Nekrasov's gauge origami and Oh–Thomas's virtual cycles

Motivation: ADHM quiver

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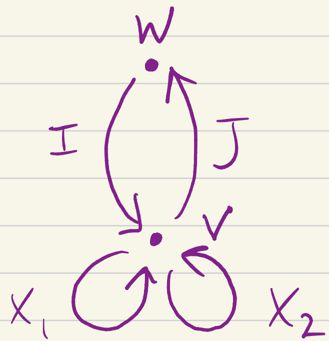
$V := \mathbb{C}^n$, $W := \mathbb{C}^r$

$M_{r,n} :=$

M



X_1, X_2, I, J linear maps s.t.

$[X_1, X_2] + IJ = 0$

stable

$\mathbb{C}\langle X_1, X_2 \rangle I W = V$

$/ GL(V)$

free

Special case:

$r = 1 \implies J = 0$, $M \cong \mathrm{Hilb}^n(\mathbb{C}^2)$

Facts:

(I) M smooth of dim. $2rn$, $T_M \cong \Omega_M$ symplectic

scale x_1, x_2 diag. action on W

(II) $T := \underbrace{(\mathbb{C}^*)^2}_{\text{scale } x_1, x_2} \times \underbrace{(\mathbb{C}^*)^r}_{\text{diag. action on } W} \curvearrowright M$, M^T 0-dim. and reduced

non-compact

$$\sum_n q^n \underbrace{\int_{M_{r,n}} 1}_{} = \sum_{p \in M_{r,n}^T} \frac{1}{e(T_M|_p)}$$

Nekrasov partition function

Atiyah-Bott

(III) sheaf description:

$\ell_\infty \subseteq \mathbb{P}^2$ line

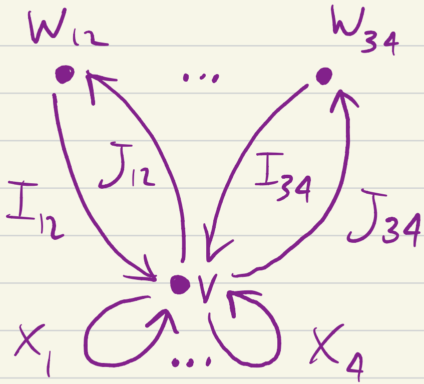
$$M_{r,n} \cong \left\{ (E, \phi) \;\middle|\; \begin{array}{l} E \text{ rk } r \text{ torsion free sheaf } / \mathbb{P}^2 \\ \phi : E|_{\ell_\infty} \cong \mathcal{O}^r, \; c_2(E) = n \end{array} \right\} / \cong$$

Barth

Nekrasov $\quad \underline{6} := \{\{a,b\} \mid 1 \leq a < b \leq 4\}, \quad \underline{4} := \{1,2,3,4\}$

$$\forall A \in \underline{6}: \quad \bar{A} = \underline{4} \setminus A, \quad V := \mathbb{C}^n, \quad \{W_A := \mathbb{C}^{r_A}\}_{A \in \underline{6}}$$

$M_{\underline{r},n} :=$



origami quiver

X_a, I_A, J_A linear maps s.t.

$$[X_a, X_b] + I_A J_A = 0$$

$$J_{\bar{A}} I_A = 0$$

$$X_{\bar{a}} I_A = 0 = J_A X_{\bar{a}} \qquad \forall A = \{a<b\} \in \underline{6}, \ \bar{a} \in \bar{A}$$

stable

$$\sum_A \mathbb{C}\langle X_1, \dots, X_4 \rangle I_A W_A = V$$

$/ \, GL(V)$

free

Related space: $\mathrm{Hilb}^n(\mathbb{C}^4) := \left\{ \begin{tikzcd} \mathbb{C} \arrow[d, "I"] \\ V \arrow[loop left, "X_1"] \arrow[loop right, "X_4"] \end{tikzcd} \quad \begin{array}{l} X_a, I \quad \text{s.t.} \\ [X_a, X_b] = 0 \quad \forall a,b \\ \text{stab.} \end{array} \right\} / GL(V)$

$\text{stab.} \Leftrightarrow \mathbb{C}\langle X_1, \dots, X_4\rangle \cdot I\,\mathbb{C} = V$

$V = \mathbb{C}^n$

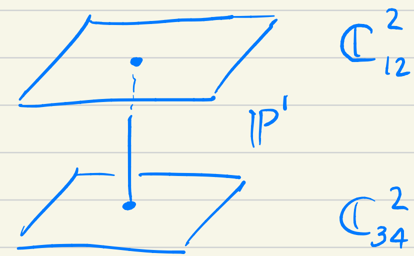
<u>Goal</u>: find analogs I, II, III

Rep. thy. direction: Kimura, Rapčák-Soibelman-Yang-Zhao

<u>Issue</u>: spaces very singular

<u>Ex. 1</u> $\underbrace{r_{12} = r_{34} = 1, \text{ else } r_A = 0}$, $n = 1$:

$\Rightarrow \quad J_A = 0 \quad \forall A$



<u>Ex. 2</u> $\mathrm{Hilb}^{\geq 4}(\mathbb{C}^4)$ all singular

$\downarrow$ j/w J. Rennemo

Extra structure $M_{r,n}$, $\mathrm{Hilb}^n(\mathbb{C}^4)$

(E, q) * quadratic vb., $q : E \otimes E \to \mathcal{O}_{\mathcal{A}}$ non-deg., symm.

$\downarrow \circlearrowright s$ * $\mathrm{rk}\, E =$ even, s isotropic section $q(s,s) = 0$

$\mathcal{A}$ * smooth conn. variety, $M := Z(s)$, $I := I_{M/\mathcal{A}}$

$$\begin{tikzcd}
T_{\mathcal{A}}|_M \arrow[r, "ds"] & E|_M \arrow[r, "q", "\sim"'] & E|_M^* \arrow[r, "(ds)^*"] \arrow[d, "s^*"] & \Omega_{\mathcal{A}}|_M \arrow[d, equal] & \Omega_M^{vir} \arrow[d] \\
& & I/I^2|_M \arrow[r, "d"] & \Omega_{\mathcal{A}}|_M & \mathbb{L}_M
\end{tikzcd}$$

Ω_M^{vir} cx. of vbs.

$\tau^{\geq -1}\, \mathbb{L}_M$ cotangent cx.

$T_M^{vir} := (\Omega_M^{vir})^\vee$

$T_M^{vir}[2] \overset{\theta}{\cong} \Omega_M^{vir}$ 3-term symmetric obstruction theory

$\theta^\vee[2] = \theta$ "-2 symplectic"

$\mathcal{A}$: moduli of reps of same quiver w/o relations

$s : \mathcal{A} \to E$: relations

Claim : $\exists\, q$ s.t. $q(s,s) = 0$

Ex. For $P = [(X_1, \cdots, X_4, I)] \in \mathcal{A}$

$$E|_P \simeq \bigoplus_{1 \leq a < b \leq 4} \mathrm{End}(V) \quad , \quad s_P = \{[X_a, X_b]\}_{a<b}$$

$$q_P(\{v_{ab}\}_{a<b}) = \mathrm{tr}\left(v_{12}v_{34} - v_{13}v_{24} + v_{14}v_{23}\right)$$

$\rightsquigarrow M_{r,n}$, $\mathrm{Hilb}^n(\mathbb{C}^4)$ "-2 symplectic" $\leftarrow$ analog I

No Behrend-Fantechi virtual cycle...

Borisov-Joyce / Oh-Thomas virtual cycle:

pick $\det E|_M \overset{o}{\cong} \mathcal{O}_M$ s.t. $o \otimes o = \det q|_M$ orientation

e.g. $\forall \underset{\text{max isotropic}}{\Lambda \subseteq E}$: $0 \to \Lambda \to E \to \Lambda^* \to 0$

$\rightsquigarrow \det E|_M \overset{o_\Lambda}{\cong} \mathcal{O}_M$ orientation induced by Λ

then $\exists\ [M]^{vir} \in CH_{vd}(M)_{\mathbb{Q}}$

depending on o!

$vd := \operatorname{rk} T_M^{vir}/2$

for $o = o_\Lambda$: $\iota_*[M]^{vir} = e(\Lambda) \cap [\mathcal{A}]$ $\iota: M \subseteq \mathcal{A}$

But $M_{\underline{r},n}$, $\mathrm{Hilb}^n(\mathbb{C}^4)$ non-compact... back to general setup:

suppose: $*$ $\exists\, T \curvearrowright \mathcal{A}$: E, s T-equivariant; q T-invariant

$*$ M^T 0-dim. and reduced

Oh-Thomas

$$\int_{[M]^{vir}} 1 = \sum_{P \in M^T} \frac{1}{\sqrt{(-1)^{vd} e\left(T_M^{vir}|_P\right)}}$$

T-equiv. max. isotropic

where $\sqrt{\cdot}$ induced by ori. ; suppose $E = \Lambda \oplus \Lambda^*$

$\rightsquigarrow$ ori. o_Λ ; $\sqrt{(-1)^{vd} e\left(T_M^{vir}|_P\right)} = \pm\, e\left(T_{\mathcal{A}}|_P - \Lambda|_P\right)$!

$\underset{\sim\sim\sim}{\text{Note}}$: $\Lambda|_p^{fix} \subseteq E|_p^{fix} \xhookleftarrow{ds} T_{\mathcal{A}}|_p^{fix}$ $\underline{\underline{\text{both}}}$ max. isotropic

$\underset{\sim\sim\sim}{\text{Fact}}$: $\pm = \begin{cases} + & \text{if } \Lambda|_p^f,\ T_{\mathcal{A}}|_p^f \text{ in same component } OGr(E|_p^f, q|_p) \\ - & \text{else} \end{cases}$

$\Rightarrow$ $\boxed{\pm = (-1)^{\dim \operatorname{cok}(pr_{\Lambda|_p} \circ ds|_p)^f}}$ K-Rennemo

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$$T := \{(t_1,\dots,t_4) \in (\mathbb{C}^*)^4 \mid t_1t_2t_3t_4 = 1\} \times \overbrace{\prod_{A\in\underline{6}} (\mathbb{C}^*)^{r_A}}^{T^{fr}} \circlearrowright M_{\underline{r},n}$$

$t \cdot X_a = t_a X_a$

then q T-inv.!

diag. action on W_A coordinates $w_{A,\alpha}$

Easy to find explicit $\Lambda \subset E$ maximal isotropic

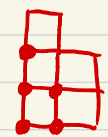
Ex. For $P = [(X_1,\dots,X_4,I)] \in \mathcal{A}$: $\quad \Lambda|_P = \bigoplus_{a<b=4} \mathrm{End}(V)$

!

$\mathrm{Hilb}^n(\mathbb{C}^4)^T$: 4D partitions $\pi \subseteq \mathbb{Z}^4_{\geq 0}$ s.t. $|\pi| = n$

$M^T_{\underline{r},n}$: 2D part. $\underline{\lambda} = \{\lambda_A = \{\lambda_{A,\alpha} \in \mathbb{Z}^2_{\geq 0}\}_{\alpha=1}^{r_A}\}_{A\in\underline{6}}$
s.t. $|\underline{\lambda}| = n$

0-dim. reduced



$\lambda_{A,\alpha}$

Define: $\quad Z_\pi := \sum_{(i,j,k,l)\in\pi} t_1^i t_2^j t_3^k t_4^l$

$$Z_{\lambda_A} := \sum_{\alpha=1}^{r_A} \sum_{(i,j)\in\lambda_{A,\alpha}} t_a^i t_b^j w_{A,\alpha} \quad \forall A = \{a<b\} \in \underline{6}$$

<u>Thm.</u> (Cao-Zhao-Zhou, K-Rennemo, in physics: Nekrasov-Piazzalunga)

$$\sum_n q^n \int_{[\mathrm{Hilb}^n(\mathbb{C}^4)]^{vir}} 1 \overset{\dagger}{=} e^{-\frac{(s_1+s_2)(s_1+s_3)(s_2+s_3)}{s_1 s_2 s_3 s_4} q}$$

$$\overset{\text{✯}}{=} \sum_\pi q^{|\pi|} (-1)^{\mu_\pi} e(-V_\pi)$$

vd = n

w/ $*$ $\mu_\pi := |\pi| + |\{(a,a,a,b) \in \pi \mid a<b\}|$

$t_a \mapsto t_a^{-1}$ $\quad s_1+s_2+s_3+s_4=0$

$*$ $V_\pi := Z_\pi + \frac{(1-t_1)(1-t_2)(1-t_3)}{t_1 t_2 t_3} Z_\pi Z_\pi^*$, $\quad s_a = c_1^T(t_a)$

$*$ ✯ conj. by Nekrasov-Piazzalunga $\leftarrow$ SUSY Yang-Mills thy. $\mathbb{C}^4$

$*$ $\dagger$ by degeneration: Cao-Zhao-Zhou (no sign formula)

$*$ analogue for $\mathrm{Hilb}^n(X)$, X compact CY4: Bojko, Park

Note $\mathrm{vd}(M_{\underline{r},n}) = -\sum_{A\in \underline{3}} r_A r_{\bar{A}}$, $\underline{3} = \{\{1,2\},\{1,3\},\{2,3\}\}$

perturbative term: $C := \int_{[M_{\underline{r},0}]^{vir}} 1$

following Nekrasov...

more notation: $\forall \underline{\lambda} \in M^T_{\underline{r},n}$ $\forall A = \{a<b\} \in \underline{6}$:

$\rightsquigarrow$ Z_{λ_A}, $N_A := \sum_{\alpha=1}^{r_A} w_{A,\alpha}$, $\phi(A) := \min \bar{A}$

$P_A := (1-t_a)(1-t_b)$, $P_{1234} := (1-t_1)(1-t_2)(1-t_3)(1-t_4)$

$T_{\lambda_A} := N_A Z^*_{\lambda_A} + t_a t_b N^*_A Z_{\lambda_A} - P_A Z_{\lambda_A} Z^*_{\lambda_A}$

$t_a t_b \times$ (tangent rep. ADHM moduli space at λ_A!)

__Thm.__ (Arbesfeld-K-Lim, in physics: Nekrasov)

$$\frac{1}{C} \sum_n q^n \int_{[M_{\underline{r},n}]^{vir}} 1 = \sum_{\underline{\lambda}} q^{|\underline{\lambda}|} e(-V_{\underline{\lambda}})$$

w/ $\ast$ $V_{\underline{\lambda}} = \sum_{A \in \underline{6}} \left(P_{\phi(A)} T_{\lambda_A} + P_{\bar{A}} N_A \sum_{B \neq A} Z^*_{\lambda_B} \right) - P_{1234} \sum_{A<B} Z_{\lambda_A} Z^*_{\lambda_B}$

$\ast$ weight $e(-V_{\underline{\lambda}})$: in Nekrasov's BPS/CFT papers

~ 2017

"origami partition function" $Z_{\underline{r}}(q)$ ⟵ analog II

$\ast$ ⚠ sign always +

$\ast$ AKL: K-theoretic version

Key ingredient: for $P = [(X_1, \dots, X_4, \{I_A, J_A\}_{A \in \underline{6}})] \in M^T_{\underline{r}, n}$

$$\operatorname{cok}\left(T_{\mathcal{A}}|_P \xrightarrow{ds} E|_P = \Lambda|_P \oplus \Lambda|_P^* \xrightarrow{pr_\Lambda} \Lambda|_P \right)^{fix} =$$

$$\operatorname{cok}\left(\bigoplus_{a=1}^{3} \operatorname{End}(V) \longrightarrow \bigoplus_{A = \{a<b<4\} \in \underline{6}} \operatorname{End}(V) \right)^{fix}$$

calculated mod 2 by K-Rennemo

$$\{Y_a\}_{a=1}^3 \longmapsto \{[X_a, Y_b] + [Y_a, X_b]\}_{A = \{a<b<4\}}$$

<u>Rem.</u> Similar reduction to signs $\mathrm{Hilb}^n(\mathbb{C}^4)$: tetrahedron inv. Fasola-Monavari, Pomoni-Yan-Zhang

(classical) ADHM moduli
w/ equiv. parameters
$t_1, t_2, w_{12,\alpha}, \alpha = 1, \dots, r$

Back to origami ...

Ex. 1 Take $r_{12} = r$, $r_A = 0$ else; $\mathcal{M}_{\underline{r}, n} = \mathcal{M}_{r,n}$

$$\Rightarrow \quad Z_{\underline{r}}(q) = \sum_n q^n \int_{\mathcal{M}_{r,n}} e\left(T_{\mathcal{M}_{r,n}} \otimes t_3^{-1}\right)$$

equiv. Euler char. of
ADHM moduli
"Vafa-Witten inv. $\mathbb{C}^2_{x_1,x_2}$"

Ex. 2 Take $r_{a4} = 0$ $\forall a = 1,2,3$

$\exists$ $\mathcal{A}_0 \xrightarrow{\Phi} \mathbb{C}$ sm. variety w/ regular function: $M_{\underline{r},n} \overset{\text{RSYZ}}{=} \mathrm{Crit}(\Phi)$

Fact: $\displaystyle\int_{[M_{\underline{r},n}]^{vir}_{OT}} 1 = \int_{[M_{\underline{r},n}]^{vir}_{BF}} 1$

Behrend-Fantechi virtual cycle

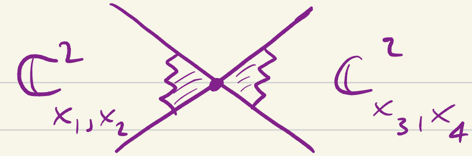
Restrict to: $T_0 := \{(t_1,\ldots,t_4, \{w_{A,\alpha}\}_{A,\alpha}) \in T \mid t_1 t_2 t_3 = 1\}$

Φ T_0-inv.

COHA: Rapčák-Soibelman-Yang-Zhao

On T_0: $Z_{\underline{r}}(q) = \bar{\eta}(q)^{-r}$, $r := \sum_{A \in \underline{6}} r_A$

normalized Dedekind eta, $\bar{\eta}(q) = \prod_{n>0} (1-q^n)$



<u>Thm.</u> (Arbesfeld-K-Lim)

Let $r_{12} = r_{34} = 1$, $r_A = 0$ else

$$Z_{\underline{r}}(q) = \bar{\eta}(q)^{-\frac{(s_2+s_3)(s_1+s_3)}{s_1 s_2}} \cdot \bar{\eta}(q)^{-\frac{(s_3+s_1)(s_4+s_1)}{s_3 s_4}}$$

Key ingredient: $M_{\underline{r},n} \xrightarrow[\text{projective}]{} M^{\text{semisimple}}_{\underline{r},n} \cong \mathrm{Sym}^n(\mathbb{C}^4)$

Towards sheaf description:

X smooth proj. 4-fold s.t. $\exists D > 0 : \ 2D \in |-K_X|$

$v = (0, 0, \gamma, *, *) \in H^{2*}(X, \mathbb{Q})$, fix F pure 1-dim. sheaf on D

$$M_X^{fr}(v) = \left\{ (E, \phi) \ : \ \begin{array}{l} E \text{ pure 2-dim. sheaf on } X \\ \phi : E|_D \xrightarrow{\sim} F \\ ch(E) = v \end{array} \right\} \Big/ \cong$$

M^{fr}

quasi-proj. scheme w/ univ. family

use Huybrechts-Lehn stab. of pairs

$\exists$ 3-term symmetric obstruction theory:

$$\Omega^{vir}_{M^{fr}} \longrightarrow \tau^{\geq -1} \mathbb{L}_{M^{fr}}$$

s.t. $\boxed{T^{vir}_{M^{fr}}\Big|_{(E,\phi)} \cong \mathrm{RHom}_X(E, E(-D))[1]}$ ← selfdual

3-term b/c $\mathrm{Hom}(E, E(-D)) = 0$!

each choice of orientation gives $[M^{fr}]^{vir} \in CH_{vd}(M)_{\mathbb{Q}}$

$vd = -\frac{1}{2}\gamma^2$

[this uses:

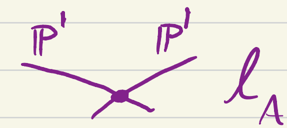
Spaide's work $\Rightarrow$ isotropic condition for normal cone]

we consider T-equivariant settings ...

Take $X := \mathbb{P}^1_{x_i} \times \mathbb{P}^1_{y_i} \times \mathbb{P}^1_{z_i} \times \mathbb{P}^1_{w_i}$, $D := \{ x_0 y_0 z_0 w_0 = 0 \}$

$$\left.\begin{array}{ll} S_{12} = \{ z_1 = w_1 = 0 \} & S_{23} = \{ x_1 = w_1 = 0 \} \\ S_{13} = \{ y_1 = w_1 = 0 \} & S_{24} = \{ x_1 = z_1 = 0 \} \\ S_{14} = \{ y_1 = z_1 = 0 \} & S_{34} = \{ x_1 = y_1 = 0 \} \end{array}\right\} \text{ all } \cong \mathbb{P}^1 \times \mathbb{P}^1$$

$\ell_A = S_A \cap D$, $F_{\underline{r}} := \bigoplus_{A \in \underline{6}} \mathcal{O}_{\ell_A}^{r_A}$, $\underline{r} := \{ r_A \}_{A \in \underline{6}}$



$v := \sum_{A \in \underline{6}} r_A \, \mathrm{ch}(\mathcal{O}_{S_A}) - n \cdot [pt]$, $vd = - \sum_{A \in \underline{3}} r_A r_{\bar{A}}$ acts on framing $F_{\underline{r}}$

note: $T \circlearrowright M^{fr}_{\underline{r},n} := M_X(v)$, $T \cong (\mathbb{C}^*)^3 \times T^{fr}$

acts on (X, D)

Fact: sheaf side quiver side

$$\mathcal{M}^{fr}_{\underline{r},n} \hookleftarrow \left(\mathcal{M}^{fr}_{\underline{r},n}\right)^{T^{fr}} \cong \left(\mathcal{M}_{\underline{r},n}\right)^{T^{fr}} \hookrightarrow \mathcal{M}_{\underline{r},n}$$

let $\mathring{\mathcal{M}}^{fr}_{\underline{r},n}$, $\mathring{\mathcal{M}}_{\underline{r},n}$ connected comp. containing T^{fr}-fx loci

Expectation: $\exists\ \mathring{\mathcal{M}}^{fr}_{\underline{r},n} \xrightarrow{\sim} \mathring{\mathcal{M}}_{\underline{r},n}$ T-equiv. iso. analog III

compatible w/ virtual structures & above inclusions

Prop. (AKL)

$\forall\ p \in \left(\mathcal{M}^{fr}_{\underline{r},n}\right)^T \cong \left(\mathcal{M}_{\underline{r},n}\right)^T$:
$$\boxed{T^{vir}_{\mathcal{M}^{fr}_{\underline{r},n}}\Big|_p = T^{vir}_{\mathcal{M}_{\underline{r},n}}\Big|_p \in K_0^T(pt)}$$

Compact case: (X,H) smooth polarized Calabi-Yau 4-fold

$v=(0,0,\gamma,*,*)\in H^{2*}(X,\mathbb{Q})$ algebraic

M

$$M_X^H(v) := \left\{ \begin{array}{l} E \quad \text{Gieseker } H\text{-stable pure 2-dim. sheaf}/X \\ ch(E)=v \end{array} \right\} \Big/ \cong$$

assume: stable = semistable, $\rightsquigarrow$ M projective scheme

$T_M^{vir}[2] \overset{\theta}{\cong} \Omega_M^{vir}$, $\theta^\vee[2]=\theta$

$\exists$ 3-term symm. obstruction theory: $\Omega_M^{vir} \longrightarrow \tau^{\geq -1}\mathbb{L}_M$

$$\Omega_M^{vir} = \tau^{[-2,0]}\left(R\mathcal{H}om_{\pi_M}(\mathcal{E},\mathcal{E})[1]\right)^\vee$$

$\leftsquigarrow$ $Hom(E,E)\cong Ext^4(E,E)^*\cong\mathbb{C}$

$Ext^1(E,E)\cong Ext^3(E,E)^*$

$Ext^2(E,E)$ selfdual

univ. sheaf on $M\times X$

Borisov-Joyce / Oh-Thomas : $\exists\ [M]^{vir} \in CH_{vd}(M)_{\mathbb{Q}}$

$$vd := 1 - \tfrac{1}{2}\gamma^2$$

Ingredients : * M is Zariski locally an isotropic zero locus (Brav-Bussi-Joyce, building on PTVV)

* $\exists\ \det T^{vir}_M \underset{o}{\cong} \mathcal{O}_M$: $o \otimes o = \det \theta$

(existence orientations: Cao-Gross-Joyce, Joyce-Upmeier)

... but often $[M]^{vir} = 0$:

Ex. X smooth sextic CY4 containing $\mathbb{P}^2 \subseteq X$

$$\gamma = [\mathbb{P}^2] \in H^{2,2}(X) \cap H^4(X,\mathbb{Q})$$

$\rightsquigarrow \quad vd = -\dfrac{19}{2}$

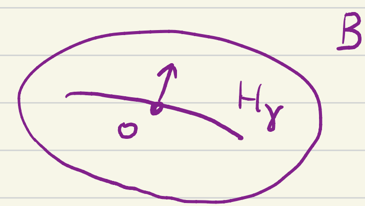
Explanation: let
$$\begin{tikzcd} X \arrow[r] \arrow[d, phantom, "\cap"] & 0 \arrow[d, phantom, "\cap"] \\ \mathcal{X} \arrow[r] & B \end{tikzcd}$$
the family of smooth sextics

after restricting to contractible ngh. 0:

$$H_\gamma := \{ b \in B \mid \gamma \in H^4(\mathcal{X}_b, \mathbb{Q}) \text{ has type } (2,2) \}$$
$$\begin{matrix} H_\gamma \\ \cap| \\ B \end{matrix}$$

Hodge locus γ, $\operatorname{cod} H_\gamma = 19$!

$[M]^{vir} = 0$ by deformation invariance



Back to any $X, H, v = (0,0,\gamma,*,*)$ ...

$$\begin{array}{ccc} \mathcal{X} & \supset & X \\ \downarrow & & \downarrow \\ B & \ni & 0 \end{array}$$

sm. proj. family, B smooth, KS_0 iso.

Bae-K-Park

"reducing": remove $\mathcal{O}^{\mathrm{cod}(H_\gamma)}$ from $Ob = h^1(T_M^{vir})$

get 3-term symm. obstruction theory $\Omega_M^{red} \longrightarrow \tau^{\geq -1} \mathbb{L}_M$

and $[M]^{red} \in CH_{rvd}(M)_{\mathbb{Q}}$

$$\boxed{rvd = 1 - \frac{1}{2}\gamma^2 + \frac{1}{2}\,\mathrm{cod}(H_\gamma)}$$

deformation invariant along Hodge locus H_γ !

Compare to "reducing" for counting curves on K3 surface

Q: what is the significance in physics of $[M]^{red}$?