. Moreover, it is possible to extend the results to manifolds with $b_2^+ > 1$.

UV SQCD on X

The Q-fixed equations are the non-Abelian Seiberg-Witten equations:

$$\left(F^{a}_{\dot\alpha\dot\beta}\right) + \frac{i}{2}\sum_{j=1}^{N_f} \bar{M}^{j}_{\dot\alpha} T^{a} M^{j}_{\dot\beta} = 0$$

$$\not{D} M^{j} = 0$$

Witten (1994); Labastida, Marino (1995); Labastida, Lozano (1998),...

Equations are invariant under $U(1)^{(f)}$ symmetry: $M_{\dot\alpha} \to e^{i\varphi} M_{\dot\alpha}$

$M_{\dot\alpha}$ is a spinor $\Rightarrow$ X is spin, or coupling to a Spin^c structure $\mathfrak{s}$ required.

Fixed point locus

Moduli space of solutions $\mathcal{M}_k^Q$

The $U(1)^{(f)}$ fixed point locus consists of two components:

- Instanton component $\mathcal{M}_k^{\mathrm{i}}$: $M_{\dot\alpha} = 0$ and $F^+ = 0$
- Abelian component $\mathcal{M}_k^{\mathrm{a}}$: F diagonal, and $M_{\dot\alpha}$ strictly upper or lower triangular

Then a correlator $\langle \mathcal{O} \rangle$ reduces to an integral of differential forms $\mathcal{M}^Q$:

$$\langle \mathcal{O} \rangle = \sum_k \Lambda^{\mathrm{vdim}(\mathcal{M}_k^Q)} \int_{\mathcal{M}_k^Q} \mathrm{Eul}(Cok(\not{D}))\, \omega_{\mathcal{O}}$$

with $\mathrm{Eul}(Cok(\not{D}))$ the equivariant Euler class of the cokernel bundle. This can be expanded in terms of Chern classes c_ℓ of the index bundle W_k

$$\langle \mathcal{O} \rangle = \sum_k \Lambda^{\mathrm{vdim}(\mathcal{M}_k^Q)} m^{-\mathrm{rk}(W_k)} \times \left[\int_{\mathcal{M}_k^{\mathrm{i}}} \sum_\ell \frac{c_\ell}{m^\ell}\, \omega_{\mathcal{O}} + \int_{\mathcal{M}_k^{\mathrm{a}}} \sum_\ell \frac{c_\ell}{m^\ell}\, \omega_{\mathcal{O}} \right],$$

Losev, Nekrasov, Shatashvili (1998),... In the $m \to \infty$ limit, only $\ell = 0$ contributes, while in the limit $m \to 0$ only c_{top} contributes

Mathematicians (Göttsche, Kool,...) refer to these intersection numbers as "Segre numbers".

Explicit results for $\mathbb{P}^2$

Four-manifold $\mathbb{P}^2$:

- $b_2 = b_2^+ = 1$
- non-spin
- SW-invariants vanish $\Rightarrow$ u-plane gives full result

Explicit results for $\mathbb{P}^2$

Aspman, Furrer, JM analyzed topological correlators of these theories with generic masses.

ℓ	$\Phi_{1/2}[u^\ell]$ for $N_f = 1$
0	$\frac{m}{\Lambda_1}$
1	$-\frac{7}{2^6}\frac{\Lambda_1^2 m^2}{m^2}$
2	$\frac{19}{2^6}\frac{\Lambda_1^2 m^2}{m^0}$
3	$-\frac{21}{2^8}\frac{\Lambda_1^5 m^3}{m^2}$
4	$\frac{85}{2^9}\frac{\Lambda_1^5 m^3}{m^0} + \frac{1093}{2^{18}}\frac{\Lambda_1^8 m^4}{m^4}$

ℓ	$\Phi_{1/2}[u^\ell]$ for $N_f = 2$
0	$\frac{m^2}{\Lambda_2^2} + \frac{3}{64}\frac{m^4}{m^4}$
1	$-\frac{7}{2^5}\frac{m^4}{m^2}$
2	$\frac{19}{2^6}m^4 + \frac{23}{2^7}\frac{\Lambda_2^2 m^6}{m^4} + \frac{53}{2^{18}}\frac{\Lambda_2^4 m^8}{m^8}$
3	$-\frac{21}{2^7}\Lambda_2^2\frac{m^6}{m^2} - \frac{421}{2^{16}}\frac{\Lambda_2^4 m^8}{m^6}$
4	$\frac{85}{2^8}\Lambda_2^2 m^6 + \frac{7263}{2^{17}}\frac{\Lambda_2^4 m^8}{m^4} + \frac{2161}{2^{21}}\frac{\Lambda_2^6 m^{10}}{m^8} + \frac{1811}{2^{30}}\frac{\Lambda_2^8 m^{12}}{m^{12}}$

- Vanishing background fluxes for flavor group
- Same result for large and small mass.

Explicit results for $\mathbb{P}^2$

ℓ	$k_1 = 1/2$	$k_1 = 3/2$	$k_1 = 5/2$
0	$-\frac{3}{4\sqrt{2}}\frac{m^1}{m^1}$	$-\frac{9}{4\sqrt{2}}\frac{\Lambda_1^2}{m^2}$	$-\frac{15}{4\sqrt{2}}\frac{\Lambda_1^6}{m^6}$
1	0	$-\frac{31}{64\sqrt{2}}\frac{\Lambda_1^5}{m^3}$	$-\frac{155}{64\sqrt{2}}\frac{\Lambda_1^9}{m^5}$
2	$-\frac{13}{64\sqrt{2}}\Lambda_1^3 m$	$-\frac{39}{64\sqrt{2}}\frac{\Lambda_1^5}{m} - \frac{567}{2^{12}\sqrt{2}}\frac{\Lambda_1^8}{m^4}$	$-\frac{65}{64\sqrt{2}}\frac{\Lambda_1^9}{m^5}$
3	$\frac{113}{2^{13}\sqrt{2}}\Lambda_1^6 + \frac{50175}{2^{23}\sqrt{2}}\frac{\Lambda_1^9}{m^3}$	$-\frac{867}{2^{12}\sqrt{2}}\frac{\Lambda_1^8}{m^2}$	$-\frac{1225}{2^{10}\sqrt{2}}\frac{\Lambda_1^{12}}{m^6}$
4	$-\frac{879}{2^{13}\sqrt{2}}\Lambda_1^6 m^2$	$-\frac{2637}{2^{13}\sqrt{2}}\Lambda_1^8 - \frac{7305}{2^{17}\sqrt{2}}\frac{\Lambda_1^{11}}{m^3}$	$-\frac{4395}{2^{13}\sqrt{2}}\frac{\Lambda_1^{12}}{m^4}$

ℓ	$k_1 = 1$	$k_1 = 2$	$k_1 = 3$
0	1	$\frac{\Lambda_1^3}{m^3} + \frac{15}{64}\frac{\Lambda_1^6}{m^6}$	$\frac{\Lambda_1^8}{m^8} + \frac{45}{32}\frac{\Lambda_1^{11}}{m^{11}}$
1	0	$\frac{21}{64}\frac{\Lambda_1^6}{m^4}$	$\frac{7}{8}\frac{\Lambda_1^{11}}{m^9}$
2	$\frac{19}{64}\frac{\Lambda_1^3 m}{m^0}$	$\frac{19}{64}\frac{\Lambda_1^6}{m^2}$	$\frac{19}{64}\frac{\Lambda_1^{11}}{m^7} + \frac{201}{256}\frac{\Lambda_1^{14}}{m^{10}}$
3	$-\frac{11}{2^9}\Lambda_1^6$	0	$\frac{237}{2^9}\frac{\Lambda_1^{14}}{m^8}$
4	$\frac{85}{2^9}\Lambda_1^6 m^2$	$\frac{85}{2^9}\frac{\Lambda_1^9}{m}$	$\frac{85}{512}\frac{\Lambda_1^{14}}{m^6} + \frac{64775}{2^{17}}\frac{\Lambda_1^9}{m}$

Large mass calculation of $\Phi_{1/2}$ with background fluxes k_1 for $N_f = 1$

Explicit results for $K3$

Four-manifold $K3$:

- $b_2 = 22,\ b_2^+ = 3 \Rightarrow$ u-plane does *not* contribute
- spin
- non-vanishing SW-invariant $SW(c) = 1$ for $c = 0$

Explicit results for $K3$

ℓ	$Z_0[u^\ell]$ for $N_f = 1$
0	$64\frac{m^2}{\Lambda_1^2}$
1	$64\frac{m^4}{\Lambda_1^2} + 8\Lambda_1 m$
2	$64\frac{m^6}{\Lambda_1^2} + 16\Lambda_1 m^3 + \frac{9}{4}\Lambda_1^4$
3	$64\frac{m^8}{\Lambda_1^2} + 24\Lambda_1 m^5 + \frac{9}{2}\Lambda_1^4 m^2$
4	$64\frac{m^{10}}{\Lambda_1^2} + 32\Lambda_1 m^7 + \frac{31}{4}\Lambda_1^4 m^4 + \frac{27}{16}\Lambda_1^7 m$
5	$64\frac{m^{12}}{\Lambda_1^2} + 40\Lambda_1 m^9 + 12\Lambda_1^4 m^6 + \frac{45}{16}\Lambda_1^7 m^3 - \frac{243}{1024}\Lambda_1^{10}$
6	$64\frac{m^{14}}{\Lambda_1^2} + 48\Lambda_1 m^{11} + \frac{69}{4}\Lambda_1^4 m^8 + \frac{9}{2}\Lambda_1^7 m^5 + \frac{1215}{1024}\Lambda_1^{10} m^2$

- vanishing background fluxes
- sum of 3 strong coupling singularities, u_1^*, u_2^* and u_3^*
- polynomials in m

Explicit results for $K3$

The singularities u_1^* and u_2^* are associated to the instanton component and u_3^* to the abelian component. If we consider $Z_{0,1}[u^\ell] + Z_{0,2}[u^\ell]$,

$$\begin{aligned}
\ell = 0: \quad & -\frac{3}{4}\frac{\Lambda_1^4}{m^4} - \frac{5}{16}\frac{\Lambda_1^7}{m^7} - \frac{63}{512}\frac{\Lambda_1^{10}}{m^{10}} - \frac{99}{2048}\frac{\Lambda_1^{13}}{m^{13}} + \dots, \\
\ell = 1: \quad & -\frac{\Lambda_1^4}{m^2} - \frac{7}{16}\frac{\Lambda_1^7}{m^5} - \frac{175}{1024}\frac{\Lambda_1^{10}}{m^8} - \frac{273}{4096}\frac{\Lambda_1^{13}}{m^{11}} + \dots, \\
\ell = 2: \quad & -\frac{5}{8}\frac{\Lambda_1^7}{m^3} - \frac{245}{1024}\frac{\Lambda_1^{10}}{m^6} - \frac{189}{2048}\frac{\Lambda_1^{13}}{m^9} - \frac{4719}{131072}\frac{\Lambda_1^{16}}{m^{12}} + \dots,
\end{aligned}$$

This reproduces "Segre numbers" for $K3$, matches with "universal functions"

Göttsche, Kool (2020), Göttsche (2021), Oberdieck (2022), ...

Analysis for 5d theory

In collaboration with Kim, Moore, Tao and Zhang.

Two approaches:

1. reduction to S^1: supersymmetric sigma model with target space the moduli space of instantons
2. reduction to X

Flux $\boldsymbol{n}_I$ for global $U(1)$ flavor symmetry $\Rightarrow$ induces a bundle $\mathcal{L}_{\boldsymbol{n}_I} \to \mathcal{M}_k$

The partition function becomes a generating function of indices of the Dirac operator coupled to $\mathcal{L}_{\boldsymbol{n}_I}$ over $\mathcal{M}_k$

$$\begin{aligned} Z_{\boldsymbol{\mu}}(\mathcal{R}, \boldsymbol{n}) &= \sum_{k \geq 0} \mathrm{Ind}(\not{D}_{A^I})\, \mathcal{R}^{4k} \\ &= \sum_{k \geq 0} \int_{\mathcal{M}_k} \hat{A}(T\mathcal{M}_k)\, e^{\mu_D(n_I)}\, \mathcal{R}^{4k} \end{aligned}$$

Nekrasov (1997)

If X is complex, this can be related to the holomorphic Euler characteristic $\chi_h(X, \mathcal{L}'_{\boldsymbol{n}_I})$.

Localization in supersymmetric quantum mechanics

If X is a toric four-manifold, we can localize with respect to the $\mathbb{C}^*$ action, and include equivariant weights ϵ_1 and ϵ_2.

The partition function is of the form

$$Z_{\boldsymbol{\mu}}^J(\mathcal{R}, \boldsymbol{n}) = \sum_k \int \frac{dh}{h} \int da \wedge d\bar{a}\, \partial_{\bar{a}} g_{k,\boldsymbol{\mu}}^J(a, \bar{a}, h)$$

At the BPS locus, $h = 0$, $g_{k,\boldsymbol{\mu}}^J(a, \bar{a}, 0)$ can be expressed in terms of Nekrasov's partition function on $\mathbb{R}^4$.

Nekrasov (2006), Gottsche, Nakajima, Yoshioka (2006), Hosseini et al, Crichigno et al, Bonelli et al (2015 & 2020)

$\Rightarrow$ Explicit equivariant wall-crossing formula and equivariant partition functions

Alternatively, we can carry out the the U-plane integral for this KK theory on X coupled to $\boldsymbol{n}$.

$$\Phi_{\boldsymbol{\mu},\boldsymbol{n}}(\mathcal{R}) = K_U \int_{\mathcal{F}_{\mathcal{R}}} d\tau \wedge d\bar{\tau}\, \nu_{\mathcal{R}}(\tau)\, C^{\boldsymbol{n}^2}\, \Psi^J_{\boldsymbol{\mu}}(\tau,\bar{\tau}, v\boldsymbol{n}, \bar{v}\boldsymbol{n})$$

where

$$\nu_{\mathcal{R}} = \frac{\vartheta_4(\tau)^{13-b_2}}{\eta(\tau)^9}\frac{1}{U}$$

$$v = -\frac{1}{2\pi i}\partial_a \partial_{m_l}\mathcal{F}, \qquad C = \frac{\vartheta_4(\tau, v)}{\vartheta_4(\tau)}$$

The integrand can be shown to be invariant under monodromies. Leads to the same wall-crossing formula.

Center symmetry anomaly

Path integral changes sign under the 1-form symmetry T:

$$T: \qquad \Phi_{\boldsymbol{\mu},\boldsymbol{n}} \mapsto (-1)^{B(2\boldsymbol{\mu},K_X-\boldsymbol{n})}\Phi_{\boldsymbol{\mu},\boldsymbol{n}}$$

$\Rightarrow$ Topological terms of the mixed Chern-Simons action $S_{\text{mixed}\,CS}$ is only well-defined for $(-1)^{B(2\boldsymbol{\mu},K_X-\boldsymbol{n})}=1$. In fact, the path integral vanishes otherwise, since the contributions from the T-images are identical up to the sign.

This anomaly has a natural counterpart for the SQM. For $\boldsymbol{n}=0$, this theory is anomalous if $\mathcal{M}_k$ is not a spin manifold since the fermion determinant is then not globally well-defined. We claim that for $(-1)^{B(2\boldsymbol{\mu},K_X)}=-1$, $\mathcal{M}_k$ is not a spin manifold, and that for $(-1)^{B(2\boldsymbol{\mu},K_X-\boldsymbol{n})}=1$, the fermions are coupled to a suitable Spin^c structure such that the anomaly is absent.

Indeed, $(-1)^{B(2\boldsymbol{\mu},K_X)}=\pm 1$ determines whether $\mathcal{M}_k$ is spin or not.

Hopkins, Freed, Moore (to appear)

Explicit results for $\mathbb{P}^2$

For the evaluation, we first expand in $\mathcal{R}$ and then determine the q^0 term. For example for $X = \mathbb{P}^2$, we obtain

$$\Phi_{1/2,n}(\mathcal{R}) = \begin{cases} 1 + O(\mathcal{R}^{13}), & n = \pm 1, \\ 1 + \mathcal{R}^4 + \mathcal{R}^8 + \mathcal{R}^{12} + \dots, & n = \pm 3, \\ 1 + 6\mathcal{R}^4 + 21\mathcal{R}^8 + 56\mathcal{R}^{12} + \dots, & n = \pm 5, \\ 1 + 21\mathcal{R}^4 + 210\mathcal{R}^8 + 1401\mathcal{R}^{12} + \dots, & n = \pm 7, \\ 1 + 55\mathcal{R}^4 + 1310\mathcal{R}^8 + 19432\mathcal{R}^{12} \dots, & n = \pm 9. \end{cases}$$

and 0 for even n. In agreement with Göttsche, Nakajima, Yoshioka (2006).

Explicit results for $\mathbb{P}^2$

$$\Phi_{0,n}(\mathcal{R}) = \begin{cases} \frac{15}{2}\mathcal{R} - 21\mathcal{R}^5 - 56\mathcal{R}^9 - 126\mathcal{R}^{13} \ldots, & n = -5, \\ 6\mathcal{R} - 6\mathcal{R}^5 - 10\mathcal{R}^9 - 15\mathcal{R}^{13} + \ldots, & n = -4, \\ \frac{9}{2}\mathcal{R} - \mathcal{R}^5 - \mathcal{R}^9 - \mathcal{R}^{13} + \ldots, & n = -3, \\ 3\mathcal{R} + O(\mathcal{R}^{17}), & n = -2, \\ \frac{3}{2}\mathcal{R} + O(\mathcal{R}^{17}), & n = -1, \\ O(\mathcal{R}^{17}), & n = 0, \\ -\frac{3}{2}\mathcal{R} + O(\mathcal{R}^{17}), & n = 1, \\ -3\mathcal{R} + O(\mathcal{R}^{17}), & n = 2, \\ -\frac{9}{2}\mathcal{R} + \mathcal{R}^5 + \mathcal{R}^9 + \mathcal{R}^{13} + \ldots, & n = 3, \\ -6\mathcal{R} + 6\mathcal{R}^5 + 10\mathcal{R}^9 + 15\mathcal{R}^{13} + \ldots, & n = 4, \\ -\frac{15}{2}\mathcal{R} + 21\mathcal{R}^5 + 56\mathcal{R}^9 + 126\mathcal{R}^{13} + \ldots, & n = 5, \\ -9\mathcal{R} + 56\mathcal{R}^5 + 230\mathcal{R}^9 + 770\mathcal{R}^{13} + \ldots, & n = 6, \end{cases}$$

In agreement with GNY, except for $O(\mathcal{R})$ coefficient. We attribute this to reducible connections or strictly semi-stable bundles.

Explicit results for X with $b_2^+(X) \geq 3$

Similarly to before, also SW contributions can be determined using wall-crossing. In this way we give a physical derivation of results by Göttsche, Kool, Williams (2019) for the K-theoretic Donaldson invariants of X

$$\frac{2^{2-\chi_{\mathrm{h}}(X)+K_X^2}}{(1-\mathcal{R}^2)^{(\boldsymbol{n}-K_X)^2/2+\chi_{\mathrm{h}}}} \sum_c \mathrm{SW}(c)\,(-1)^{\mu\,(c+K_X)} \left(\frac{1+\mathcal{R}}{1-\mathcal{R}}\right)^{c(K_X-\boldsymbol{n})/2}$$

with K_X the canonical class of X, and $\mathrm{SW}(c)$ the Seiberg-Witten invariant for the basic class c.

Conclusion

- We have explicitly evaluated and analyzed the partition function & correlators of
  1. the $\mathcal{N} = 2$ $SU(2)$ theory with fundamental matter.
  2. 5d $\mathcal{N} = 1$ theory on $X \times S^1$, which gives rise to K-theoretic Donaldson invariants
- New results on Coulomb branch geometries, gauge theoretic moduli spaces and their invariants
- Analysis motivates the study of more general theories

Thank you!