Mock modularity of Calabi-Yau threefolds

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BPS indices in Type IIA/CY

Type IIA string theory on
a Calabi-Yau threefold $\mathfrak{Y}$

Effective theory in 4d — N=2 SUGRA

BPS indices $\Omega(\gamma) = \mathrm{Tr}_{\mathcal{H}(\gamma)}(-1)^F$

electro-magnetic charge
due to $b_2(\mathfrak{Y})+1$ gauge fields
from b_2 vector multiplets + graviphoton

$\gamma = (p^0, p^a, q_a, q_0)$
$a = 1, \dots, b_2(\mathfrak{Y})$

Physics		Mathematics
#states of ½ BPS black holes (bound states of D6, D4, D2, D0-branes wrapping 6, 4, 2, 0-dimensional cycles)	**=**	**generalized Donaldson-Thomas invariant of CY**

While for *non-compact* CYs there are various techniques to compute BPS indices (localization, quivers, relation to Vafa-Witten topological theory, …), it is a tremendous problem to compute them for *compact* CYs.

D4-D2-D0 BPS states

We consider D4-D2-D0 bound states, i.e. no D6-brane charge $(p^0 = 0)$

$\Omega(\gamma)$ — rank 0 DT invariant

Due to "*spectral flow*" symmetry it depends only on:

- D4-brane charge p^a
- residue class $\mu \in H_2(\mathbb{Z})/H_4(\mathbb{Z})$ (runs over $\det \kappa_{ab}$ values)
- invariant D0-charge $\hat{q}_0 \equiv q_0 - \frac{1}{2}\kappa^{ab} q_a q_b$

$\Rightarrow \Omega(\gamma) = \Omega_{p^a,\mu_a}(\hat{q}_0)$

$\kappa_{ab} = \kappa_{abc} p^c$ — intersection numbers

$$\overline{\Omega}(\gamma) := \sum_{d|\gamma} \frac{1}{d^2}\, \Omega(\gamma/d)$$

generating functions

$$h_{p,\mu}(\tau) = \sum_{\hat{q}_0 \le \hat{q}_0^{\max}} \overline{\Omega}_{p,\mu}(\hat{q}_0)\, e^{-2\pi \mathrm{i} \hat{q}_0 \tau}$$

possess nice **modular properties!!!**

The simplest case: D4-brane wraps an *irreducible* divisor $\Gamma_4 = p^a \gamma_a \subset \mathfrak{Y}$

$h_{p,\mu}(\tau)$ — weakly holomorphic vector valued modular form of weight $-\frac{1}{2} b_2 - 1$

[Maldacena,Strominger,Witten '97]

$$h_{p,\mu}(\tau) = \sum_{n \ge n_{\min}} c_{p,\mu}(n)\, q^n \qquad q = e^{2\pi \mathrm{i}\tau}, \quad n_{\min} < 0$$

$$h_{p,\mu}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-\frac{1}{2} b_2 - 1} \sum_{\nu} M_{\mu\nu} h_{p,\nu}(\tau)$$

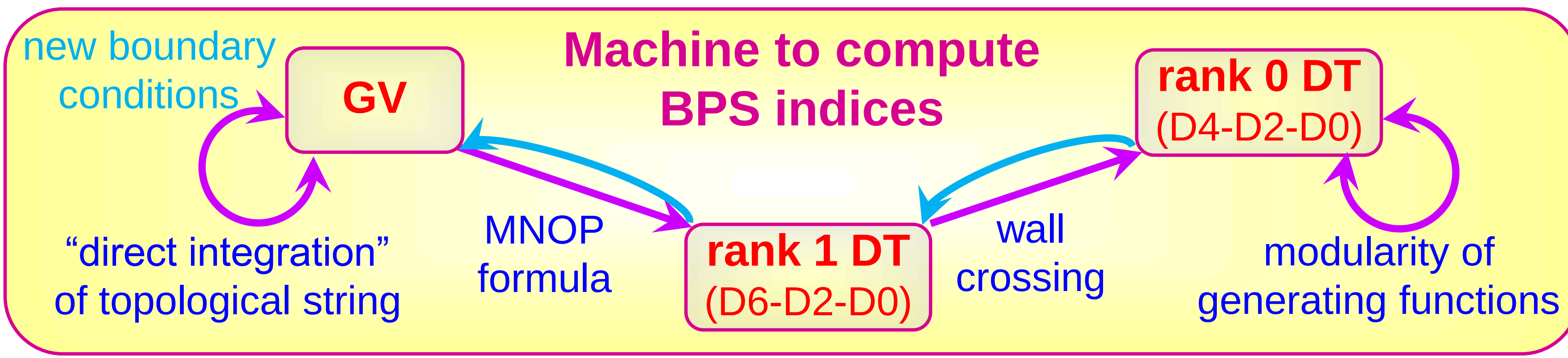
Modular bootstrap

The space of weakly holomorphic modular forms is finite dimensional!

$\dim \mathcal{M}_w(\mathfrak{Y}) \leq$ **# polar terms** — terms with $n < 0$

It is sufficient to find only the polar coefficients

[Gaiotto,Strominger,Yin ’06, Gaiotto,Yin ‘07, Collinucci,Wyder ‘08, Van Herck,Wyder ‘09]



[S.A.,Feyzbakhsh,Klemm, Pioline,Schimannek ’23]

Examples:

quintic	decantic
$g_{\rm int} = 53$	$g_{\rm int} = 50$
$g_{\rm avail} = 64$	$g_{\rm avail} = 71$

What is the story for arbitrary divisors?

The plan of the talk

1. The origin of modularity and modular anomaly

2. Reducible divisors: Mock modular case
   Solution of the anomaly, polar terms and results for BPS indices

3. Reducible divisors: Higher depth case
   Solution of the anomaly via indefinite theta series

4. Conclusions

The origin of modularity

Why should $h_p(\tau)$ possess modular properties?

S-duality

T-duality

modular group

Type IIB $/_{CY\times S^1}$ **Type IIA** $/_{CY \times S^1}$ = **M-theory** $/_{CY \times \underbrace{S^1 \times S^1}_{T^2}}$

Effective theory in 3d — **SUSY non-linear sigma model with target space** $\mathcal{M}$

After compactification on the circle

D4-brane $/_{\Gamma_4 \times S^1}$ — pointlike object (*instanton*)

$\mathcal{M}$ carries an isometric action of the modular group

- Affects the metric on $\mathcal{M}$ by instanton corrections
- Each instanton is weighted by the BPS index $\Omega(\gamma)$
- All instantons are explicitly known from *twistorial* construction of $\mathcal{M}$

[S.A.,Pioline,Saueressig,Vandoren '08, S.A. '09]

Restriction on (the generating function of) BPS indices $\Omega(\gamma)$

S.A.,Pioline '18

Modular anomaly

The restriction specifies a function that must transform as a vector valued *modular form* of weight $-\frac{1}{2} b_2 - 1$

$$\widehat{h}_{p,\mu}(\tau,\bar\tau) = h_{p,\mu}(\tau) + \sum_{n=2}^{n_{\max}} \sum_{\sum_{i=1}^n p_i = p} \sum_{\{\mu_i\}} R^{(\{p_i\})}_{\mu,\{\mu_i\}}(\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i,\mu_i}(\tau)$$

- $R^{(\{p_i\})}_{\mu,\{\mu_i\}}(\tau,\bar\tau)$ — non-holomorphic indefinite theta series with kernels constructed from (derivatives of) *generalized error functions*
- the sum over n is bounded by $n_{\max} = \#\text{irreducible components in } \Gamma_4 \equiv r$

If the divisor is *reducible*, $h_p(\tau)$ has a *modular anomaly* cancelled in $\widehat{h}_p(\tau,\bar\tau)$

modular completion

$h_p(\tau)$ are (higher depth) *mock* modular forms

Mixed Mock modular form — *a holomorphic* function which has a modular anomaly controlled by another modular form (*shadow*)

such that its (*non-holomorphic*) modular *completion* is given by

$$\widehat{h}(\tau,\bar\tau) = h(\tau) - \int_{\bar\tau}^{-i\infty} dz \, \frac{\overline{g(\bar z)}}{(\tau - z)^w} f(\tau)$$

Zwegers '02

Mock modular case

From now on, restrict to one-modulus CY $(b_2(\mathfrak{Y})=1)$.
Then D4-charge $p=r$ degree of reducibility.

The case $r=2$: $h_{2,\mu}$ — *mixed mock* modular form of weight -3/2 with a completion having the holomorphic anomaly:

$$\partial_{\bar\tau}\widehat{h}_{2,\mu} = \frac{\sqrt{\kappa}}{16\pi \mathrm{i}\tau_2^{3/2}} \sum_{\mu_1=0}^{\kappa-1} \overline{\theta^{(2\kappa)}_{\mu-2\mu_1+\kappa}}(\bar\tau)\, h_{1,\mu_1}(\tau)\, h_{1,\mu-\mu_1}(\tau)$$

where
$\mu = 0,\dots,2\kappa-1$
κ — intersection number
$$\theta^{(2\kappa)}_{\mu}(\tau) = \sum_{k\in 2\kappa\mathbb{Z}+\mu} e^{\frac{\pi \mathrm{i}\tau}{2\kappa}\, k^2}$$

For *mixed mock* modular functions, the polar terms are not enough to fix them uniquely. We first need to solve the modular anomaly:

$$h_{2,\mu} = h^{(0)}_{2,\mu} + h^{(\mathrm{an})}_{2,\mu}$$

usual modular form fixed by polar terms

mock modular form with the above modular anomaly

The form of the completion $\widehat{h}_{2,\mu}$ implies that $h^{(\mathrm{an})}_{2,\mu} = \sum_{\mu_1=0}^{\kappa-1} G^{(\kappa)}_{\mu-2\mu_1+\kappa}\, h_{1,\mu_1}\, h_{1,\mu-\mu_1}$
where $G^{(\kappa)}_{\mu}$ is a *usual* mock modular form with completion $\widehat{G}^{(\kappa)}_{\mu}$ having the holomorphic anomaly

$$\partial_{\bar\tau}\widehat{G}^{(\kappa)}_{\mu} = \frac{\sqrt{\kappa}}{16\pi \mathrm{i}\tau_2^{3/2}}\, \overline{\theta^{(2\kappa)}_{\mu}}$$

Solution constructed in [Dabholkar,Murthy,Zagier '12]

Mock modular forms of optimal growth

[Dabholkar,Murthy,Zagier '12] constructed a class of mock modular forms, with shadows given by $\theta_\mu^{(2\kappa)}(\tau)$, distinguished by the *slowest growth* of their Fourier coefficients

$$G_\mu^{(\kappa)} \sim \sum_{\substack{d|\kappa \\ \mu(d)=1}} \left(\mathcal{V}_{\kappa/d}^{(d)}[\mathcal{Q}_d]\right)_\mu$$

Möbius function $\mu(d) = \begin{cases} +1 & \text{if } d \text{ is a square-free with an even \# of prime factors} \\ -1 & \text{if } d \text{ is a square-free with an odd \# of prime factors} \\ 0 & \text{if } d \text{ has a squared prime factor} \end{cases}$

$\mathcal{V}_r^{(d)}$ — Hecke-like operator (rescales the lattice of theta series by r)

$\mathcal{Q}_d$ — a set of "seed" functions: $\mathcal{Q}_1,\ \mathcal{Q}_6,\ \mathcal{Q}_{10},\ \mathcal{Q}_{14},\ \mathcal{Q}_{15},\ \dots$

$\mathcal{Q}_1$ coincides with the generating series of *Hurwitz class numbers*, which is also the (normalized) generating function of *SU(2) Vafa-Witten invariants* on $\mathbb{CP}^2$.

It is enough to account for all κ which are *powers of prime number*.

The problem reduces to finding polar terms

Polar terms

The polar terms can be found using *wall-crossing* — jumps of BPS indices across co-dimension 1 walls in the moduli space.

It happens because $\Omega(\gamma)$ counts not only single centered black holes, but also their bound states which become stable/unstable after crossing a wall.

Stability condition is determined by the central charge $Z_\gamma = q_\Lambda z^\Lambda - p^\Lambda F_\Lambda$

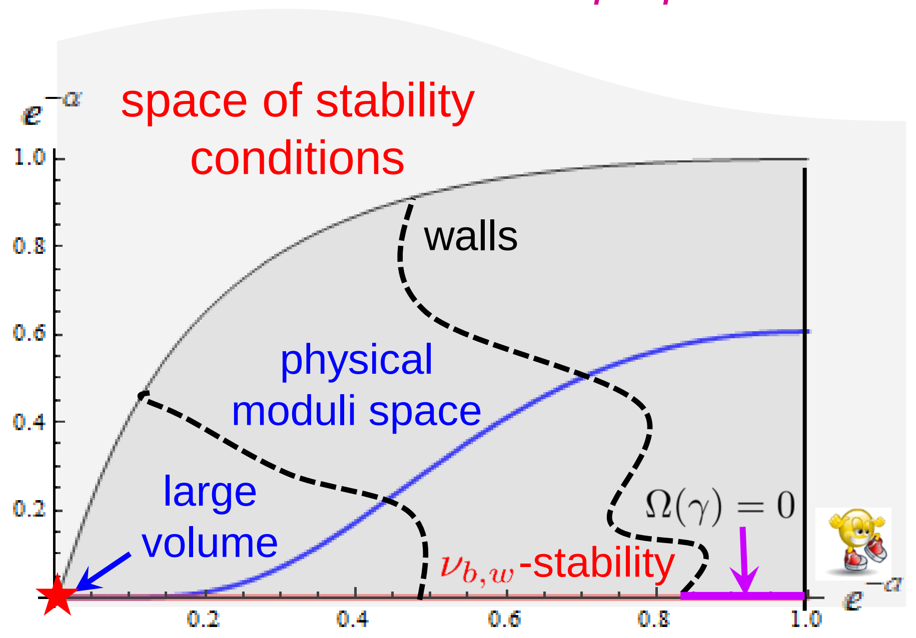
where $F_\Lambda(z) = \partial_{z^\Lambda} F(z)$ — *holomorphic prepotential*
$\Lambda = 0, \dots, b_2(\mathfrak{Y})$

Idea: take F_Λ to be independent parameters

For $\nu_{b,w}$-stability there is a chamber where (for some γ) BPS indices vanish

Recipe: start in this chamber and using wall-crossing formulae, go to the large volume chamber where the $\nu_{b,w}$-stability coincides with the physical one

[Feyzbakhsh '22]



Explicit formula for rank 0 DT in terms of rank 1 DT invariants

Can be computed from GV invariants using MNOP formula

Results

S.A.,Feyzbakhsh,Klemm, Pioline '23

Problem: one needs to know GV invariants at large genus.

So far we have found generating functions of $\overline{\Omega}_{2,\mu}(\hat{q}_0)$ for two CICY: X_{10} and X_8

degree 10 hypersurface in $\mathbb{P}_{5,2,1,1,1}$ with $\kappa = 1$

degree 8 hypersurface in $\mathbb{P}_{4,1,1,1,1}$ with $\kappa = 2$

$$h_1 = \frac{203E_4^4 + 445E_4E_6^2}{216\,\eta^{35}} = q^{-\frac{35}{24}}\left(3 - 575q + 271955q^2 + 206406410q^3 + 21593817025q^4 + \cdots\right)$$

polar terms: $3 - 575q$

$$h_{2,\mu} = \frac{5397523E_4^{12} + 70149738E_4^9E_6^2 - 12112656E_4^6E_6^4 - 61127530E_4^3E_6^6 - 2307075E_6^8}{46438023168\eta^{100}}\,\theta_\mu^{(2)}$$
$$+ \frac{-10826123E_4^{10}E_6 - 14574207E_4^7E_6^3 + 20196255E_4^4E_6^5 + 5204075E_4E_6^7}{1934917632\eta^{100}}\,D_{1/2}\theta_\mu^{(2)}$$
$$+ H_\mu(\tau)\,h_1(\tau)^2$$

$H_\mu(\tau)$ — generating series of Hurwitz class numbers (**mock modular**)

$E_4,\ E_6$ — Eisenstein series

$D_w = q\partial_q - \frac{w}{12}E_2$ — Serre derivative

$$\begin{cases} h_{2,0} = q^{-\frac{19}{6}}\left(7 - 1728q + 203778q^2 - 13717632q^3 - 23922034036q^4 + \dots\right) \\ h_{2,1} = q^{-\frac{35}{12}}\left(-\frac{21}{4} + 1430q - \frac{4344943}{4}q^2 + 208065204q^3 - \frac{199146131237}{4}q^4 + \dots\right) \end{cases}$$

S.A.,Bendriss '24

Higher ranks

Modular completion in the one-modulus case:

$$\widehat{h}_{r,\mu}(\tau,\bar\tau) = h_{r,\mu}(\tau) + \sum_{n=2}^{r} \sum_{\sum_{i=1}^n r_i = r} \sum_{\{\mu_i\}} R^{(\{r_i\})}_{\mu,\{\mu_i\}}(\tau,\bar\tau) \prod_{i=1}^{n} h_{r_i,\mu_i}(\tau)$$

Use the 2-step procedure as in the $r=2$ case:

Can we solve the modular anomaly for any *r*?

$$h_{r,\mu} = h^{(0)}_{r,\mu} + h^{(\mathrm{an})}_{r,\mu}$$

usual modular form fixed by polar terms

mock modular form of depth $r-1$ with the above modular anomaly

Problem: the anomaly depends on all $h^{(0)}_{r_i,\mu_i}$ with $r_i < r$ remaining unknown

Express $h^{(\mathrm{an})}_{r,\mu}$ in terms of $h^{(0)}_{r_i,\mu_i}$ and find coefficients

$$h^{(\mathrm{an})}_{r,\mu}(\tau) = \sum_{n=2}^{r} \sum_{\sum_{i=1}^n r_i = r} \sum_{\{\mu_i\}} g^{(\{r_i\})}_{\mu,\{\mu_i\}}(\tau) \prod_{i=1}^{n} h^{(0)}_{r_i,\mu_i}(\tau)$$

What are these functions?

anomalous coefficients

Anomalous coefficients

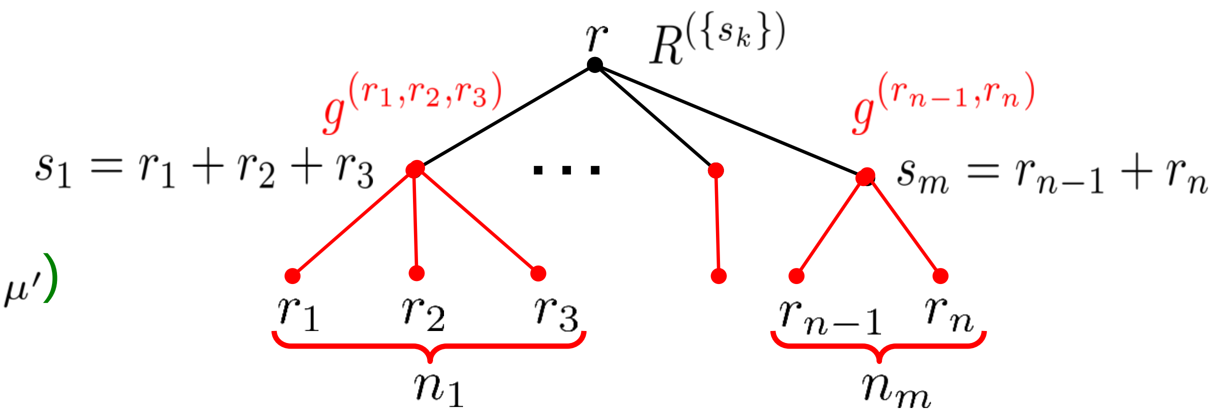
Anomaly equation for $h^{(\mathrm{an})}_{r,\mu}$

$g^{(\{r_i\})}_{\mu,\{\mu_i\}}$ are depth $n-1$ mock modular with completion

$$\widehat{g}^{(\{r_i\})}_{\mu,\{\mu_i\}} = g^{(\{r_i\})}_{\mu,\{\mu_i\}} + \mathrm{Sym}\left\{ \sum_{m=2}^{n} \sum_{\sum_{k=1}^{m} n_k = n} \sum_{\{\nu_k\}} R^{(\{s_k\})}_{\mu,\{\nu_k\}} \prod_k g^{(r_{j_k+1},\dots,r_{j_{k+1}})}_{\nu_k,\mu_{j_k+1},\dots,\mu_{j_{k+1}}} \right\}$$

The sum goes over trees of depth 2 with *n* leaves:

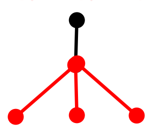
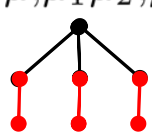
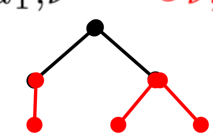
(for one charge $g^{(r)}_{\mu,\mu'} \equiv \delta_{\mu,\mu'}$)



Examples:

$$\widehat{g}^{(r_1,r_2)}_{\mu,\mu_1,\mu_2} = g^{(r_1,r_2)}_{\mu,\mu_1,\mu_2} + R^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}$$

$$\widehat{g}^{(r_1,r_2,r_3)}_{\mu,\mu_1,\mu_2,\mu_3} = g^{(r_1,r_2,r_3)}_{\mu,\mu_1,\mu_2,\mu_3} + R^{(r_1,r_2,r_3)}_{\mu,\mu_1\mu_2,\mu_3} + \frac{2}{3}\left[\sum_{\nu=0}^{\kappa(r_2+r_3)-1} R^{(r_1,r_2+r_3)}_{\mu,\mu_1,\nu}\, g^{(r_2,r_3)}_{\nu,\mu_2,\mu_3} + \begin{matrix}\text{cyclic}\\ \text{perm.}\end{matrix} \right]$$

Two charges

Find a mock modular form satisfying

$$\widehat{g}^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}(\tau,\bar\tau) = g^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}(\tau) + R^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}(\tau,\bar\tau)$$

It is sufficient to know that

$$\partial_{\bar\tau} R^{(r_1,r_2)}_{\mu,\mu_1,\mu_2} = \delta^{(\kappa r_0)}_{\mu-\mu_1-\mu_2}\,\frac{r_0\sqrt{\kappa_0}}{16\pi \mathrm{i}\tau_2^{3/2}}\,\overline{\theta^{(2\kappa_0)}_{\mu_0}}$$

$$r_0 = \gcd(r_1,r_2)$$

$$\kappa_0 = \frac{\kappa}{2r_0^2}\, r_1 r_2(r_1+r_2)$$

$$\delta^{(\kappa)}_x = \delta_{x \bmod \kappa}$$

$$g^{(r_1,r_2)}_{\mu,\mu_1,\mu_2} = r_0\,\delta^{(\kappa r_0)}_{\mu-\mu_1-\mu_2}\, G^{(\kappa_0)}_{\mu_0}$$

DMZ mock modular form of optimal growth

Unit charges and Vafa-Witten

$\kappa = r_1 = r_2 = 1 \;\Rightarrow\; \kappa_0 = 1 \;\Rightarrow\; g_\mu^{(1,1)} = H_\mu$

generating series of Hurwitz class numbers

||

(normalized) generating function of *SU(2)* Vafa-Witten invariants on $\mathbb{CP}^2$

[Vafa,Witten '94]

$$h_{2,\mu}^{\rm VW}[\mathbb{CP}^2] = 3\left(h_1^{\rm VW}[\mathbb{CP}^2]\right)^2 H_\mu$$

where $h_1^{\rm VW}[\mathbb{CP}^2] = \eta^{-3}(\tau)$

This is not an accident!

The anomaly equation for $g_\mu^{\overbrace{(1,\dots,1)}^{n}} \equiv g_{n,\mu}$ and $\kappa = 1$ coincides with the anomaly equation for the normalized generating functions of *SU(n)* Vafa-Witten invariants on $\mathbb{CP}^2$

$$g_{n,\mu} = 3^{1-n}\,\frac{h_{n,\mu}^{(\rm VW)}[\mathbb{CP}^2]}{\left(h_1^{(\rm VW)}[\mathbb{CP}^2]\right)^n}$$

Neither of these solutions generalizes to more general cases

Indefinite theta series

We construct a general solution in terms of *indefinite theta series*

Typical indefinite theta series $\sum_{k\in\Lambda+\mu} q^{\frac12 Q(k)}\,\Phi(\sqrt{2\tau_2}\,k)$

$Q(k)=Q_+(k)-Q_-(k)$ is a quadratic form of signature (n_+,n_-)

Two ways to achieve convergence

$\Phi(x)$ exponentially decaying along negative directions

Ex.: Seigel theta series

$$\Phi(x)\sim \exp\left(-\pi Q_-(x)\right)$$

modular, but not holomorphic

$\Phi(x)$ piecewise constant

$$\Phi(x)\sim \prod_{i=1}^{n_-}\Big(\mathrm{sgn}(v_{i,1}\cdot x)-\mathrm{sgn}(v_{i,2}\cdot x)\Big)$$

holomorphic, but not modular

mock (of depth n_-)

How to construct completion?

Recipe: [S.A.,Banerjee,Manschot,Pioline '16]

replace

$$\prod_{i=1}^{n}\mathrm{sgn}(v_i\cdot x)\mapsto E_n\big(\{v_i\};x\big)\equiv\int_{\mathbb{R}^n}\mathrm{d}x'\,e^{-\pi(x-x')^2}\prod_{i=1}^{n}\mathrm{sgn}(v_i\cdot x')$$

generalized error functions

building blocks of the coefficients $R^{(\{r_i\})}_{\mu,\{\mu_i\}}$

Lattice extension

Consider again the case $n=2$:

$$\widehat{g}^{(r_1,r_2)}_{\mu,\mu_1,\mu_2} = g^{(r_1,r_2)}_{\mu,\mu_1,\mu_2} + R^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}$$

where

$$R^{(r_1,r_2)}_{\mu,\mu_1,\mu_2} \sim \sum_{k\in\Lambda+\mu'} q^{\frac{1}{2}Q(k)} \left(E_1\left(\sqrt{2\tau_2}\,\frac{v\cdot k}{||v||}\right) - \mathrm{sgn}(v\cdot k)\right)$$

$n_- = 1$

$$g^{(r_1,r_2)}_{\mu,\mu_1,\mu_2} \sim \sum_{k\in\Lambda+\mu'} q^{\frac{1}{2}Q(k)} \Big(\mathrm{sgn}(v\cdot k) - \mathrm{sgn}(w\cdot k)\Big)$$

needed for convergence

where $w\in\Lambda$ and $Q(w)=0$

i.e. w Is a ***null vector***

$$\left(E_1(x) \underset{x\to\infty}{\longrightarrow} \mathrm{sgn}(x)\right)$$

Problem: in our case $\Lambda=\mathbb{Z}$ with $Q(k)=-2\kappa_0 k^2$ does not have null vectors

Solution: *Extend* the lattice by multiplying by Jacobi theta functions

$$\widehat{\check{g}}^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}(\tau,z_1,z_2) = \check{g}^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}(\tau,z_1,z_2) + R^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}(\tau)\,\theta_1^{\kappa r_1}(\tau,z_1)\,\theta_1^{\kappa r_2}(\tau,z_2)$$

Each theta function adds 1 dimension to the lattice, but does *not* change Λ^*/Λ

In general case:

$$\Lambda = \mathbb{Z}_-^{n-1} \mapsto \Lambda_{\rm ext} = \mathbb{Z}_-^{n-1} \oplus \Big(\mathop{\oplus}_{i=1}^{n} \mathbb{Z}_+^{\kappa r_i}\Big)$$

$$g^{(\{r_i\})}_{\mu,\{\mu_i\}}(\tau) = \left(\prod_{i=1}^{n} \frac{\mathcal{D}^{(\kappa r_i)}(z_i)}{\left(-2\pi\eta^3(\tau)\right)^{\kappa r_i}}\right) \check{g}^{(\{r_i\})}_{\mu,\{\mu_i\}}(\tau,\{z_i\})\Big|_{z_i=0}$$

solve extended anomaly eq. → $\check{g}^{(\{r_i\})}_{\mu,\{\mu_i\}}$ → extract solution of the original eq.

Refinement

We also need to introduce *refinement* $z=\alpha-\tau\beta \qquad \alpha,\beta\in\mathbb{R}$

Physically, it corresponds to switching on *Ω-background*

$$g^{(\{r_i\})}_{\mu,\{\mu_i\}}(\tau)\mapsto g^{(\{r_i\})\mathrm{ref}}_{\mu,\{\mu_i\}}(\tau,z)$$

— mock Jacobi-like form

under $\tau\mapsto\dfrac{a\tau+b}{c\tau+d},\ z\mapsto\dfrac{z}{c\tau+d}$

- simplifies functions $R^{(\{r_i\})}_{\mu,\{\mu_i\}}$
- regularizes divergences due to null vectors

$$\left.\begin{array}{r} k=k_v v+k_w w+k_\perp \\ w^2=0\end{array}\right\} \Rightarrow \left.\begin{array}{r} w\cdot k=0 \\ k_\perp=0\end{array}\right\} \Longrightarrow k_v=0,\ k^2=0$$

$$\sum_{k\in\Lambda+\mu'} q^{\frac12 k^2}\Big(\mathrm{sgn}(v\cdot k)-\mathrm{sgn}(w\cdot k)\Big)\supset\sum_{k_w\in\mathbb{Z}}\mathrm{sgn}((v\cdot w)\,k_w)$$

divergent

refinement

$y=e^{2\pi\mathrm{i}z}$

$$\sum_{k\in\Lambda+\mu'} q^{\frac12 k^2}\,y^{\theta\cdot k}\Big(\mathrm{sgn}(v\cdot k)-\mathrm{sgn}(w\cdot k+\beta)\Big)$$

$$\supset\sum_{k_w\in\mathbb{Z}} y^{(\theta\cdot w)k_w}\Big(\mathrm{sgn}((v\cdot w)\,k_w)-\mathrm{sgn}(\beta)\Big)$$

geometric progression

convergent, but *poles* at $z=0$

Results

Goal: Find mock Jacobi-like forms on the extended lattice that are regular at $z=0$ and then take the unrefined limit

$$g^{(\{r_i\})\mathrm{ref}}_{\mu,\{\mu_i\}}(\tau,z)=\mathrm{Sym}\left\{\sum_{m=1}^{n}\sum_{\sum_{k=1}^{m}n_k=n}\sum_{\{\nu_k\}}\vartheta^{(\{s_k\})}_{\mu,\{\nu_k\}}\prod_k\phi^{(r_{j_k+1},\dots,r_{j_{k+1}})}_{\nu_k,\mu_{j_k+1},\dots,\mu_{j_{k+1}}}\right\}$$

theta series on Λ_{ext} with kernel

$$\prod_{i=1}^{n-1}\Big(\mathrm{sgn}(v_i\cdot k)-\mathrm{sgn}(w_i\cdot k+\beta)\Big)$$

Jacobi-like form needed to cancel poles at $z=0$

One can take $\phi\sim\dfrac{1}{z^{n-1}}$

unrefined limit $z\to 0$ — the most non-trivial step

Explicit expressions for $g^{(r_1,r_2)}_{\mu,\mu_1,\mu_2}$ and $g^{(r_1,r_2,r_3)}_{\mu,\mu_1,\mu_2,\mu_4}$

Conclusions

- We derived modular properties of generating functions of D4-D2-D0 BPS indices (rank 0 DT invariants).

- Using these properties, wall-crossing and direct integration of topological string, we computed the generating functions for one-parameter compact CY threefolds → mock modular forms
  → new boundary conditions for the direct integration of topological string

- Solution of the modular anomaly for $r > 2$ (consistency of different solutions)
  → reduces the problem to finding just a finite number of polar terms

Possibles extensions:

- compute polar terms for $r > 2$ → new strategy?
- CYs with two and more moduli
- elliptic and K_3 fibrations …
- DT invariants of higher rank?

Thank you!