Non-toric brane webs, Calabi–Yau 3-folds, and 5d SCFTs

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Introduction

- **Alexeev–Argüz–B** (AAB): Non-toric brane webs, Calabi–Yau 3-folds, and 5d SCFTs, arXiv:2410.04714
- Physics interpretation/application of a series of recent mathematical works around mirror symmetry for log Calabi-Yau surfaces:
  - Gross–Hacking–Keel, Engel–Friedman, Hacking–Keel–Yu
  - Alexeev–Argüz–B: The KSBA moduli space of log Calabi–Yau surfaces, arXiv:2402.15117

5d SCFTs

- Among the many remarkable predictions of string/M-theory:
  - The existence of interacting 5-dimensional quantum field theories
  - 5-dimensional $\mathcal{N} = 1$ superconformal field theories [Seiberg 1996, Morrison–Seiberg 1996, Douglas–Katz–Vafa 1996, ...]
- Two main constructions:
  1) Singular geometry: M-theory on a canonical 3-fold singularity $\overline{\mathcal{X}}$:
  $$\mathbb{R}^{1,4} \times \overline{\mathcal{X}}\,.$$
  2) Intersecting branes in IIB string theory on $\mathbb{R}^{1,9}$:
  $$\mathbb{R}^{1,4} \times \mathbb{R}^5$$
  Several possible ingredients: 5-branes, 7-branes, orientifolds, S-folds.

Questions

- Basic question: relation between these two main constructions?
  - The M-theory construction seems more general.
- More precise questions:
  - For every system of intersecting branes in IIB string theory engineering a 5d SCFT, can one realize the same 5d SCFT as M-theory on a canonical 3-fold singularity $\overline{\mathcal{X}}$?
  - Can one give an algebro-geometric description of the class of canonical 3-fold singularities $\overline{\mathcal{X}}$ with a dual IIB string theory description?
- Known results:
  - If only 5-branes, duality with M-theory on toric Calabi–Yau 3-fold $\overline{\mathcal{X}}$ [Aharony–Hanany–Kol, Leung–Vafa, 1997]
  - Particular configurations of 5-branes and 7-branes are dual to M-theory on cones over del Pezzo surfaces.
  - Most recent progress: [Bourget, Collinuci, Schafer-Nameki, 2023], [Arias-Tamargo, Franco, Rodríguez-Gómez, 2024].

Overview of results (AAB)

- From intersecting branes to Calabi–Yau 3-folds:

> Every **consistent** web of 5-branes with 7-branes is dual to M-theory on a canonical 3-fold singularity $\overline{\mathcal{X}}$.

- Key point: clarify what **consistent** means.

Overview of results (AAB)

- From Calabi–Yau 3-folds to intersecting branes.

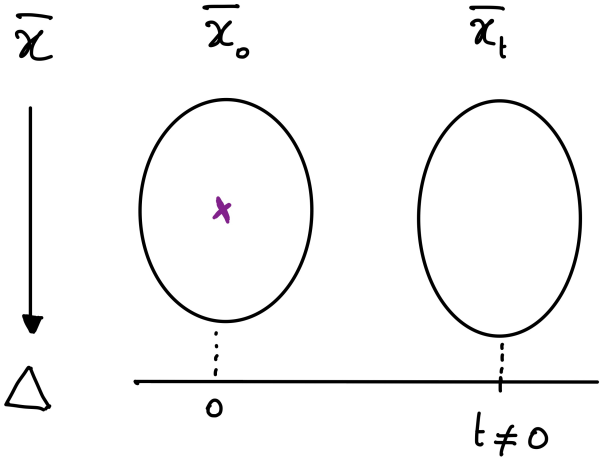
M-theory on a canonical 3-fold singularity $\overline{\mathcal{X}}$ admits a IIB brane dual description in terms of a web of 5-branes with 7-branes $\iff$ there exists a surjective holomorphic map $\pi : \overline{\mathcal{X}} \to \Delta$, where

$$\Delta = \{z \in \mathbb{C} \,|\, |z| < \epsilon\}$$

is a small disk around $0 \in \mathbb{C}$, such that:

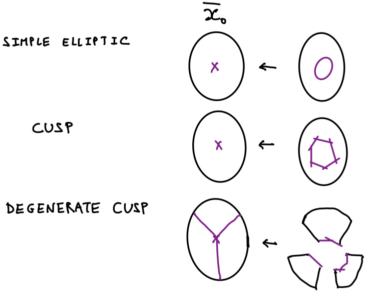
- for every $t \neq 0$, the general fiber $\overline{\mathcal{X}}_t := \pi^{-1}(t)$ is a smooth surface
- the central fiber $\overline{\mathcal{X}}_0 := \pi^{-1}(0)$ is either:
  - a simple elliptic singularity
  - a cusp singularity
  - a degenerate cusp singularity
- There exists a semistable reduction of $\pi : \overline{\mathcal{X}} \to \Delta$.

Results


$\overline{\mathcal{X}}$
$\overline{\mathcal{X}}_0$
$\overline{\mathcal{X}}_t$
Δ
0
$t \neq 0$

Overview of results

- Simple elliptic, cusp, and degenerate cusp singularities are examples of semi–log–canonical (slc) singularities [Kollár, Shepherd-Barron, 1988].



  - Difficult: given the central fiber $\overline{\mathcal{X}}_0$, classify all the possible smoothings $\pi : \overline{\mathcal{X}} \to \Delta$.

- If one allows orientifolds and S-folds, expect a similar result involving $\mathbb{Q}$-Gorenstein slc singularities (all finite quotients of the above).

Plan of the talk

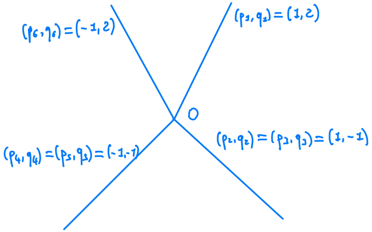
- Webs of 5-branes and toric mirror symmetry.
- Webs of 5-branes with 7-branes and polarized log Calabi–Yau surfaces.
- The dual Calabi–Yau 3-fold from mirror symmetry.
- Worldsheet instantons corrections as 4d $\mathcal{N} = 2$ BPS states.

Webs of 5-branes and toric mirror symmetry

- Review of a classical story [Aharony–Hanany–Kol, Leung–Vafa, 1997]
- Webs of 5-branes:
  - In IIB string theory, (p, q) 5-branes for every coprime $(p, q) \in \mathbb{Z}^2$.
  - Pick (p_i, q_i) such that $\sum_i (p_i, q_i) = 0$.
  - IIB string theory on

$$\mathbb{R}^{1,4} \times \mathbb{R}^2 \times \mathbb{R}^3$$

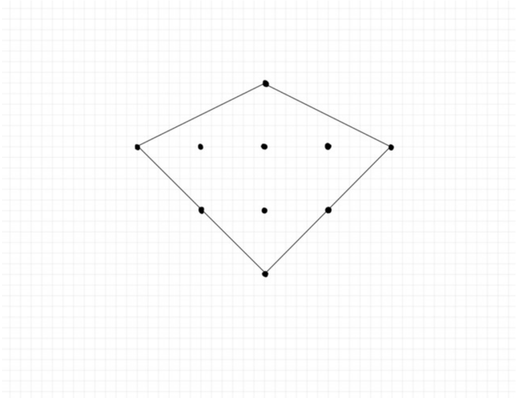
  - 5-branes on $\mathbb{R}^{1,4} \times \overline{W}$, where $\overline{W} \subset \mathbb{R}^2$ is the web:



- 5d SCFT on the common intersection $\mathbb{R}^{1,4} \times \{0\}$ of the branes.

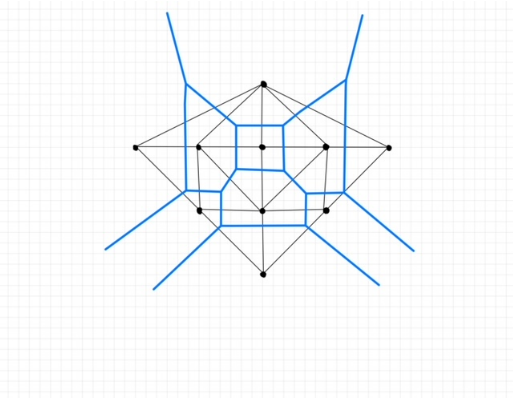
Webs of 5-branes and toric mirror symmetry

- Dual polytope P to $\overline{W}$:

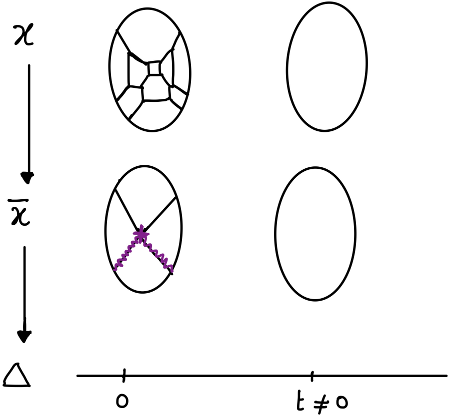


- Dual toric Calabi–Yau 3-fold $\overline{\mathcal{X}}$ with fan the cone over P.
  - Toric morphism $\pi : \overline{\mathcal{X}} \to \mathbb{C}$
  - General fiber $\pi^{-1}(t) = (\mathbb{C}^\star)^2$ for $t \neq 0$.
  - $\overline{\mathcal{X}}_0$: degenerate cusp singularity.

- General (3-valent) perturbations of the web of 5-branes.
  - Regular maximal triangulations of P.
  - Crepant resolutions $\mathcal{X} \to \overline{\mathcal{X}}$.



- Toric morphism $\mathcal{X} \to \mathbb{C}$:
  - central fiber $\mathcal{X}_0$ with irreducible components in one-to-one correspondence with the faces of the perturbed web


$\mathcal{X}$
$\overline{\mathcal{X}}$
Δ
0
$t \neq 0$

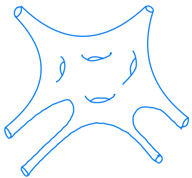
Physics derivation of the duality in the toric case

- By a series of string dualities, the 5d SCFT constructed from the brane web $\overline{W}$, is dual, after compactification to S^1, to IIB string theory on the non-compact Calabi–Yau 3-fold

$$Z : uv = f(x,y)$$

with mirror IIA string theory on $\mathcal{X}$:

  - $f(x,y)$ Laurent polynomial with Newton polygon P.
  - Curve $C^\circ = \{f = 0\} \subset (\mathbb{C}^\star)^2$ is the Seiberg–Witten curve of the resulting $4d$ $\mathcal{N} = 2$ theory.
  - Tropically reduces to the deformed brane web.



  - Natural compactication $C \in |L|$ in the polarized toric variety (Y, D, L) with momentum polytope P.

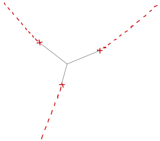
- IIB string theory on

$$\mathbb{R}^{1,4} \times \mathbb{R}^2 \times \mathbb{R}^3$$

with (p,q) 7-branes on

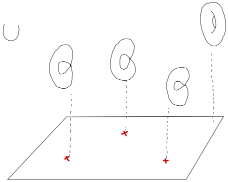
$$\mathbb{R}^{1,4} \times \{point\} \times \mathbb{R}^3$$

- Question: classification of configurations of 7-branes up to Hanany–Witten moves?



Claim (AAB): Configurations of 7-branes up to Hanany–Witten moves are in one-to-one correspondence with interiors U of log Calabi–Yau surface.

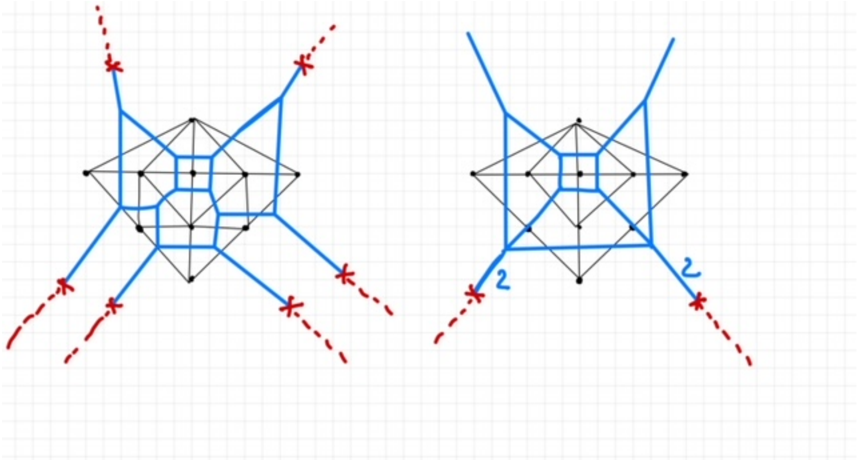
- $U = Y \setminus D$, Y: smooth projective surface, D: singular normal crossing anticanonical divisor.
  - U has a Lagrangian torus fibration over $\mathbb{R}^2$, with nodal singular fibers over the positions of 7-branes.



  - U is a cluster variety: $(\mathbb{C}^\star)^2$ charts related by mutations corresponding to Hanany–Witten moves.
  - Link between HW moves and combinatorial mutations: [Arias-Tamargo, Franco, Rodríguez-Gómez, 2024]
- Configuration of 7-branes = F-theory on U.

Webs of 5-branes with 7-branes

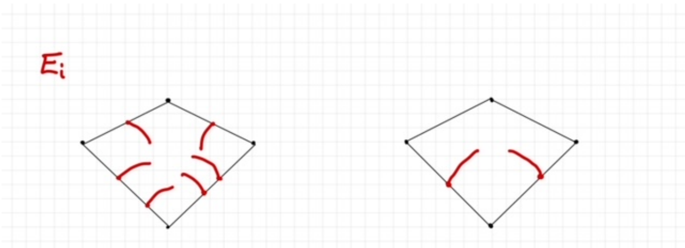
- (p, q) 5-branes can end on a (p, q) 7-brane.
  - Single (p, q) 7-brane ending on a (p, q) 7-branes [DeWolfe, Hanany, Iqbal, Katz, 1999]
  - Several (p, q) 7-branes ending on the same (p, q) 7-brane [Benini, Benvenuti, Tachikawa, 2009].



Consistent webs of 5-branes with 7-branes

- Problem: not all choices are consistent/supersymmetric (s-rule, r-rule,...)
- Our algebro-geometric proposal (AAB):
  - $(\overline{Y}, \overline{D}, \overline{L})$: polarized toric surface, defined by polytope $\overline{P}$ corresponding to the original web of 5-branes.
  - (Y, D): log Calabi–Yau surface obtained by a non-toric blow-up on $\overline{D}$ for each 7-brane.
  - The interior $U = Y \setminus D$ characterizes the configuration of 7-branes.
  - Exceptional curve E_i: if a_i 5-branes end on the corresponding 7-brane,

$$L := \overline{L} - \sum_i a_i E_i$$



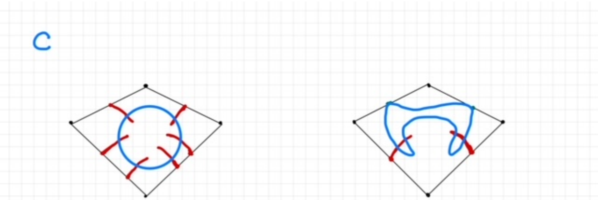
Consistent webs of 5-branes with 7-branes

Claim (AAB): The web of 5-branes with 7-branes is consistent $\iff$ L is nef ($L \cdot S \geq 0$ for every effective curve S in Y) and

- either $L^2 > 0$,
- or $L^2 = 0$ and $L = kE$, $k \geq 1$, E smooth elliptic curve such that $E \cdot D = 0$.

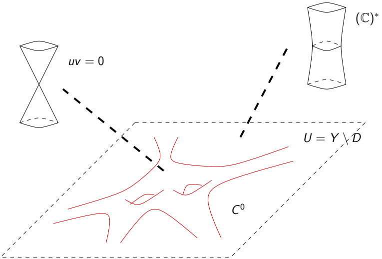
Lemma (AAB)

If the web of 5-branes with 7-branes is consistent, then there exists a smooth curve $C \in |L|$ in the associated polarized log Calabi–Yau surface.



Physics interpretation of C

- The 4d $\mathcal{N}=2$ theory obtained by compactifying the 5d SCFT on S^1 is dual to IIB string theory on the Calabi–Yau 3-fold $Z : uv = f$,
  - $f = 0$ is the section of $L|_U$ defining the curve $C^\circ = C \cap U$ in U.
  - $C^\circ \subset U$ is the Seiberg–Witten curve of this 4d $\mathcal{N}=2$ theory.



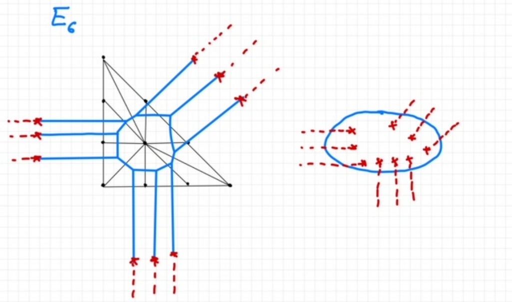
- As in the toric situation, the dual M-theory description of the 5d SCFT will be given by the mirror $\overline{\mathcal{X}}$ of Z.
  - Problem: how to describe the mirror in this non-toric situation?

Pushing 7-branes

Lemma (Alexeev-Argüz-B)

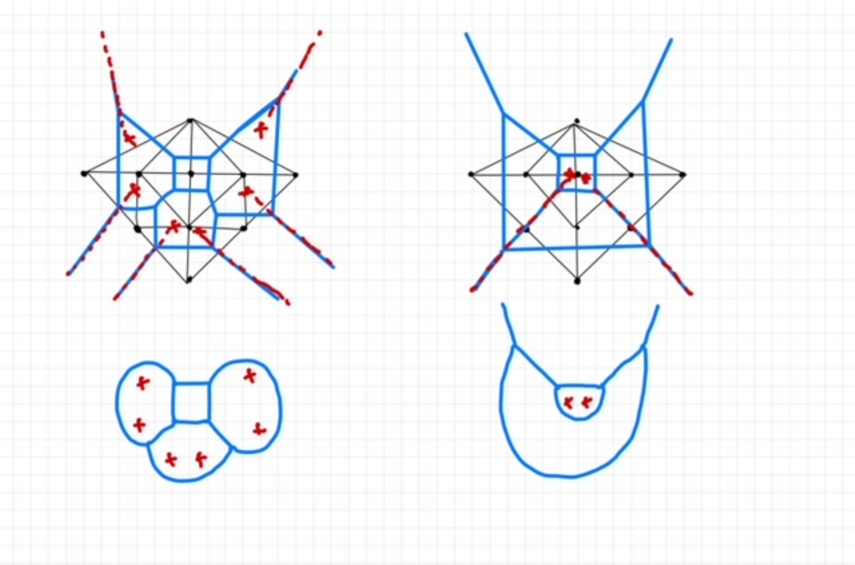
If the web of 5-branes with 7-branes is consistent, then there is no obstruction to "push in" the 7-branes along their monodromy invariant directions until they are no longer attached to any 5-branes.

- Reformulation of a result in birational geometry (existence of a minimal model after a sequence of flops).
- Known example of local del Pezzo surfaces [DeWolfe, Hanany, Iqbal, Katz, 1999]



Pushing 7-branes

- More examples:



Construction of the mirror to Z

- The initial web W of 5-branes defines a (possible singular) toric Calabi–Yau 3-fold $\mathcal{X}_W \to \mathbb{C}$.
  - Irreducible components of the central fiber $\mathcal{X}_{W,0}$ in 1:1 correspondence with the faces of the web.

Theorem (Alexeev-Argüz-B)

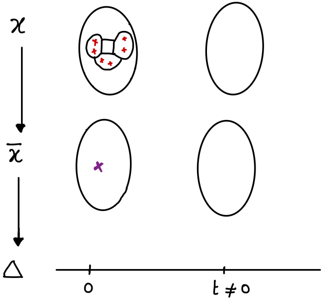
Pushing in the 7-branes induces a non-toric deformation $\mathcal{X}_0$ of $\mathcal{X}_{W,0}$, with irreducible components in 1:1 correspondence with the faces of the deformed web.

Theorem (Alexeev-Argüz-B, Engel–Friedman in the "cusp case")

There exists a smoothing $\mathcal{X} \to \Delta$ of $\mathcal{X}_0$ (the "mirror" to Z), and a contraction $\mathcal{X} \to \overline{\mathcal{X}}$, where $\overline{\mathcal{X}}$ is an affine canonical 3-fold singularity.

- The central fiber $\overline{\mathcal{X}}_0$ is:
  - a simple elliptic singularity if $L^2 = 0$.
  - a degenerate cusp singularity if $L^2 > 0$ and $\deg L|_D > 0$
  - a cusp singularity if $L^2 > 0$ and $\deg L|_D = 0$.

Construction of the mirror to Z



Conversely, we have the following:

Theorem (Alexeev-Argüz-B)

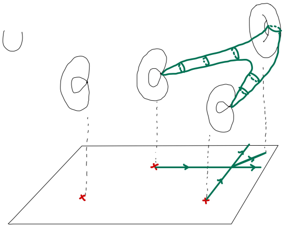
Associated to any $\mathcal{X} \to \overline{\mathcal{X}} \to \Delta$, there exists a log Calabi–Yau surface (Y, D) with a nef line bundle L, and so a web of 5-branes with 7-branes.

Explicit equations for the mirror?

Theorem (AAB)

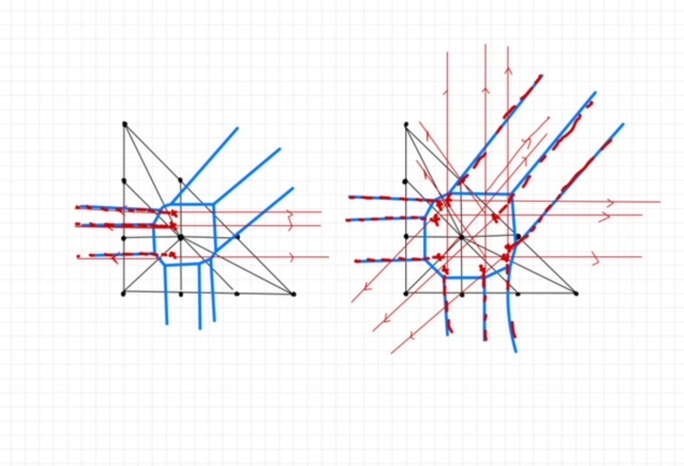
The equations of the mirror $\mathcal{X} \to \Delta$ can be obtained algorithmically from a scattering diagram with initial rays coming out of the 7-branes.

- Scattering diagram in mirror symmetry: Kontsevich–Soibelman, Gross–Siebert, Gross–Hacking–Keel, ...
- Physics interpretation: tropicalizations of worldsheet instantons, holomorphic discs in U coming out of the singular fibers.
- Algebro-geometrically: punctured Gromov–Witten invariants [Abramovich–Chen–Gross–Siebert]. See Mark Gross' talk tomorrow.



Explicit equations for the mirror?

- If all 5-branes ending on 7-branes are parallel,
  - Walls are parallel, no scattering.
  - Recovers the explicit results of [Bourget, Collinuci, Schafer-Nameki, 2023].
- In general, scattering can be arbitrarily complicated.



Worldsheet instantons as $\mathcal{N} = 2$ 4d BPS states.

- Consider the 4d $\mathcal{N} = 2$ theory on the worldvolume of a D3-brane probing the 7-branes.
  - Rank 1 theory (possibly not UV complete) with Coulomb branch the base B of the torus fibration on U.
  - Equivalently: worldvolume of an M5-brane wrapping around a torus fiber of $U \to B$.
- The previous worldsheet instantons can be viewed as BPS states of this 4d $\mathcal{N} = 2$ theory:
  - String junctions between the D3-brane and the 7-branes.
  - M2-brane in U with boundary on a torus fiber of $U \to B$.
- Scattering diagram = Kontsevich–Soibelman wall-crossing formula for BPS states of this 4d $\mathcal{N} = 2$ theory.

BPS quivers

- M-theory on $\mathcal{X}$: 5d SCFT.
  - Compactify on S^1: $\mathcal{N} = 2$ 4d theory at low energy.
  - BPS spectrum, derived category $D^b(\mathcal{X})$.

Conjecture (AAB)

Let $\mathcal{X}$ be a crepant resolution of a canonical 3-fold singularity $\overline{\mathcal{X}}$ that is M-theory dual to a web of 5-branes with 7-branes. Then, there exists a quiver with potential (Q, W) such that

$$D^b(\mathcal{X}) \simeq D^b(Q, W)\,.$$

- True in the toric case (dimer model)
- True when all the pushed 7-branes are contained in a common face of the web of 5-branes: $D^b(\mathcal{X}) = D^b(K_S)$, S: orbifold del Pezzo surface.

Thank you for your attention !