Winter School in Mathematical Physics
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Quantum chaos, random matrices and low-dimensional quantum gravity

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1 Lyapunov Exponent of the Logistic Map

Consider the logistic map

$$x_{n+1} = r x_n (1 - x_n) \ . \tag{1.1}$$

(a) Study the fixed points x^* of the evolution. Starting from $x_{n+1} = f(x_n)$, these can be found solving $x^* = f(x^*)$. Show that the only fixed point for $r < 1$ is $x^* = 0$, while for $r > 1$ the fixed point is $x^* = 1 - (1/r)$. Show that this fixed point is stable only if $r < 3$. What happens then?

For $r > 3.57\ldots$, the system becomes chaotic. Computing its Lyapunov exponent is generally hard, but it can be done for the special case $r = 4$. Under this assumption, consider the change of variables

$$x_n = \sin^2(\pi y_n) \ , \tag{1.2}$$

with the constraint $0 < y_n \leq 1$.

(b) What is the Logistic map in the y_n coordinates? In the following we will choose the branch

$$y_{n+1} = \begin{cases} 2y_n, & 0 < y_n \leq 1/2 \\ 2y_n - 1, & 1/2 < y_n \leq 1 \end{cases} \tag{1.3}$$

In the y_n coordinates, the logistic map can simply be rewritten as

$$\sin^2(\pi y_{n+1}) = \sin^2(2\pi y_n) \tag{1.4}$$

The positive branch of equation (1.4) gives

$$\pi y_{n+1} = 2\pi y_n \qquad \text{or} \qquad \pi y_{n+1} = \pi - 2\pi y_n \ , \tag{1.5}$$

while the negative branch gives

$$\pi y_{n+1} = 2\pi - 2\pi y_n \qquad \text{or} \qquad \pi y_{n+1} = 2\pi y_n - \pi \tag{1.6}$$

The domains of definition are chosen so that both y_n and y_{n+1} are defined on $(0, 1]$. Whatever brach we choose, the underlying dynamics will be chaotic. However, since the original logistic map was concave, it makes sense to choose the branch

$$y_{n+1} = \begin{cases} 2y_n, & 0 < y_n \leq 1/2 \\ 2y_n - 1, & 1/2 < y_n \leq 1 \end{cases} \tag{1.7}$$

(c) As you have seen in the lecture, for any dynamical map $y_{n+1} = f(y_n)$ the Lyapunov exponent can be written as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left|f'(y_i)\right| \ . \tag{1.8}$$

Trivially show that for $r = 4$, the Lyapunov exponent of (1.3) is $\lambda = \log(2)$.
No matter the value of y_n, the absolute value of the derivative of f is always 2, and therefore the Lyapunov exponent is $\lambda = \log(2)$.

Let's try to visualise this result. Write a $y_n \in \mathbb{R}$ in base 2.

(d) Show that the Logistic map (1.3) acts as $y_{n+1} = 2y_n (\mathrm{mod}\, 2)$. This means that, in base two, it shifts the whole digit expression to the left, such that $0.abcd\ldots$ becomes $0.bcd\ldots$, with $a, b, c, d, \dots \in \{0, 1\}$.
Clearly, if $y_n < 1/2$, it can be written as $y_n = 0.0bcd$, so that $y_{n+1} = 2y_n = 0.bcd$. On the other hand, if $y_n \leq 1/2$, it can be written as $y_n = 0.1bcd$, so that $y_{n+1} = 2y_n - 1 = 0.bcd$. Therefore, the new logistic map shifts the number written in base two to the left, canceling the first digit.

(e) Can you convince yourself that this map is chaotic? Why is the Lyapunov exponent $\log(2)$?
Assume that you start with two different initial conditions, that differ by δy_0, a very small number. Assume for concreteness that $\delta y_0 \approx 2^{-N}$, so that in the base-2 digit expression it has $N - 1$ zeros and then a one. This means that for $n < N$, the evolution of the two starting points will differ by $\delta y_n = 2^n \delta y_0 = e^{\log(2)n} \delta y_0$. Of course, for $n \geq N$, this will not be true anymore, but in the limit $\delta y_0 \to 0$ this pattern will continue arbitrarily long. Thus, the Lyapunov exponent of this map is $\lambda = \log(2)$.

(f) What are the periodic orbits (*i.e.* a set of points connected by the action of the map, with a finite period) of this map? Is it an infinite set? Is it dense in $(0,1]$?
Rational numbers are periodic orbits. They are a measure-zero, dense set in $(0,1]$.

2 Random Matrices and Spectral Statistics

Consider a Gaussian matrix model made of real Hermitian (*i.e.* symmetric) matrices of dimension 2, so that

$$P(H)\,\mathrm{d}H = \frac{\sqrt{2}}{(2\pi)^{\frac{3}{2}}}\, e^{-\frac{1}{2}\operatorname{Tr}[H^2]}\,\mathrm{d}H\ , \qquad \text{with} \qquad \mathrm{d}H = \prod_{i\le j} \mathrm{d}H_{ij}\ . \tag{2.1}$$

(a) What is the generic form of H? What are its eigenvalues $\lambda_\pm$?
The generic form of the matrix is

$$H = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \tag{2.2}$$

and its eigenvalues are

$$\lambda_\pm = \frac{x_1+x_2}{2} \pm \sqrt{\left(\frac{x_1-x_2}{2}\right)^2 + x_3^2}\ . \tag{2.3}$$

(b) Compute the probability distribution of the difference between the eigenvalues. To do so, find $\Delta\lambda = \lambda_+ - \lambda_-$ in terms of the generic parameters of H, and then compute

$$P(\omega) = \int \mathrm{d}H\, P(H)\, \delta(\omega - \Delta\lambda)\ . \tag{2.4}$$

Clearly

$$\Delta\lambda_\pm = \sqrt{(x_1-x_2)^2 + 4x_3^2}\ , \tag{2.5}$$

and thus

$$P(\omega) = \int \mathrm{d}^3x\, \frac{\sqrt{2}}{(2\pi)^{\frac{3}{2}}}\, e^{-\frac{x_1^2+x_2^2}{2} - x_3^2}\, \delta\left(\omega - \sqrt{(x_1-x_2)^2 + 4x_3^2}\right) \tag{2.6}$$

Let's do this tedious computation. We first integrate over x_3 to get rid of the delta, which localises to

$$x_3 = \pm\frac{1}{2}\sqrt{\omega^2 - (x_1-x_2)^2}\ . \tag{2.7}$$

The integral above is symmetric under $x_3 \to -x_3$, so we choose the positive branch and multiply by two at the end. Then, calling $f(x_3) = \omega - \Delta\lambda$, we can transform the delta into

$$\delta\left(\omega - \sqrt{(x_1 - x_2)^2 + 4x_3^2}\right) = \frac{\delta\left(x_3 - \frac{1}{2}\sqrt{\omega^2 - (x_1 - x_2)^2}\right)}{|f'(x_3)|} , \tag{2.8}$$

where the above equation is valid in a distributional sense. Putting everything together, we arrive at

$$P(\omega) = \int_{\Omega_\omega} \mathrm{d}^2x \, \frac{\sqrt{2}}{(2\pi)^{\frac{3}{2}}} \, e^{-\frac{(x_1+x_2)^2+\omega^2}{4}} \frac{\omega}{\sqrt{\omega^2 - (x_1 - x_2)^2}} , \tag{2.9}$$

where the two-dimensional integration contour is

$$\Omega_\omega = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid \omega^2 - (x_1 - x_2)^2 > 0\right\} . \tag{2.10}$$

The above integral can be performed with the change of variables

$$\xi = \frac{x_1 + x_2}{2} , \qquad \chi = x_1 - x_2 , \tag{2.11}$$

where then

$$P(\omega) = \frac{\sqrt{2}}{(2\pi)^{\frac{3}{2}}} \omega \, e^{-\frac{\omega^2}{4}} \int_{-\infty}^{+\infty} \mathrm{d}\xi \, e^{-\xi^2} \int_{-\omega}^{\omega} \frac{\mathrm{d}\chi}{\sqrt{\omega^2 - \chi^2}} = \frac{\omega}{2} \, e^{-\frac{\omega^2}{4}} , \tag{2.12}$$

(c) Compute the mean level spacing

$$\Delta = \int_0^\infty \mathrm{d}\omega \, \omega P(\omega) . \tag{2.13}$$

Find $P(s)$, with $s = \omega/\Delta$. This distribution is normally called Wigner's surmise (for the GOE random matrix class).

A straightforward computation gives $\Delta = \sqrt{\pi}$. Normally we write the distribution in the coordinate $s = \omega/\Delta$, so that

$$P(s) = \frac{\pi s}{2} \, e^{-\frac{\pi s^2}{4}} \qquad \text{(GOE)} . \tag{2.14}$$

It is straightforward to check that this distribution is normalised, and the first moment is also one. This is commonly called *Wigner' surmise*. This is the statistics for matrices which have also a time-reversal symmetry. For generic hermitian matrices it is instead

$$P(s) = \frac{32 s^2}{\pi^2} e^{-\frac{4s^2}{\pi}} \qquad \text{(GUE)} . \tag{2.15}$$

Instead, for hermitian-quaternion matrices we have

$$P(s) = \frac{2^{18} s^4}{3^6 \pi^3} e^{-\frac{64 s^2}{9\pi}} \qquad \text{(GSE)} \tag{2.16}$$

We now want to look at the distribution of the eigenvalues, for large matrices. We consider then $N \times N$ hermitian matrices, distributed as

$$P(H)\mathrm{d}H = e^{-\frac{N}{2g^2}\,\mathrm{Tr}\left[H^2\right]} \ . \tag{2.17}$$

Here, the parameter g^2 sets the energy units, while the factor of N ensures that the large-N limit is well defined. One then defines, in analogy with statistical physics, the partition function

$$\mathcal{Z} = \int \mathrm{d}H e^{-V(H)} = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{k<l}^{N} |\lambda_k - \lambda_l|^2 \, e^{-\frac{N}{2g^2}\sum_i \lambda_i^2} \ . \tag{2.18}$$

Let's unpack this expression. on the RHS of the first equality, we have the integral of the probability distribution, as we would in statistical mechanics. Then, this integral is written in terms of the eigenvalues of the matrix H. Notice the interesting measure, which is called *Vandermonde determinant*[1]. We can rewrite this partition function into the form

$$\mathcal{Z} = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, e^{-N^2 S_{\text{eff}}} \tag{2.19}$$

where

$$S_{\text{eff}} = \frac{1}{2Ng^2}\sum_{i=1}^{N} \lambda_i^2 - \frac{2}{N^2}\sum_{i<j} \log|\lambda_i - \lambda_j| \ . \tag{2.20}$$

Notice that S_{eff} is $\mathcal{O}(1)$ in the large-N limit, justifying our previous choice of (2.17). Moreover, sending $N \to \infty$, the partition function localises on the saddle.

(d) Show that the saddle-point equations for the potential above are

$$\frac{1}{Ng^2}\lambda_i = \frac{2}{N^2}\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} \ . \tag{2.21}$$

Equation (2.21) is derived simply setting

$$\frac{\mathrm{d}S_{\text{eff}}}{\mathrm{d}\lambda_i} = 0 \ . \tag{2.22}$$

(e) The density of eigenvalues is defined as

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \delta(\lambda - \lambda_i) \ , \qquad \text{so that} \qquad \frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) = \int \mathrm{d}\lambda \, \rho(\lambda) f(\lambda) \ . \tag{2.23}$$

[1] Unfortunately we won't have time to discuss its derivation in details, but if you are interested you can read about it in many textbooks on Random Matrix Theory.

We also define the resolvent as

$$\omega_0(p) = \int \mathrm{d}\lambda \, \frac{\rho(\lambda)}{p-\lambda} \ . \tag{2.24}$$

Show that

$$\rho(\lambda) = \lim_{\varepsilon\to 0^+} -\frac{1}{2\pi i}\Big(\omega_0(\lambda+i\varepsilon)-\omega_0(\lambda-i\varepsilon)\Big) \ . \tag{2.25}$$

The last equation can be found using the distributional identity

$$\lim_{\varepsilon\to 0^+} \frac{1}{x\pm i\varepsilon} = \mathcal{P}\,\frac{1}{x} \mp i\pi\delta(x) \ . \tag{2.26}$$

Using this fact in the definition of ω_0 gives equation (2.25).

(f) Multiply (2.21) by $1/(\lambda_i - p)$, and sum over the index i. Show that the resulting equation can be written as

$$\omega_0^2(p) - \frac{p\,\omega_0(p)}{g^2} + \frac{1}{g^2} = 0 \ . \tag{2.27}$$

Solving the quadratic equation for $\omega_0(p)$, show that

$$\rho(\lambda) = \frac{1}{\pi g}\sqrt{1-\frac{\lambda^2}{4g^2}} \ . \tag{2.28}$$

This is the famous *Wigner' semicircle law.*
The LHS of (2.21) becomes

$$\frac{1}{Ng^2}\sum_{i=1}^{N}\frac{\lambda_i}{\lambda_i-p} = \frac{1}{Ng^2}\sum_{i=1}^{N}\frac{\lambda_i-p}{\lambda_i-p} + \frac{1}{Ng^2}\sum_{i=1}^{N}\frac{p}{\lambda_i-p} = \frac{1}{g^2} - \frac{p\,\omega_0(p)}{g^2} \ . \tag{2.29}$$

The RHS becomes instead

$$\begin{aligned}\text{RHS} = \frac{2}{N^2}\sum_{i\neq j}\frac{1}{(\lambda_i-p)(\lambda_i-\lambda_j)} &= \frac{2}{N^2}\sum_{i\neq j}\frac{1}{p-\lambda_j}\left(\frac{1}{\lambda_i-p}-\frac{1}{\lambda_i-\lambda_j}\right) \\ &= -2\omega_0^2(p) - \text{RHS} \ .\end{aligned} \tag{2.30}$$

This implies that RHS $= -\omega_0^2(p)$, and combining it with the LHS gives (2.27). The solution of the quadratic is

$$\omega_0(p) = \frac{p}{2g^2} + \frac{1}{g}\sqrt{\frac{p^2}{4g^2}-1} \ . \tag{2.31}$$

Looking at the discontinuity across the branch cut of the square root we get

$$\rho(\lambda) = -\frac{2}{2\pi i g}\sqrt{\frac{\lambda^2}{4g^2}-1} = \frac{1}{\pi g}\sqrt{1-\frac{\lambda^2}{4g^2}} \tag{2.32}$$

3 Anti-de Sitter Space

Anti-de Sitter space is the maximally symmetric solution of the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \ , \tag{3.1}$$

with negative cosmological constant ($\Lambda < 0$). They can be obtained as the equations of motions of the Einstein-Hilbert action

$$S = \frac{1}{16\pi G_N}\int \mathrm{d}^{d+1}x\sqrt{g}\Big(R - 2\Lambda\Big) \ . \tag{3.2}$$

A very nice parametrisation of Λ is

$$\Lambda = -\frac{d(d-1)}{L^2} \ , \tag{3.3}$$

and in this convention the solution of (3.1) can be embedded into a flat space of dimension $(d+2)$ with signature $(-,+,\dots,+)$ through the hyperboloid

$$-X_0^2 + \sum_{i=1}^{d+1} X_i^2 = -L^2 \ , \tag{3.4}$$

equipped with a Minkowski metric

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 + \sum_{i=1}^{d+1}\mathrm{d}X_i^2 \tag{3.5}$$

In the following, we conveniently assume $L = 1$ and we will only look at the case of AdS_3.

(a) Consider

$$X_0 = \sqrt{1+r^2}\cosh(\tau) \ , \qquad X_1 = r\cos(\theta) \ , \tag{3.6}$$

$$X_3 = \sqrt{1+r^2}\sinh(\tau) \ , \qquad X_2 = r\sin(\theta) \ . \tag{3.7}$$

Show that it is a parametrisation of the surface (3.4), and that the induced metric is

$$\mathrm{d}s^2 = (1+r^2)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{1+r^2} + r^2\mathrm{d}\theta^2 \ . \tag{3.8}$$

These are normally called *global coordinates*[2].

This is a simple but tedious check. It can be also done with softwares like *Mathematica*.

[2] In the lecture, Julian used the parameterization $\mathrm{d}s^2 = \cosh^2(\rho)\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2(\rho)\mathrm{d}\theta^2$. The two can be connected via $r = \sinh(\rho)$.

(b) Consider

$$X_0 = \frac{1+\tau^2+y^2}{2y}\cosh(\rho)\,, \qquad X_1 = \sinh(\rho)\,, \tag{3.9}$$

$$X_2 = \frac{1-\tau^2-y^2}{2y}\cosh(\rho)\,, \qquad X_3 = \frac{\tau}{y}\sinh(\rho)\,. \tag{3.10}$$

Show that it is a parametrisation of the surface (3.4), and that the induced metric is

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \cosh^2(\rho)\left(\frac{\mathrm{d}\tau^2+\mathrm{d}y^2}{y^2}\right)\,. \tag{3.11}$$

This parametrisation is also written through the coordinates

$$z = y\,\mathrm{sech}(\rho)\,, \qquad x = y\tanh(\rho)\,, \tag{3.12}$$

and the associated metric is

$$\mathrm{d}s^2 = \frac{\mathrm{d}\tau^2+\mathrm{d}z^2+\mathrm{d}x^2}{z^2}\,, \tag{3.13}$$

which is normally called the *Poincaré patch* or *Poincaré slicing* of AdS_3.
Same answer as point (a).

(c) Let's focus on the Poincaré slicing (3.13), and let's find geodesics. We want to minimise the lagrangian

$$\mathcal{L} = \int \mathrm{d}s = \int \mathrm{d}\lambda\, \frac{\sqrt{\dot\tau^2(\lambda)+\dot z^2(\lambda)+\dot x^2(\lambda)}}{z(\lambda)}\,. \tag{3.14}$$

There is a rotational symmetry in the (τ, x) plane, so that we can just look at geodesics at fixed τ. Try to compute the Euler-Lagrange equations of motion, and show that the geodesic the connects the points $x = \sigma_{1,2}$ on the boundary $(z = 0)$ is

$$\left(x - \frac{\sigma_1+\sigma_2}{2}\right)^2 + z^2 = \left(\frac{\sigma_1-\sigma_2}{2}\right)^2 \tag{3.15}$$

The system is also translationally symmetric with respect to x, so we can also choose $\sigma_1 = -\sigma_2 \equiv \sigma$. In this case, the solution we want to check is

$$x^2 + z^2 = \sigma^2\,. \tag{3.16}$$

A good parametrisation is then

$$x(\lambda) = x\,, \qquad \text{and} \qquad z(\lambda) = z(\lambda(x))\,. \tag{3.17}$$

Therefore, the Lagrangian becomes

$$\mathcal{L} = \int \mathrm{d}x\, \frac{\sqrt{\dot z^2(x)+1}}{z(x)}\,, \tag{3.18}$$

and the equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\dot{z}}{z\sqrt{\dot{z}^2+1}}\right)+\frac{\sqrt{\dot{z}^2+1}}{z^2}=0\ . \tag{3.19}$$

It is a tedious but simple computation to check that

$$z=\sqrt{\sigma^2-x^2} \tag{3.20}$$

is a solution of the differential equation, for any σ.

(d) The length of this geodesic is strictly speaking divergent. To regularise it, we introduce a cutoff at $z=\varepsilon$. Show that the length of the regularised geodesics is

$$\ell=2\log\left(\frac{|\sigma_1-\sigma_2|}{\varepsilon}\right)\ . \tag{3.21}$$

If we introduce a cutoff at $z=\varepsilon$, the domain of x is in the interval $\pm\sigma\mp\varepsilon^2/2\sigma$ (up to first non-trivial order in ε). Moreover,

$$\frac{\sqrt{\dot{z}^2(x)+1}}{z(x)}=\frac{\sigma}{(\sigma^2-x^2)}\ . \tag{3.22}$$

All in all, the length integral is

$$\ell=2\int_0^{\sigma-\frac{\varepsilon^2}{2\sigma}}\frac{\sigma\,\mathrm{d}x}{(\sigma^2-x^2)}\ , \tag{3.23}$$

where we have used that the integrand is even to start from $x=0$, adding the two in front. This integral is elementary and the result is

$$\ell=\log\left(\frac{\sigma+x}{\sigma-x}\right)\Bigg|_0^{\sigma-\frac{\varepsilon^2}{2\sigma}}=2\log\left(\frac{2\sigma}{\varepsilon}\right)\ . \tag{3.24}$$

In getting this result we have also neglected a subleading ε^2. Because of translation invariance, this can also be written as

$$\ell=2\log\left(\frac{|\sigma_1-\sigma_2|}{\varepsilon}\right)\ . \tag{3.25}$$

(e) Suppose that $\sigma_{1,2}$ are the locations of the insertions of two operators. We can compute the correlation function between them in the *geodesic approximation* through

$$\langle\mathcal{O}(\sigma_1)\mathcal{O}(\sigma_2)\rangle\sim e^{-m\ell}\ , \tag{3.26}$$

where m is the mass of the dual field in the bulk. What result do you get from (3.26)? Is it the one you would expect from a CFT? What is the dictionary between

the mass m of the dual field and the scaling dimension Δ of the operator $\mathcal{O}$, in this approximation?
Using the geodesic approximation, we get

$$\langle \mathcal{O}(\sigma_1)\mathcal{O}(\sigma_2)\rangle \sim \frac{\varepsilon^{2m}}{|\sigma_1-\sigma_2|^{2m}} \ . \tag{3.27}$$

For scalar operators in a CFT we would get

$$\langle \mathcal{O}(\sigma_1)\mathcal{O}(\sigma_2)\rangle = \frac{1}{|\sigma_1-\sigma_2|^{2\Delta}} \ , \tag{3.28}$$

which instructs us that, in order to obtain CFT correlators from bulk geodesics, we need to compensate the divergence, and the dictionary is $m = \Delta$. This is actually just an approximate result, since the correct result (not in the geodesic approximation) is $m^2 = \Delta(\Delta - d)$.

Bonus : For later purposes, and for your own knowledge, you can also have fun showing that

$$X_0 = \frac{r}{r_+}\cosh(r_+\theta) \ , \qquad X_1 = \sqrt{\frac{r^2}{r_+^2}-1}\,\cos(r_+\tau) \ , \tag{3.29}$$

$$X_2 = \frac{r}{r_+}\sinh(r_+\theta) \ , \qquad X_3 = \sqrt{\frac{r^2}{r_+^2}-1}\,\sin(r_+\tau) \ , \tag{3.30}$$

which gives the (non-rotating) BTZ black hole metric, namely

$$\mathrm{d}s^2 = (r^2-r_+^2)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{r^2-r_+^2} + r^2\mathrm{d}\theta^2 \ . \tag{3.31}$$

In turn, this means that the BTZ black hole is locally equivalent to AdS_3. This is funny, isn't it?

4 Maldacena-Maoz Wormholes

The Maldacena-Maoz construction of two-boundary Wormholes in AdS_{d+1} considers generic metrics of the form

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \cosh^2(\rho)\,\mathrm{d}\Sigma_d^2 \ , \qquad \text{with} \qquad \rho \in (-\infty, +\infty) \ . \tag{4.1}$$

The geometry (4.1) has a constant negative curvature if and only if $\mathrm{d}\Sigma_d^2$ has a constant negative curvature. We call two-boundary wormhole a geometry with the topology $[0,1]\times\Sigma_d$. Not all metrics of the form (4.1) have this topology.

(a) Let's focus on three-dimensional wormholes, and consider

$$\mathrm{d}\Sigma_2^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z}}{(\mathrm{Im}\, z)^2}\ , \qquad \text{with} \qquad \mathrm{Im}\, z > 0\ , \tag{4.2}$$

namely $\Sigma_2 = \mathbb{H}_2$, the hyperbolic upper-half plane. Do you remember that this is just a different parametrisation of the Poincaré patch of AdS_3?
This is the same metric as (3.11).

The idea of the construction is to use the simplicity of the hyperbolic upper-half plane to find geometries with topology $[0,1]\times\Sigma_2$. In order to do that, we quotient the upper-half plane $\mathbb{H}_2$ with a group Γ, such that $\Sigma_2 = \mathbb{H}_2/\Gamma$. We take Γ to be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, namely the group of matrices γ with $\det(\gamma) = 1$. Such matrices γ act on z, the complex coordinate of $\mathbb{H}_2$ as

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ , \qquad \text{such that} \qquad \gamma z \equiv \frac{az+b}{cz+d}\ . \tag{4.3}$$

(b) A well known example is

$$\gamma = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\ , \qquad \text{with} \qquad q \in \mathbb{R}_+\ . \tag{4.4}$$

What is $\mathbb{H}_2/\Gamma$ in this case? Looking at the various parametrisation of Exercise 3, can you see that this construction is the BTZ black hole?
The matrix γ acts on z as

$$\gamma z = qz\ , \tag{4.5}$$

thus rescales its absolute value. Therefore, $\mathbb{H}_2/\Gamma$ can be described by the points $1 < |z| < q$ (assuming WLOG $q > 1$). On the boundary this has two periodicities, one in the argument of z and the other in its absolute value. Therefore, the boundary has the topology of a torus, and the corresponding geometry is a BTZ black hole. This can also be seen from the parametrisations of AdS_3 given in Exercise 3.

The idea of the last part of this exercise is to show that it is possible to construct wormholes such that Σ_2 is a two-dimensional sphere with three conical defects. These geometries give informations about the statistics of the OPE structure constants C_{ijk}. We focus on matrices with $|\,\mathrm{Tr}(\gamma)| < 2$, which are called *elliptic*.

(c) Consider

$$\gamma_1 = \begin{pmatrix} \cos(\phi_1) & -\sin(\phi_1) \\ \sin(\phi_1) & \cos(\phi_1) \end{pmatrix}\ . \tag{4.6}$$

Show that the fixed point of this map is $z = i$.
Simply

$$\gamma_1 i = \frac{i\cos(\phi_1) - \sin(\phi_1)}{i\sin(\phi_1) + \cos(\phi_1)} = i\frac{e^{i\phi_1}}{e^{i\phi_1}} = i\ . \tag{4.7}$$

(d) What is $\mathbb{H}_2/\Gamma_1$, with Γ_1 the discrete group generated by γ_1? You should find it generates a conical defect.

PSL$(2,\mathbb{R})$ generate conformal transformation that map the upper half plane into itself, and also circles into circles. To understand what is $\mathbb{H}_2/\Gamma$, it is useful to start from the locus of points $z = iy$, with $y \in \mathbb{R}_+$, and see where is it mapped to. This locus is a straight (half) line, and it is therefore mapped to a (half) circle. This half circle is the one that passes through the three points

$$\begin{aligned} z_1 &= i \ , \\ z_2 &= \frac{0\cdot\cos(\phi_1) - \sin(\phi_1)}{0\cdot\sin(\phi_1)+\cos(\phi_1)} = -\tan(\phi_1) \ , \\ z_3 &= \frac{\infty\cdot\cos(\phi_1) - \sin(\phi_1)}{\infty\cdot\sin(\phi_1)+\cos(\phi_1)} = \cot(\phi_1) \ . \end{aligned}$$

Moreover, it is not hard to show that the locus of points $z = iy$ and the circle just defined meet at the point $z = i$ with a relative angle of $2\phi_1$. This procedure can also be reiterated, and we can ask where is the circle we just defined mapped to under the action of γ_1. The answer is clearly to another circle, defined by the three numbers

$$\begin{aligned} w_1 &= i \ , \\ w_2 &= \frac{z_2\cos(\phi_1) - \sin(\phi_1)}{z_2\sin(\phi_1)+\cos(\phi_1)} = -\tan(2\phi_1) \ , \\ w_3 &= \frac{z_3\cos(\phi_1) - \sin(\phi_1)}{z_3\sin(\phi_1)+\cos(\phi_1)} = \cot(2\phi_1) \ . \end{aligned}$$

It is not hard to foresee that the same happens with $-\tan(n\phi_1) \mapsto -\tan((n+1)\phi_1)$ and $\cot(n\phi_1) \mapsto \cot((n+1)\phi_1)$, for $n \in \mathbb{N}$. Therefore, if we want Γ to be a (finite) group, we can demand $\phi_1 = \pi/m$. for $m \in \mathbb{N}_+$, and $\mathbb{H}_2/\Gamma$ has a conical defect at $z = i$ of opening $2\phi_1$.

(e) Consider

$$\gamma_2 = \begin{pmatrix} \cos(\phi_2) & e^{-\alpha}\sin(\phi_2) \\ -e^{\alpha}\sin(\phi_2) & \cos(\phi_2) \end{pmatrix} \ , \tag{4.8}$$

with $\alpha \in \mathbb{R}_+$. Show that the fixed point of this map is $z = e^{-\alpha} i$.

Simply

$$\gamma_2 e^{-\alpha} i = \frac{e^{-\alpha} i\cos(\phi_2) + e^{-\alpha}\sin(\phi_2)}{-i\sin(\phi_2)+\cos(\phi_2)} = e^{-\alpha} i \ . \tag{4.9}$$

The group Γ that gives a sphere with three defects is the discrete group generated by γ_1, γ_2 and $\gamma_3 = \gamma_2\gamma_1$. These quotients are also called Schwarz triangles[3].

[3]https://en.wikipedia.org/wiki/Schwarz_triangle

5 Universal Asymptotics for 2D CFTs

Let's consider a two-dimensional CFT on a circle of length 2π, with Hamiltonian

$$H = L_0 + \bar{L}_0 - \frac{c}{12} \ . \tag{5.1}$$

The two Virasoro generators commute with each other, so that the eigenvalues of the Hamiltonian are $|h, \bar{h}\rangle$, thus labeled by the eigenvalues of $L_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle$ and $\bar{L}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle$, with energy

$$E_{h,\bar{h}} = h + \bar{h} - \frac{c}{12} \ . \tag{5.2}$$

(a) A holographic CFT has a spectral gap between the vacuum and thermal states (black holes in AdS_3). For example, there are no eigenvalues between the vacuum and the first excited state which has $h = c/24$ (and similarly for $\bar{h}$). Show that, in the low temperature regime ($\beta \to \infty$), the thermal partition function is

$$\lim_{\beta\to\infty} Z(\beta) = e^{\beta c/12} \ . \tag{5.3}$$

Since there's a gap, we have

$$Z(\beta) = \sum_{i\geq 0} e^{-\beta E_i} = e^{-\beta E_0}\left(1 + \sum_{i>0} e^{-\beta(E_i - E_0)}\right) \to e^{-\beta E_0} = e^{c\beta/12} \ . \tag{5.4}$$

(b) Any thermal observable of the system is defined through a torus of the form $\mathbb{S}^1_{2\pi} \times \mathbb{S}^1_{\beta}$. Moreover, the conformal invariance of the system demands that observables should depend only on the parameter $\mathtt{t} = \beta/2\pi$. However, a different identification of the thermal and spatial circle implies invariance under the *modular transformation* $\mathtt{t} \to 1/\mathtt{t}$. Thus, show that the minimal modular completion of the partition function (5.3) is

$$Z(\beta) = e^{\beta c/12} + e^{\pi^2 c/3\beta} \ . \tag{5.5}$$

This is rather striking, as the low-energy states severely constrain the distribution of the high energy states.
The modular transformation of the term coming from the vacuum is

$$e^{\beta c/12} \to e^{\pi^2 c/3\beta} \ . \tag{5.6}$$

Therefore the minimal modular completion of the partition function (which is modular invariant is)

$$Z(\beta) = e^{\beta c/12} + e^{\pi^2 c/3\beta} \ . \tag{5.7}$$

Let's consider the high-energy spectrum, looking at the high-temperature limit $\beta \to 0$ and approximating the partition function to $Z(\beta) = e^{\pi^2 c/3\beta}$.

(c) Compute the thermal energy E_β and thermal entropy S_β of the system.
The thermal energy is

$$E_\beta = -\partial_\beta Z(\beta) = \frac{\pi^2 c}{3\beta^2} \ . \tag{5.8}$$

Through AdS/CFT, this can be shown to be the mass of a BTZ black hole. The thermal entropy instead is

$$S_\beta = \beta E_\beta + \log(Z(\beta)) = \frac{\pi^2 c}{3\beta} + \frac{\pi^2 c}{3\beta} = \frac{2\pi^2 c}{3\beta} \ . \tag{5.9}$$

(d) Consider the euclidean BTZ black hole,

$$\mathrm{d}s^2 = (r^2 - r_+^2)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{r^2 - r_+^2} + r^2 \mathrm{d}\theta^2 \ , \tag{5.10}$$

which is the geometry dual to the CFT thermal state.

- Look at the near-horizon ($r = r_+ + \varepsilon$) limit, and show that the geometry does not have a conical singularity unless τ is periodic with period $\beta = 2\pi/r_+$.
In the near-horizon coordinates, we have

$$\mathrm{d}s^2 = 2r_+\varepsilon \mathrm{d}\tau^2 + \frac{\mathrm{d}\varepsilon^2}{2r_+\varepsilon} + \dots \ . \tag{5.11}$$

In the equation above we have also omitted the angular part in $\mathrm{d}\theta^2$ since it will not play any role. Let's change coordinates into

$$\frac{\mathrm{d}\varepsilon}{\sqrt{2r_+\varepsilon}} = \mathrm{d}\eta \qquad \Rightarrow \qquad \varepsilon = \frac{r_+}{2}\eta^2 \ . \tag{5.12}$$

In this coordinates, the near-horizon geometry becomes

$$\mathrm{d}s^2 = r_+^2 \eta^2 \mathrm{d}\tau^2 + \mathrm{d}\eta^2 + \dots \ . \tag{5.13}$$

To avoid a conical singularity (since at the horizon the black hole geometry is regular), we need to have $r_+\tau \simeq r_+\tau + 2\pi$, which mean that τ is periodic with period $\beta = \frac{2\pi}{r_+}$. This is the the temperature of the black hole.

- With this information, show that

$$S_\beta = \frac{c\pi r_+}{3} = \frac{cA}{6} \ , \tag{5.14}$$

where A is the area of the BTZ black hole. Notice that this is the famous $S_\beta = A/4G_N$ Bekenstein-Hawking entropy, provided $c = 3/2G_N$ in AdS-length units.
Clearly then

$$S_\beta = \frac{2\pi^2 c}{3\beta} = \frac{2\pi r_+ c}{6} = \frac{cA}{6} \ . \tag{5.15}$$

(e) Taking the inverse Laplace transform of (5.5), it is possible to show that the high-energy density of states is given by the *Cardy formula*

$$\rho(E) \sim e^{2\pi\sqrt{\frac{c}{3}E}} \tag{5.16}$$

Do that, or show (in a saddle-point approximation) that with the ansatz (5.16), you get the high-temperature partition function.
Let's follow the second route which is easier. The thermal partition function is

$$Z(\beta) = \int_0^\infty \rho(E) e^{-\beta E} \mathrm{d}E = \int_0^\infty e^{2\pi\sqrt{\frac{c}{3}E} - \beta E} \mathrm{d}E \tag{5.17}$$

The saddle-point equation in the energy is

$$\pi\sqrt{\frac{c}{3E}} - \beta = 0 \qquad \text{solved by} \qquad E = \frac{\pi^2 c}{3\beta^2} \ . \tag{5.18}$$

Notice that this is also the thermal entropy E_β. Substituting this value we get

$$Z(\beta) \approx e^{\pi^2 c/3\beta} \ , \tag{5.19}$$

the same high-temperature partition function computed above.

We can actually do something similar to the OPE structure constants. Consider the four point function (expanded in different channels)

$$\begin{aligned} \langle \mathcal{O}(0)\mathcal{O}(x)\mathcal{O}(1)\mathcal{O}(\infty)\rangle &= \sum_{h_s,\bar{h}_s} |C_{\mathcal{O}\mathcal{O}\mathcal{O}_s}|^2 x^{h_s - 2h_\mathcal{O}} \bar{x}^{\bar{h}_s - 2\bar{h}_\mathcal{O}} \\ &= \sum_{h_t,\bar{h}_t} |C_{\mathcal{O}\mathcal{O}\mathcal{O}_t}|^2 (1-x)^{h_t - 2h_\mathcal{O}} (1-\bar{x})^{\bar{h}_t - 2\bar{h}_\mathcal{O}} \end{aligned} \tag{5.20}$$

The contribution of the identity in the t-channel ($h_t = 0$) has a pole at $x = \bar{x} = 1$, which has to be reproduced by the terms of the s-channel with h_s and $\bar{h}_s$ large. In particular, we can expand

$$\frac{1}{(1-x)^{2h_\mathcal{O}}} = \sum_{n=0}^{\infty} \binom{2h_\mathcal{O} + n - 1}{n} x^n \ . \tag{5.21}$$

Remembering also the fact that $C_{\mathcal{O}\mathcal{O}\mathbb{1}} = 1$, we have that the scaling of the square of the structure constants is

$$|C_{\mathcal{O}\mathcal{O}\mathcal{O}_s}|^2 \sim \binom{h_s - 1}{h_s - 2h_\mathcal{O}} \binom{\bar{h}_s - 1}{\bar{h}_s - 2\bar{h}_\mathcal{O}} \sim \frac{h_s^{2h_\mathcal{O}-1}}{\Gamma(2h_\mathcal{O})} \frac{\bar{h}_s^{2h_\mathcal{O}-1}}{\Gamma(2\bar{h}_\mathcal{O})} \ , \qquad \text{for} \qquad h_s, \bar{h}_s \to \infty \ . \tag{5.22}$$

This information is captured by the Maldacena-Maoz wormhole we studied in the previous exercise.