Universal Virasoro Constraints for Linear Categories

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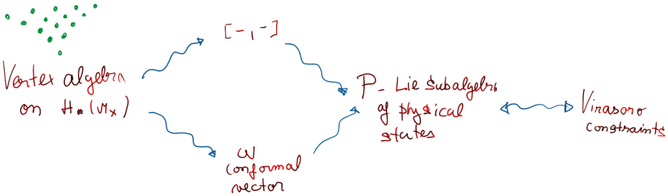

Structure of the talk.

1. History and background
2. Geometric formulation of Virasoro constraints and the main claim
3. Reformulation in terms of vertex algebras
    3.1 Joyce's construction of VA's
    3.2 The conformal element
    3.3 Virasoro constraints make virtual fundamental classes into physical states
4. Main results for quivers and varieties

Gromov–Witten side

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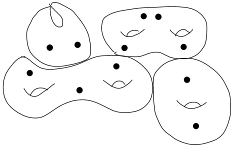

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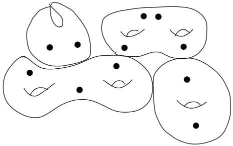
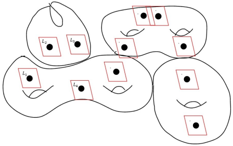


   with $f : C \to X$ such that $f_*([C]) = \beta \in H_2(X)$.
3. Line bundles $L_i \to \overline{\mathcal{M}}_{g,n}$ given by $T^*_C|_{x_i}$ at each C:



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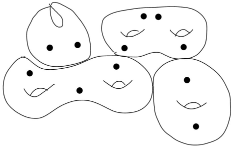

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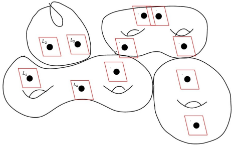

4. Denote the powers of the first Chern classes by $\tau_d := \psi_i^d := c_1(L_i)^d$

Gromov–Witten potential

[1] Here ev_j is the evaluation map for the j'th marked point.

Gromov–Witten potential

1. Fix a basis $B = \{v\} \subset H^*(X)$ with $1 \in B$ for the generator of $H^0(X)$ and define the classes [1]

$$\tau_k(v) = (\psi_j)^k \mathrm{ev}_j^*(v)\,, \quad \text{for all} \quad k \geq 0, v \in B\,.$$

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$$F^X(\vec{t}) = \sum_{\substack{g \geq 0 \\ \beta \in H_2(X)}} \left\langle \exp\left[\sum_{\substack{k \geq 0 \\ v \in B}} \tau_k(v) t_{k,v}\right]\right\rangle_{\beta,g}^X q^\beta \lambda^{2g-2}$$

where

$$\left\langle - \right\rangle_{g,\beta}^X = \sum_{n \geq 0} \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} (-)\,.$$

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where
$$\left\langle - \right\rangle_{g,\beta}^X = \sum_{n \geq 0} \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} (-)\,.$$
It collects the invariants
$$\langle \tau_{k_1}(v_1)^{a_1} \tau_{k_2}(v_2)^{a_2} \cdots \tau_{k_l}(v_l)^{a_l} \rangle_{\beta,g}^X = \int_{\left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}}} \tau_{k_1}(v_1)^{a_1} \tau_{k_2}(v_2)^{a_2} \cdots \tau_{k_l}(v_l)^{a_l}$$
under the condition that $\sum_i a_i = n$.

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Virasoro constraints

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$$T_k = \frac{\lambda^2}{2} \sum_{m=1}^{k} \frac{\partial^2}{\partial t_{m-1} \partial t_{k-m}} \,, \qquad R_k = \sum_{m=0}^{\infty} (m + \frac{1}{2}) t_m \frac{\partial}{\partial t_{m+k}} \,,$$

and $T_0 = \frac{1}{16}$, $T_{-1} = \frac{1}{8}\lambda^{-2} t_0^2$.

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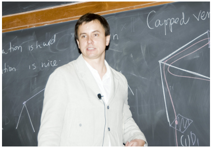

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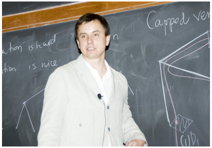
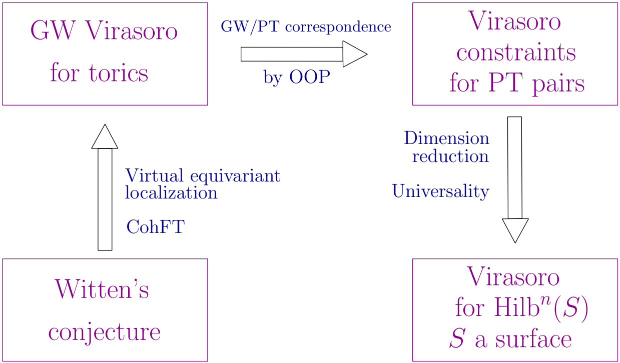


3. Virasoro constraints have been proved by Okounkov–Pandharipande (03') for curves $X = C$ and by Givental (01') and Teleman (12') for toric X (more generally X with semisimple quantum cohomology). Givental's formalism uses Kontsevich's result for $\overline{\mathcal{M}}_{g,n}$, equivariant localization, and cohomological field theories.

Transposing to the sheaf side

1. A rather long roundabout proof:



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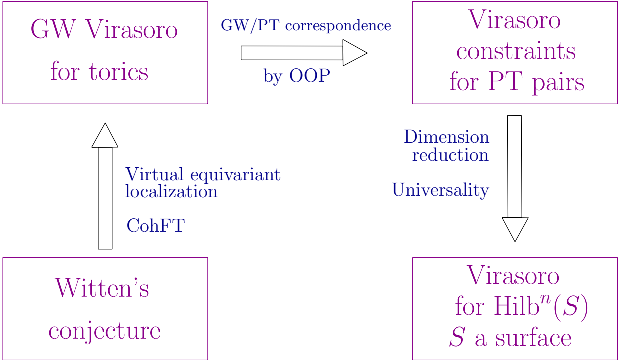

2. Virasoro constraints for toric 3-folds X were transported to stationary descendents of PT stable pairs using the GW–PT correspondence by Moreira–Oblomkov–Okounkov–Pandharipande (20'). Dimensional reduction was used to prove these constraints for $\mathrm{Hilb}^n(S)$.

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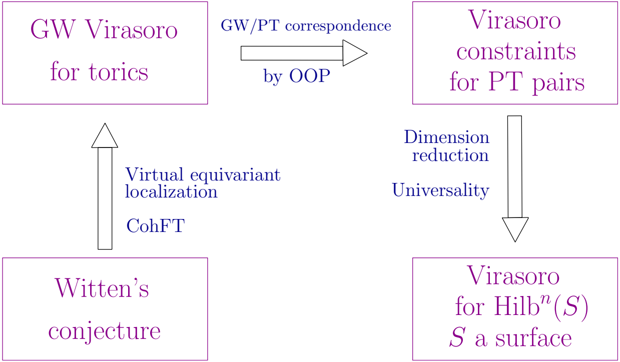

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3. Using quivers, I will give a more direct proof.

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3. Fixing a stability condition σ and a class α, e.g. a dimension vector or a Chern character, Virasoro constraints are stated for each virtual fundamental class $[M_\alpha^\sigma]^{\mathrm{vir}}$ separately (M_α^σ parametrizes σ-stable objects in class α)

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4. I will just write M when σ, α are not important. Assume that M is fine for now.

Quivers and sheaves

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2. Some examples to see how much variety the theory offers:

	Without framing	With framing
Sheaves	Gieseker stable torsion-free sheaves on curves or surfaces, dimension 1 sheaves on surfaces, Fano 3-folds, CY fourfolds[2]	Bradlow pairs on curves or surfaces, DT/PT pairs on $\leq$ 4-folds, Quot schemes
Quivers with relations (quasi-smooth, CY4)	Bridgeland stable quiver representations	Framed quiver representations: e.g., Grassmanians and Flag varieties

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Quivers with relations and frozen vertices

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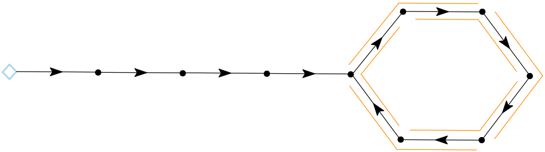

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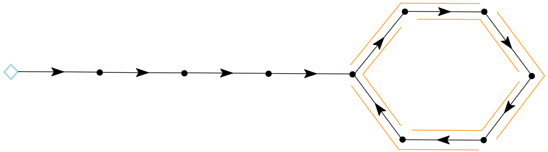

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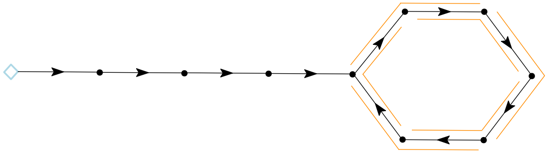

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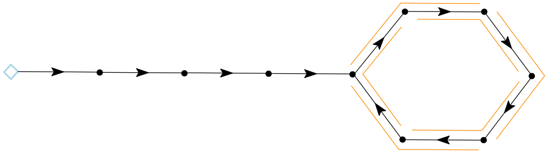

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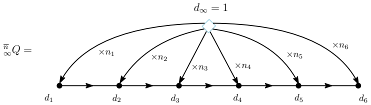
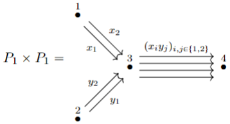


Flag

$$(x_i y_k) \circ x_j = (x_j y_k) \circ x_i \,, \qquad (x_k y_i) \circ y_j = (x_k y_j) \circ y_i$$



$D^b(\mathbb{P}^1 \times \mathbb{P}^1)$

Defining Virasoro constraints for a category $\mathcal{A}$

Quivers | Sheaves

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depends on a choice of a universal object.

Quivers:

$\tau_i(v) \mapsto \mathrm{ch}_i(\mathbb{U}_v)$
$\mathbb{U}_v$ is a universal vector space at v

Sheaves:

$\tau_i^H(v) = \pi_{2,*}\Big(\pi_1^*(\bar{v})\mathrm{ch}_{i+p}(\mathbb{G})\Big)$
$\mathbb{G}$ universal sheaf on $X \times M$
$v \in H^{p,q}(X)$, $\overline{v}$ its Poincaré dual

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4. Euler pairing $\chi : \Lambda(\mathcal{A}) \times \Lambda(\mathcal{A}) \to \mathbb{C}$

$\chi(v, w) = \delta_{v,w} - A_{v,w} + S_{v,w}$	$\chi(v, w) = \int_X v^\vee \cdot w \cdot \mathrm{td}(X)$

Virasoro operators

5. Quadratic terms from the diagonal pushforward:
$\tau_i\tau_j\big(\Delta_*\mathrm{td}(\mathcal{A})\big) := \sum_{v,w\in B} \chi(v,w)\tau_i(w)\tau_j(v)\,.$

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6. Virasoro operators $\mathrm{L}_k = \mathrm{T}_k + \mathrm{R}_k$: R_k are always the same
$$\mathrm{R}_k\big(\tau_i^{(H)}(v)\big) = i^{(k+1)}\tau_{i+k}^{(H)}(v)$$
where $i^{(k+1)} = i(i+1)\cdots(i+k)$ is the rising factorial.

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$$\mathrm{ch}\big(T^{\mathrm{vir}}\big) = -\sum_{i,j}(-1)^i\tau_i\tau_j\big(\Delta_*\mathrm{td}(\mathcal{A})\big)(+1)$$

$$T_k = \sum_{i+j=k} i!j!\tau_i\tau_j\big(\Delta_*\mathrm{td}(\mathcal{A})\big)(+1)$$

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When $K^1_{\mathrm{top}}(X) \neq 0$, set $\chi^H(v,w) = (-1)^p \int_X v \cdot w \cdot \mathrm{td}(X)$ and
$$\tau_i^H \tau_j^H \big(\Delta_* \mathrm{td}(X)\big) = \sum_{v,w \in B} \chi^H(v,w) \tau_i^H(w) \tau_j^H(v)\,.$$

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1. The operators L_k satisfy the dual Virasoro commutation relations

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3. Instead, we need to make up for the non-uniqueness of the choice of a universal object. Use another operator S^k compatible with the universal object. It can also absorb fixing determinants of sheaves.

Claim (B.–Lim–Moreira(22'), B.(23'))

Let M be a fine moduli space of stable objects with a virtual fundamental class, then it often satisfies Virasoro constraints

$$\int_{[M]^{\mathrm{vir}}} (\mathsf{L}_k + \mathsf{S}_k)(D) = 0 \qquad \text{for} \quad k \geq 0, D \in \mathbb{D}^{\mathcal{A}}\,.$$

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1. To avoid talking about S_k, we introduced the weight-zero operator

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2. This formulation a) is independent of the choice of the universal object when it exists, b) can be defined formally without the universal object.

Flag varieties

1. The partial flag variety $\mathrm{Flag}(d_1, d_2, \ldots, d_l)$ for $d_1 > d_2 \cdots d_{l-1} > d_l$ parametrizes sequences of quotients

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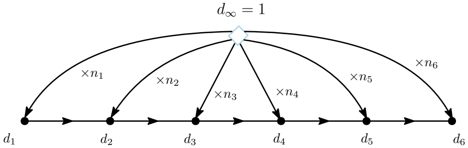

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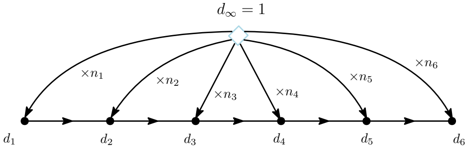

and choosing the right stability condition,obtain the flag variety. Also identify $\mathcal{Q}_v = \mathbb{U}_v$.
3. The only non-constant descendents are $\tau_i(v) = \mathrm{ch}_i(\mathcal{Q}_v)$ for $v \neq \infty, 1$. Then
$$\mathsf{T}_k = \sum_{\substack{i+j=k \\ v \geq 2}} \tau_i(v)\big(\tau_j(v) - \tau_j(v+1)\big) - d_1 \tau_k(2)\,,$$
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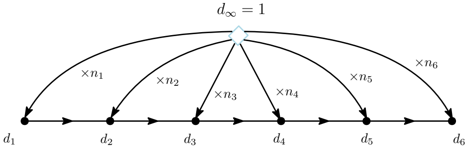

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4. Choosing a polynomial D in $\mathrm{ch}_i(\mathcal{Q}_v)$ for $v \geq 2$, Virasoro constraints tell us that

$$\int_{\mathrm{Flag}} (\mathsf{T}_k + \mathsf{R}_k)(D) = 0 \qquad \text{for} \quad k \geq 0\,.$$

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5. Define homology classes $t_{k,v}$ as duals of $\tau_k(v)$:

$$\tau_k(v) \cap (-) = \frac{\partial}{\partial t_{k,v}} .$$

6. The homology of $\mathcal{M}_{\overline{d}}$ is the polynomial algebra

$$H_*(\mathcal{M}_{\overline{d}}) = e^{\overline{d}} \otimes \mathrm{Sym}[t_{i,v}, i > 0, v \in V] .$$

Recovering a GW-like formulation of Virasoro constraints

1. Let us introduce the notation $\langle \tau_{k_1}(v_1) \cdots \tau_{k_n}(v_n) \rangle_M = \int_{[M]^{\text{vir}}} \tau_{k_1}(v_1) \cdots \tau_{k_n}(v_n)$

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$$\begin{aligned}\iota_*[M]^{\mathrm{vir}} &= \left\langle \exp\left[\sum_{k,v}\tau_k(v)t_{k,v}\right]\right\rangle_M \\ &= \int_{[M]^{\mathrm{vir}}} \exp\left[\sum_{k,v}\tau_k(v)t_{k,v}\right],\end{aligned}$$

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4. As before T_k is a second order differential operator in t's and R_k is a degree changing operator

$$T_k = \sum_{i+j=k} i!j!\tau_i\tau_j\big(\Delta_*\mathrm{td}(Q)\big)\cap\,,$$

$$R_k = \sum_{\substack{j\geq 1\\ v\in V}} j_{(k+1)}\, t_{j-k,v}\frac{\partial}{\partial t_{j,v}}\,.$$

   Here $a_{(b)} = a(a-1)\cdots(a-b+1)$ is the falling factorial.

Virasoro constraints for sheaves: the operators

1. The homology version of Virasoro constraints:

$$(L_k + S_k)\, \iota_*[M]^{\text{vir}} = 0 \qquad \text{for} \quad k \geq 0$$

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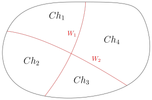
2. The operator on the second line is defined by

$$L_{\text{wt}=0} = \sum_{n \geq -1} \frac{(-1)^n}{(n+1)!}\, T^{n+1} \circ L_n\,.$$

Wall-crossing and vertex algebras

1. Use Joyce's geometric vertex algebras and wall-crossing to prove Virasoro constraints



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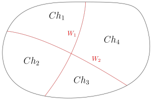

2. This wall-crossing allows us to compare virtual fundamental classes $[M^{\sigma}_{\underline{d}}]^{\mathrm{vir}}, [M^{\sigma'}_{\underline{d}}]^{\mathrm{vir}}$ of σ and σ'-stable representations in terms of some Lie bracket $[-,-]$ on the quotient $K_* = H_*(\mathcal{M}_Q)/T$.

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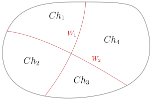

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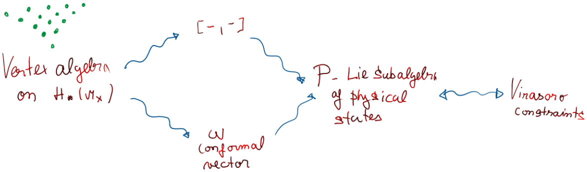

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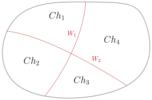

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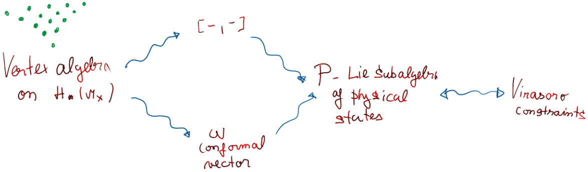

4. VFC's satisfying Virasoro constraints are physical states, and this property is preserved under changing stability conditions.

Example of a wall-crossing formula

1. We never discussed what happens when we have strictly semistables. In this case, there is a geometric way of using the framed quiver to define classes $[M^{\sigma}_{d}]^{\mathrm{in}} \in K_*$ counting semistables.

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4. This means that all the information about $[M^{\sigma}_{\overline{d}}]^{\text{in}}$ is already contained in the Lie algebra structure.

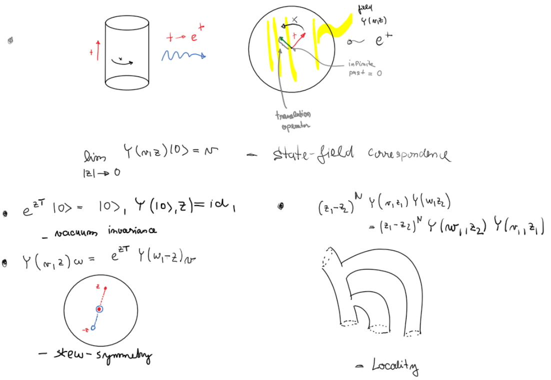
Vertex algebras

1. A vertex algebra is the data of a $\mathbb{Z}$-graded vector space V_* over $\mathbb{C}$ together with
   1.1 a vacuum vector $|0\rangle \in V_0$,
   1.2 a linear operator $T\colon V_* \to V_{*+2}$ called the translation operator,
   1.3 and a state-field correspondence which is a degree 0 linear map

$$Y\colon V_* \longrightarrow \mathrm{End}(V_*)[\![z, z^{-1}]\!]\,,$$

   denoted by $Y(a,z) := \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$, where $\deg(z) = -2$.

2. These need to satisfy some axioms that have enlightening interpretations in terms of CFTs



Sketch of the geometric construction of vertex algebras

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2. The vertex algebra V_* was shown to be a lattice vertex algebra by Joyce (17'), which is the most natural vertex algebra with the underlying graded vector space

$$V_* = \mathbb{Q}[\mathbb{Z}^V] \otimes \mathrm{Sym}[\![t_{i,v}, i > 0, v \in V]\!] .$$

The geometric construction of a Lie algebra

1. The stacky quotient of $\mathcal{M}_Q$ by $B\mathbb{G}_m$ is denoted by $\mathcal{M}_Q^{\text{rig}}$. As one would expect, there is roughly the correspondence

$$K_* = V_{*+2}/TV_* = H_{*+\text{vdim}_{\mathbb{C}}}(\mathcal{M}_Q^{\text{rig}}).$$

The geometric construction of a Lie algebra

1. The stacky quotient of $\mathcal{M}_Q$ by $B\mathbb{G}_m$ is denoted by $\mathcal{M}_Q^{\text{rig}}$. As one would expect, there is roughly the correspondence

$$K_* = V_{*+2}/TV_* = H_{*+\text{vdim}_{\mathbb{C}}}(\mathcal{M}_Q^{\text{rig}})\,.$$

2. Now there are two roles that K_* plays:
   2.1 The classes $[M]^{\text{in}}$ live in K_* even if M is not fine.
   2.2 K_* has a Lie bracket:

$$[\overline{v}, \overline{w}] = \overline{v_0 w}\,, \quad \forall v, w \in V_*\,.$$

Conformal element

1. A conformal element $\omega \in V_4$ leads to a field $Y(\omega, z) = \sum_{k\in\mathbb{Z}} L_k z^{-k-2}$ with

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2. Assume that $\chi_{\mathrm{sym}}(v,w) = \chi(v,w) + \chi(w,v)$ is non-degenerate to get $\omega \in V_*$. One can always find a larger vertex algebra containing ω such that L_k's restrict to V_*.

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3. Take the dual basis $\hat{v} \subset \hat{B}$ to B. Then

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Remark

When working with X, need to include $H^{\mathrm{odd}}(X) \cong K^1(X,\mathbb{Q})$. One can still write $\omega = \frac{1}{2}\sum_{v\in B} t^H_{1,v} t^H_{1,\hat{v}}$ where ^{H} denotes some holomorphic grading shift leading to odd degrees. The conformal charge is given by $\chi(X)$.

Physical states and the main claim

1. Note that

$$\int_{[M_{\frac{\sigma}{d}}]^{\mathrm{in}}} L_{\mathrm{wt}=0}(D) = 0 \qquad \text{for} \quad D \in H^*(\mathcal{M}_Q)$$

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Theorem (BLM(22'), B. (23'))

The condition that $[M_\alpha^\sigma]^{\mathrm{in}}$ satisfies Virasoro constraints is equivalent to it being a physical state with respect to the ω given above. I.e. $[M_\alpha^\sigma]^{\mathrm{in}} \in \check{P}$.
In particular, wall-crossing, stated in terms of iterated Lie brackets, preserves Virasoro constraints from the RHS to the LHS of the formula.

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Remark

In the second point, I used derived equivalences of the surfaces to quivers. This is the first proof of Virasoro constraints for sheaves on surfaces independent of Witten's conjecture. Using a universality of Virasoro constraints for $\mathrm{Hilb}^n(S)$, gives an independent proof of Virasoro constraints for any surface S. In particular, this establishes them as an autonomous phenomenon.