Generalized Riemann-Hilbert correspondence and wall-crossing structures

Yan Soibelman

KANSAS STATE UNIVERSITY and IHES

January 21, 2025

Conventional RH-correspondence

Riemann-Hilbert correspondence is typically understood as a correspondence between differential equations and their solutions. For regular singular differential equations (e.g. $(z\frac{d}{dz} + A)f = 0, A \in Mat(n, \mathbb{C}))$ the only invariant of the locally constant sheaf of solutions is its monodromy. The question about reconstruction of the regular singular equation from its sheaf of solutions is the Hilbert's 21st problem. It was solved in the higher-dimensional case by Deligne (for bundles with flat connections) and by Kashiwara (for holonomic D-modules). In both cases the RH-correspondence is a statement about equivalence of some subcategory of the category of D-modules and some subcategory of the category of contsructible sheaves (derived categories in the work of Kashiwara). For irregular singular equations even in dimension one the monodromy is not the only invariant: one should take into account Stokes filtrations on fibers of the local system of solutions for each singular point (Deligne-Malgrange). In higher-dimensional case the latest advance for irregular D-modules is due to Agnolo-Kashiwara.

HFT and GRHC

There are other classes of equations e.g. difference, q-difference, or elliptic difference, where solutions do not form a constructible sheaf. It is an interesting problem to formulate a universal RH-correspondence which serves those classes as well. This is what we did with Maxim Kontsevich in 2015 as a part of our program "Holomorphic Floer Theory" (HFT). Our generalized RH-correspondence (GRHC) relates two different areas of mathematics: deformation quantization and Floer theory. From this perspective constructible sheaves provide a computational tool in the Floer theory of cotangent bundles, whose deformation quantization is known to be a theory of D-modules.
Another feature of our GRHC is that it makes the RH-correspondence similar to the Homological Mirror Symmetry. The "A-side" will be some Fukaya category, while the "B-side" will be the some deformation of the derived category of coherent sheaves in the "non-commutative direction".

Applications which will not be discussed today

Today I am going discuss geometry underlying the categories involved in the generalized RH-correspondence. Before doing that I would like to mention some applications. I don't have time to discuss them today, but I think it is worthwhile to mention them.

a) Extension of the non-abelian Hodge theory (NAHT) in dimension one beyond the case of bundles with connections on curves (where it is due to Simpson). In our generalized NAHT periodic monopoles in $\mathbb{R}^3$ play the role of harmonic bundles.
b) In the generalized NAHT one has analogs of several questions typically associated with Simpson's NAHT, e.g. the generalized $P = W$ conjecture.
c) Relation to the representation theory of quantized algebras (e.g. quantum tori, rational Cherednik algebras, Sklyanin algebras etc.).
d) Adding $\mathbb{C}^*$-actions to our story one can relate it to quantized Coulomb branches.

Wall-crossing structures in GRHC

In my talk the key role is played by wall-crossing structures (WCS), the notion we introduced with Maxim in arXiv:1303.3253. WCS allow us to connect the local RH-correspondence (the one associated with a given complex Lagrangian subvariety, see later) with the global one.

This local-to-global result is related to Vladimir Arnold's question about holomorphic analog of Morse theory. The answer to his question is Picard-Lefschetz theory. In C^∞ version of Morse theory local data at the critical locus of a Morse function $f : X \to \mathbb{R}$ are related to the global topology of X via Morse complex of f. In Picard-Lefschetz theory Morse complex is trivial for a generic holomorphic Morse function f, but it is "replaced" by Picard-Lefschetz wall-crossing formulas. In the GRHC we have more general wall-crossing formulas. From this perspective Arnold's question corresponds to the D-module $\mathcal{M} = \mathcal{D}_{\mathbb{C}_\hbar} \cdot e^{f/\hbar}$, where $\mathcal{D}_{\mathbb{C}_\hbar}$ is the sheaf of differential operators on the complex line endowed with the coordinate $\hbar$.

RH-correspondence with and without quantization parameter

There are two versions of the generalized RH-correspondence: the one depending on the *formal* parameter $\hbar$ and then one which does not depend on it. In fact it is better to say that in the latter case it depends on $\hbar \in \mathbb{C}^*$ *analytically*, and we give to $\hbar$ a particular value. This is why people do not see the parameter in $\hbar$ in the conventional RH-correspondence, which corresponds to the *global* RH-correspondence in our terminology. In the *local* RH-correspondence the dependence on $\hbar$ is always formal. I will explain later how WCS can help us to get rid of $\hbar$.

Category of DQ-modules

Given a complex symplectic manifold $(M, \omega^{2,0})$, $dim_{\mathbb{C}} M = 2n$ the deformation quantization (Kontsevich, Kashiwara, Schapira,...) gives rise to the following structures:
1) A sheaf of categories over $\mathbb{C}[[\hbar]]$ which modulo $\hbar$ is equivalent to the sheaf of categories of coherent $\mathcal{O}_M$-modules.
Assume: there is a $*$-product on $\mathcal{O}_M[[\hbar]] = \{\sum_{k \geq 0} \hbar^k f_k, f_k \in \mathcal{O}_M\}$. Then we have a sheaf of algebras $\mathcal{O}_{M,\hbar} = (\mathcal{O}_M[[\hbar]], *)$ which mod $\hbar$ is isomorphic to the sheaf of Poisson algebras $\mathcal{O}_M$.
A DQ-module is a finitely generated $\mathcal{O}_{M,\hbar}$-module. More pedantically, one should quotient this category by the subcategory of $\hbar$-torsion modules.

de Rham and Betti sides of the global RH-correspondence

2) (de Rham side of the RH-correspondence) After some additional choices (they include e.g. a partial Poisson log compactification $P_{log} \supset M$) one can define the category $Hol_{glob} := Hol_{glob}(M)$ of holonomic DQ-modules. Roughly, it is the category of global sections of the sheaf of categories of finite rank $\mathcal{O}_{M,\hbar}$-modules which modulo $\hbar$ have Lagrangian support, and the support "behaves nicely" near the normal crossing divisor $D_{log} = P_{log} - M$.
Assume: the category Hol_{glob} is defined in fact over the subring $\mathbb{C}\{\hbar\} \subset \mathbb{C}[[\hbar]]$ of analytic germs at $\hbar = 0$.

3) (Betti side of the RH-correspondence) One can also define a family of partially wrapped Fukaya categories $\mathcal{F}_{glob,\hbar} := \mathcal{F}(M, \frac{1}{\hbar}(Re(\omega^{2,0}) + i\, Im(\omega^{2,0})))$. It can be thought of as a single category $\mathcal{F}_{glob}$, linear over the Novikov ring.
Assume: it is defined over the subring $\mathbb{C}\{\hbar\}[\hbar^{-1}]$ of meromorphic germs.

Global GRHC

Roughly, our global Riemann-Hilbert correspondence (2015), still conjectural in general, says that:

Over the ring $\mathbb{C}\{\hbar\}[\hbar^{-1}]$ we have the (derived) equivalence $Hol_{glob} \simeq \mathcal{F}_{glob}$. The t-structure with the heart consisting of holonomic DQ-modules corresponds to the subcategory of $\mathcal{F}_{glob}$ in which objects are supported on complex Lagrangian analytic subsets.

Short formulation: de Rham and Betti sides of the RH-correspondence are equivalent.

In fact the equivalence should come from a faithful embedding of $\mathcal{F}_{glob}$ to the category of all DQ-modules ("Lagrangian A-branes form a subset of the set of coisotropic branes")

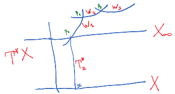
Intuition behind the partial compactification P_{log}: on the de Rham side the normal crossing divisor $D_{log} = P_{log} - M$ controls "singularities of holonomic modules", while on the Betti side it controls the "partial wrapping".

Next three slides: illustrations in $n = 1$ case of how the previously known versions of the RH-correspondence can be restated as GRHC.

Example:rational case (Deligne-Malgrange)

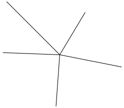
1) Meromorphic connections $(\mathcal{E}, \nabla)$ on a marked smooth complex curve $(X, x_1, ..., x_n)$ with prescribed singular behavior at each x_i. By Hukuhara-Levelt-Turrittin (HLT) theorem formally $(\mathcal{E}, \nabla) = \oplus_{i,\alpha} e^{c_\alpha^{(i)}} \otimes \nabla_\alpha^{(i),RS}$. Here finite collection of singular terms $c_\alpha^{(i)} = \sum_{\lambda \in \mathbb{Q}_{\leq 0}} c_{\alpha,\lambda}^{(i)} (x - x_i)^\lambda$ controls the irregular behavior at x_i, and $\nabla_\alpha^{(i),RS}$ are connections which are regular singular at x_i. Then P_{log} is obtained from the fiberwise compactification $\overline{T^*X}$ by a finite sequence of blow-ups and then by adding to T^*X those divisors $D_\alpha^{(i)} \subset D_{log}$ on which the symplectic form $\omega_{T^*X}^{2,0}$ has pole of order 1. Components $D_\alpha^{(i)}$ bijectively correspond to the singular terms $c_\alpha^{(i)}$. The RH-correspondence relates meromorphic connections with prescribed singular terms $(c_\alpha^{(i)})_{i,\alpha}$ with a certain subcategory of the Fukaya category of T^*X.

Figure: blow-ups



Example:trigonometric case (Ramis-Sauloy-Zhang)

2) q-difference equations $f(qx) = A(x)f(x)$. They are holonomic modules over the quantum torus $A_q(n)$, $0 < |q| < 1$, i.e. a $\mathbb{C}$-algebra with invertible generators $x_i, y_i, 1 \le i \le n$ and relations $x_i y_j = q^{\delta_{ij}} y_j x_i, 1 \le i, j \le n$. Here $q = e^{\hbar}$.
Then $M = (\mathbb{C}^*)^{2n}, \omega^{2,0} = \sum_{1 \le i \le n} \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i}$, and P_{log} is a toric variety corresponding to a "Lagrangian fan" in $\mathbb{R}^{2n}$. In other words analogs of singular terms are unions of rational Lagrangian cones in $\mathbb{R}^{2n}$ with vertices at the origin (hence the term "Lagrangian fan") .



For $n = 1$ the GRHC amounts to an equivalence of the category of coherent sheaves on $\mathbb{C}^*/q^{\mathbb{Z}}$ endowed with two anti-Harder-Narasimhan filtrations (slopes of consecutive semistable factors increase and equal to the slopes of the rays) and a certain Fukaya category associated with the corresponding toric surface.

Example: elliptic case (Rains,...)

Take $\tau = 2\pi i\hbar$, and E be the corresponding elliptic curve.
3a) (difference equations on elliptic curve) $f(x+u) = A(x)f(x)$,where $u \in E$ is fixed. Then $M = E \times \mathbb{C}^*_z, \omega^{2,0} = dx \wedge \frac{dz}{z}$, and $P_{log} = E \times \mathbb{P}^1$.
3b) (Sklyanin algebras). Here we consider holonomic modules over the elliptic algebra corresponding to the quantization of $M = \mathbb{P}^2 - E$ endowed with a simplectic form $\omega^{2,0}$ which has a pole of order 1 on the smooth cubic E. Then $P_{log} = \mathbb{P}^2$.

What means "local" RH-correspondence?

For any (possibly singular) complex Lagrangian subvariety $L \subset M$ we can define local versions of the de Rham and Betti sides, $Hol_{L,loc}$ and $\mathcal{F}_{L,loc}$. The category $\mathcal{F}_{L,loc}$ is defined over $\mathbb{Z}$ (no instanton corrections). There is a local version of the RH-correspondence which claims equivalence $Hol_{L,loc} \simeq \mathcal{F}_{L,loc}$ after extending scalars to $\mathbb{C}((\hbar))$. Informally, we can think about objects of these local categories as those "living in a small Stein neighborhood of L".

From local to global: rough idea

Notice that the quantization parameter $\hbar$ is a fixed number in all above one-dimensional examples, so both sides of the RH-correspondence depend analytically on $\hbar$. Then e.g. in the rational case we set $\hbar = 1$.

Analyticity in $\hbar$ of the global RH-correspondence is a strong assumption. It can be approached in the following way. First, we study a local version of the RH-correspondence over formal series in $\hbar$. The corresponding local Fukaya categories form a local system on $S^1_{\theta=Arg(\hbar)}$. There are also Stokes isomorphisms derived from Floer-theoretical considerations (i.e. they are defined in terms of instanton corrections). These data give rise to a WCS. If this WCS is analytic (the notion we introduced in 2020), then we can "correct" the local system of local Fukaya categories obtaining the category over meromorphic germs in $\hbar$. After that we can set $\hbar$ to be a fixed number (e.g. $\hbar = 1$ for D-modules).

From local to global on the Betti side: more details

Local Fukaya category gives rise to a *local system* of A_∞-categories over the circle of directions $\theta = Arg(\hbar)$ (or over $\mathbb{C}^*_\hbar$) via dilation $\omega^{2,0} \mapsto \omega^{2,0}/\hbar$.
The corresponding wall-crossing structure is determined by (in general countable) set of Stokes rays $\theta = Arg(\int_\gamma \omega^{2,0})$, where $\gamma \in H_2(M, L, \mathbb{Z})$, as well as by the Stokes isomorphisms for each Stokes ray. Stokes isomorphisms are equivalences of the fibers of $\mathcal{F}_{L,loc}$ slightly on the left and on the right of a Stokes ray. They are defined in terms of the virtual count of pseudo-holomorphic discs with boundary on L (instanton corrections). Generically the discs are absent, but they appear on Stokes rays. In order to make all that precise we combine some ideas from deformation theory of A_∞-categories with those from Symplectic Field Theory of Eliashberg-Givental-Hoffer. I don't have time to explain the details here.

Sometimes (e.g. if one has exponential bounds on the number of discs) the corresponding wall-crossing structure belongs to a smaller class of analytic wall-crossing structures (see our arXiv:2005.10651).
If this is the case, one can start with the local system $\mathcal{F}_{L,loc}$ then use Stokes isomorphisms and glue a new meromorphic family of categories, i.e. a new category $\mathcal{F}^{mer}_{L,loc}$ over $\mathbb{C}\{\hbar\}[\hbar^{-1}]$. It is no longer a local system. There is a faithful embedding $\mathcal{F}^{mer}_{L,loc} \hookrightarrow \mathcal{F}_{glob}$. The inductive limit over all L conjecturally coincides with $\mathcal{F}_{glob}$ (or one can simply define $\mathcal{F}_{glob}$ as such a limit).

Side remark about resurgence

A conjecture we stated in arXiv:2005.10651 says that analyticity of the wall-crossing structure implies resurgence of some related formal series. In particular, this conjecture explains resurgence of formal expansions as $\hbar \to 0$ of solutions of the equations covered by our RH-correspondence. For more details and applications to exponential integrals including the resurgence of formal expansions in the complexified Chern-Simons theory see our arXiv:2402.07343.

Next we are going to discuss the geometry of local RH-correspondence and explain why and how it should be enhanced.

Local de Rham side: naive picture

After inverting $\hbar$ one can use results of Agnolo, Kashiwara, Schapira who associated to a smooth Lagrangian L a standard DQ-module supported on L. Then objects of $Hol_{L,loc}$ are direct sums of cyclic modules such that the cyclic vector (quantum wave function) is given in local symplectic coordinates (x, p) by the WKB-expansion
$\psi(x,\hbar) = exp(\frac{S_{-1}(x)}{\hbar} + S_0(x) + \hbar S_1(x) + ...) = exp(\sum_{i\geq -1} \hbar^i S_i(x))$, where
$L = \{p = dS_{-1}(x)\}$. Notice that the series in the exponent gives a deformation of L corresponding to the formal path of closed 1-forms $\alpha_{-1} + \sum_{i\geq 1} \hbar^i \alpha_i$, where $\alpha_i = dS_i$ locally.

In fact there are more types of holonomic DQ-modules associated with a smooth L than those above. There are also holonomic DQ-modules associated with non-smooth L which are not covered by the previous theory.

Motivating example: fractional WKB expansions

Even for a smooth L one can consider a DQ-module supported on L which is a cyclic module with the generator $\psi(x,\hbar) = exp(\sum_{\lambda\in\mathbb{Q}\cap[-1,0)} \hbar^\lambda S_\lambda(x) + \sum_{i\geq 0} \hbar^i S_i(x))$.

Such solutions can appear as formal flat sections of connections $\nabla = d + \frac{A_{-1}}{\hbar} + \sum_{i\geq 0} \hbar^i A_i$, where $A_i = A_i(x)$ are holomorphic matrix-valued functions and A_{-1} is nilpotent.

Enhanced Lagrangian subvarieties

From the point of view of deformation theory this means that we should consider deformations L' of L which are more general than those corresponding to formal paths of closed 1-forms of the type $dS_{-1} + \sum_{i\geq 1} \hbar^i \alpha_i(x)$, where locally $L = graph(dS_{-1})$. For example we can consider deformations L' of L corresponding to formal paths of closed 1-forms of the type $dS_{-1} + \sum_{\lambda\in\mathbb{Q}\cap(0,1)} \hbar^\lambda \alpha_\lambda(x) + \sum_{i\geq 1} \hbar^i \alpha_i(x)$. Notice the analogy of the finite sum over $\lambda \in \mathbb{Q} \cap (0,1)$ with singular terms for meromorphic connections in the rational case of GRHC. The difference is that now we are talking about singular behavior w.r.t. $\hbar \to 0$, and not with respect to x. These new " singular terms at $\hbar = 0$" should correspond to certain divisors. I will explain later how to construct them.

Remark that $graph(dS_{-1} + \sum_{\lambda\in\mathbb{Q}\cap(0,1)} \hbar^\lambda \alpha_\lambda(x))$ is a family of Lagrangian varieties which survives when we throw away terms of the size $O(\hbar^{\geq 1})$. Thus it can be thought of as an "enhancement" of the support L of the corresponding DQ-module.

Local de Rham side: enhancements in general

Assume first that M is compact, so we can ignore P_{log}. Consider the manifold $P = M \times \mathbb{C}_\hbar$ endowed with the Poisson structure $\pi_P = \hbar(\omega^{2,0})^{-1}$. It is foliated by symplectic leaves $P_\hbar = M \times \{\hbar\} \simeq M, \hbar \neq 0$, while the fiber P_0 although isomorphic to M as a variety consists of 0-dimensional symplectic leaves.
Let me explain what are the divisors which parametrize the above-mentioned singular terms at $\hbar = 0$. Let us fix an enhancement L' of L (it is a certain deformation of L).
Then we make consecutive blow-ups of P. If L is smooth and $L' = L \times \mathbb{C}_\hbar$ (trivial deformation) the first blow-up is $Bl_{\overline{L'} \cap P_0}(P)$. After that centers of the blow-ups are intersections of exceptional divisors with proper transforms of $\overline{L'}$. Even if L is smooth we can start with a more complicated enhancement L' than the trivial family $L \times \mathbb{C}_\hbar$. If L is non-smooth we first resolve singularities of L. We continue to do blow-ups until the proper transform of $\overline{L'}$ intersects only symplectic divisors ("symplectic" means that on the complement with intersections with other divisors the Poisson structure π_P gives rise to a symplectic form).

For $n = 1$ one can show that smooth parts of symplectic divisors are twisted cotangent bundles to some smooth curves. Moreover in this case the common curve of two intersecting symplectic divisors is a log-curve for each of the two symplectic forms. We expect similar story in general.

We treat the set of symplectic divisors $\Delta_{L'}$ as the set of singular terms at $\hbar = 0$. In the above example of the fractional WKB expansion the Lagrangian subvariety $graph(\alpha_\lambda)$ intersects a unique symplectic divisor D_λ^{sympl}. The set $\Delta_{L'}$ determines the local category $Hol_{\Delta_{L'},loc}$ of holonomic DQ modules. Roughly, objects of this category are families of holonomic DQ-modules "supported" on L' such that their "limits" as $\hbar \to 0$ are certain coherent sheaves with Lagrangian supports determined by the set $\Delta_{L'}$.

If M is non-compact one adds to $P_0 := M \times \{0\}$ the divisor which is the intersection of the closure of $D_{log} \times \mathbb{C}^*_\hbar$ with P_0. Then one repeats the above construction with blow-ups. Assume that the categories $Hol_{\Delta_{L'},loc}$ are well-defined over the ring $\mathbb{C}\{\hbar\}$. Then Hol_{glob} coincides with the inductive limit of $Hol_{\Delta_{L'},loc}$ over all Lagrangian analytic subsets L and all their enhancements L' (notice that in our approach L and L' can be singular, so the categories form a filtered system: union of two singular analytic Lagrangian subsets is a singular analytic Lagrangian subset).

Local Betti side with enhancements

On the Betti side the enhancements affect the WCS. E.g. in the above-mentioned enhancement L' of L associated with the fractional WKB expansion, one can have pseudo-holomorphic discs of the area $O(e^{-\hbar^{\lambda}}), \lambda \in \mathbb{Q} \cap (0,1)$ which are discs with boundaries on $graph(\alpha_\lambda) \cup graph(\alpha_\mu), \lambda, \mu \in \mathbb{Q} \cap (0,1)$. This phenomenon leads to the new "enhanced" WCS discussed on the next slide.

Local RH-correspondence and enhanced WCS

For a given L we define the local system on $S^1 := S^1_{\theta=Arg(\hbar)}$ of local Fukaya categories $\mathcal{F}_{L,loc}$ over $\mathbb{Z}$ as before. Then we change scalars to $K = \mathbb{C}((\hbar^{\frac{1}{N}}))$, where $N \in \mathbb{Z}_{\geq 1}$ depends on L.
The central charge for the enhanced WCS is the linear map
$Z : \gamma \in H_2(M, L, \mathbb{Z}) \mapsto \sum_{\lambda=\frac{m}{N}\in\mathbb{Q}\cap(0,1)} \hbar^{\lambda} \int_{\gamma} \alpha_{\lambda}$. Stokes isomorphisms take into account the above-mentioned pseudo-holomorphic discs of areas $O(e^{-\hbar^{\lambda}})$ sitting in a small neighborhhod of L' with boundaries on the Lagrangian submanifold $\cup_{\lambda} graph(\alpha_{\lambda})$. We use this WCS to change the above-mentioned local system of categories $\mathcal{F}_{L,loc}$ to a meromorphic family of categories as we discussed before.
Since matrix coefficients of the Stokes isomorphisms are of the size $e^{-\hbar^{\alpha}}, Re(\hbar^{\alpha}) > 0$ they do not affect the formal series expansions as elements of $K = \mathbb{C}((\hbar))$ which appear in the local RH-correspondence over K.

From local to global in the enhanced setting

After that we should take into account "large discs in M with boundary on L". This is the previously mentioned part of the story which depends on SFT and deformation theory of A_∞-categories.
In the end we take the inductive limit of modified local Fukaya categories. If the arising WCS are analytic then the inductive limit is a category linear over the germs of meromorphic functions at $\hbar = 0$. Finally having global de Rham and Betti categories we can formulate the conjectural global RH-correspondence.
(Version: define first the local Fukaya categories without taking into account enhancements, and then consider the WCS which takes into account all discs at once including those coming from enhancements).

Non-archimedean geometry and skeleta

Assume for simplicity that M is algebraic. Our global Betti and de Rham categories are inductive limits over the sets which are unions of certain compactifying divisors of P (symplectic divisors serving $\hbar = 0$ and log divisors serving $\hbar \neq 0$). Each divisor gives rise to a discrete valuation on the field of rational functions on the scheme M_K obtained from M by extension of scalars. Hence it gives a point in the Berkovich analytic space M_K^{an}. Then we have a compact $\mathbb{Q}PL$-subspace ("skeleton") of M_K^{an} which is a union of simplices "spanned" by above divisorial valuations. These skeleta are similar to those which appeared in our paper on homological mirror symmetry 25 years ago.

Hope: with each skeleton one can associate de Rham and Betti categories which are equivalent via the GRHC.

Berkovich analytic space contains not only divisorial points, but e.g. generic points of polydiscs $\{(x_1, ..., x_n, y_1, ..., y_n) \in \overline{K}^{2n} | max_i |x_i| \leq r_1, max_i |y_i| \leq r_2\}$ where $r_1 r_2 = 1$. Notice that the Moyal $*$-product converges in a small neighborhood of such a polydisc, if we treat $\hbar$ as a non-archimedean parameter. Then we can consider more general types of DQ-modules and Fukaya categories associated with compact closed subsets of M_K^{an}.

Appendix: some results which can be revisited via our approach

P. Etingof, W. L. Gan, A. Oblomkov, Generalized double affine Hecke algebras of higher rank, arXiv:math/0504089.

S. Gukov, P. Koroteev, S. Nawata, Du Pei, I. Saberi, Branes and DAHA Representations, arXiv:2206.03565.

G. Bellamy, C. Dodd, K. Mcgerty, T. Nevins, Categorical cell decomposition on quantized symplectic varieties, arXiv:1311.6805.

E. Rains, The noncommutative geometry of elliptic difference equations, arXiv:1607.08876.