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Winter School in Mathematical Physics  
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Quantum chaos, random matrices and low-dimensional quantum gravity

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## 1 LYAPUNOV EXPONENT OF THE LOGISTIC MAP

Consider the logistic map

$$x_{n+1} = rx_n(1 - x_n) . \quad (1.1)$$

- (a) Study the fixed points  $x^*$  of the evolution. Starting from  $x_{n+1} = f(x_n)$ , these can be found solving  $x^* = f(x^*)$ . Show that the only fixed point for  $r < 1$  is  $x^* = 0$ , while for  $r > 1$  the fixed point is  $x^* = 1 - (1/r)$ . Show that this fixed point is stable only if  $r < 3$ . What happens then?

For  $r > 3.57\dots$ , the system becomes chaotic. Computing its Lyapunov exponent is generally hard, but it can be done for the special case  $r = 4$ . Under this assumption, consider the change of variables

$$x_n = \sin^2(\pi y_n) , \quad (1.2)$$

with the constraint  $0 < y_n \leq 1$ .

- (b) What is the Logistic map in the  $y_n$  coordinates? In the following we will choose the branch

$$y_{n+1} = \begin{cases} 2y_n, & 0 < y_n \leq 1/2 \\ 2y_n - 1, & 1/2 < y_n \leq 1 \end{cases} \quad (1.3)$$

- (c) As you have seen in the lecture, for any dynamical map  $y_{n+1} = f(y_n)$  the Lyapunov exponent can be written as

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(y_i)| . \quad (1.4)$$

Show that for  $r = 4$ , the Lyapunov exponent of (1.3) is  $\lambda = \log(2)$ .

Let's try to visualise this result. Write a  $y_n \in \mathbb{R}$  in base 2.

- (d) Show that the Logistic map (1.3) acts as  $y_{n+1} = 2y_n \pmod{2}$ . This means that, in base two, it shifts the whole digit expression to the left, such that  $0.abcd\dots$  becomes  $0.bcd\dots$ , with  $a, b, c, d, \dots \in \{0, 1\}$ .
- (e) Can you convince yourself that this map is chaotic? Why is the Lyapunov exponent  $\log(2)$ ?
- (f) What are the periodic orbits (*i.e.* a set of points connected by the action of the map, with a finite period) of this map? Is it an infinite set? Is it dense in  $(0, 1]$ ?

## 2 RANDOM MATRICES AND SPECTRAL STATISTICS

Consider a Gaussian matrix model made of real Hermitian (*i.e.* symmetric) matrices of dimension 2, so that

$$P(H) dH = \frac{\sqrt{2}}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2} \text{Tr}[H^2]} dH, \quad \text{with} \quad dH = \prod_{i \leq j} dH_{ij}. \quad (2.1)$$

- (a) What is the generic form of  $H$ ? What are its eigenvalues  $\lambda_{\pm}$ ?
- (b) Compute the probability distribution of the difference between the eigenvalues. To do so, find  $\Delta\lambda = \lambda_+ - \lambda_-$  in terms of the generic parameters of  $H$ , and then compute

$$P(\omega) = \int dH P(H) \delta(\omega - \Delta\lambda). \quad (2.2)$$

- (c) Compute the mean level spacing

$$\Delta = \int_0^{\infty} d\omega \omega P(\omega). \quad (2.3)$$

Find  $P(s)$ , with  $s = \omega/\Delta$ . This distribution is normally called Wigner's surmise (for the GOE random matrix class).

We now want to look at the distribution of the eigenvalues, for large matrices. We consider then  $N \times N$  hermitian matrices, distributed as

$$P(H) dH = e^{-\frac{N}{2g^2} \text{Tr}[H^2]}. \quad (2.4)$$

Here, the parameter  $g^2$  sets the energy units, while the factor of  $N$  ensures that the large- $N$  limit is well defined. One then defines, in analogy with statistical physics, the partition function

$$\mathcal{Z} = \int dH e^{-V(H)} = \int \prod_{i=1}^N d\lambda_i \prod_{k<l}^N |\lambda_k - \lambda_l|^2 e^{-\frac{N}{2g^2} \sum_i \lambda_i^2} . \quad (2.5)$$

Let's unpack this expression. on the RHS of the first equality, we have the integral of the probability distribution, as we would in statistical mechanics. Then, this integral is written in terms of the eigenvalues of the matrix  $H$ . Notice the interesting measure, which is called *Vandermonde determinant*<sup>1</sup>. We can rewrite this partition function into the form

$$\mathcal{Z} = \int \prod_{i=1}^N d\lambda_i e^{-N^2 S_{\text{eff}}} \quad (2.6)$$

where

$$S_{\text{eff}} = \frac{1}{2Ng^2} \sum_{i=1}^N \lambda_i^2 - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| . \quad (2.7)$$

Notice that  $S_{\text{eff}}$  is  $\mathcal{O}(1)$  in the large- $N$  limit, justifying our previous choice of (2.4). Moreover, sending  $N \rightarrow \infty$ , the partition function localises on the saddle.

(d) Show that the saddle-point equations for the potential above are

$$\frac{1}{Ng^2} \lambda_i = \frac{2}{N^2} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} . \quad (2.8)$$

(e) The density of eigenvalues is defined as

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i) , \quad \text{so that} \quad \frac{1}{N} \sum_{i=1}^N f(\lambda_i) = \int d\lambda \rho(\lambda) f(\lambda) . \quad (2.9)$$

We also define the resolvent as

$$\omega_0(p) = \int d\lambda \frac{\rho(\lambda)}{p - \lambda} . \quad (2.10)$$

Show that

$$\rho(\lambda) = \lim_{\varepsilon \rightarrow 0^+} -\frac{1}{2\pi i} \left( \omega_0(\lambda + i\varepsilon) - \omega_0(\lambda - i\varepsilon) \right) . \quad (2.11)$$

(f) Multiply (2.8) by  $1/(\lambda_i - p)$ , and sum over the index  $i$ . Show that the resulting equation can be written as

$$\omega_0^2(p) - \frac{p \omega_0(p)}{g^2} + \frac{1}{g^2} = 0 . \quad (2.12)$$

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<sup>1</sup>Unfortunately we won't have time to discuss its derivation in details, but if you are interested you can read about it in many textbooks on Random Matrix Theory.

Solving the quadratic equation for  $\omega_0(p)$ , show that

$$\rho(\lambda) = \frac{1}{\pi g} \sqrt{1 - \frac{\lambda^2}{4g^2}} . \quad (2.13)$$

This is the famous *Wigner' semicircle law*.

### 3 ANTI-DE SITTER SPACE

Anti-de Sitter space is the maximally symmetric solution of the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 , \quad (3.1)$$

with negative cosmological constant ( $\Lambda < 0$ ). They can be obtained as the equations of motions of the Einstein-Hilbert action

$$S = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{g} (R - 2\Lambda) . \quad (3.2)$$

A very nice parametrisation of  $\Lambda$  is

$$\Lambda = -\frac{d(d-1)}{L^2} , \quad (3.3)$$

and in this convention the solution of (3.1) can be embedded into a flat space of dimension  $(d+2)$  with signature  $(-, +, \dots, +)$  through the hyperboloid

$$-X_0^2 + \sum_{i=1}^{d+1} X_i^2 = -L^2 , \quad (3.4)$$

equipped with a Minkowski metric

$$ds^2 = -dX_0^2 + \sum_{i=1}^{d+1} dX_i^2 \quad (3.5)$$

In the following, we conveniently assume  $L = 1$  and we will only look at the case of  $\text{AdS}_3$ .

(a) Consider

$$X_0 = \sqrt{1+r^2} \cosh(\tau) , \quad X_1 = r \cos(\theta) , \quad (3.6)$$

$$X_3 = \sqrt{1+r^2} \sinh(\tau) , \quad X_2 = r \sin(\theta) . \quad (3.7)$$

Show that it is a parametrisation of the surface (3.4), and that the induced metric is

$$ds^2 = (1 + r^2)d\tau^2 + \frac{dr^2}{1 + r^2} + r^2d\theta^2 . \quad (3.8)$$

These are normally called *global coordinates*<sup>2</sup>.

(b) Consider

$$X_0 = \frac{1 + \tau^2 + y^2}{2y} \cosh(\rho) , \quad X_1 = \sinh(\rho) , \quad (3.9)$$

$$X_2 = \frac{1 - \tau^2 - y^2}{2y} \cosh(\rho) , \quad X_3 = \frac{\tau}{y} \sinh(\rho) . \quad (3.10)$$

Show that it is a parametrisation of the surface (3.4), and that the induced metric is

$$ds^2 = d\rho^2 + \cosh^2(\rho) \left( \frac{d\tau^2 + dy^2}{y^2} \right) . \quad (3.11)$$

This parametrisation is also written through the coordinates

$$z = y \operatorname{sech}(\rho) , \quad x = y \tanh(\rho) , \quad (3.12)$$

and the associated metric is

$$ds^2 = \frac{d\tau^2 + dz^2 + dx^2}{z^2} , \quad (3.13)$$

which is normally called the *Poincaré patch* or *Poincaré slicing* of  $\text{AdS}_3$ .

(c) Let's focus on the Poincaré slicing (3.13), and let's find geodesics. We want to minimise the lagrangian

$$\mathcal{L} = \int ds = \int d\lambda \frac{\sqrt{\dot{\tau}^2(\lambda) + \dot{z}^2(\lambda) + \dot{x}^2(\lambda)}}{z(\lambda)} . \quad (3.14)$$

There is a rotational symmetry in the  $(\tau, x)$  plane, so that we can just look at geodesics at fixed  $\tau$ . Try to compute the Euler-Lagrange equations of motion, and show that the geodesic that connects the points  $x = \sigma_{1,2}$  on the boundary ( $z = 0$ ) is

$$\left( x - \frac{\sigma_1 + \sigma_2}{2} \right)^2 + z^2 = \left( \frac{\sigma_1 - \sigma_2}{2} \right)^2 \quad (3.15)$$

(d) The length of this geodesic is strictly speaking divergent. To regularise it, we introduce a cutoff at  $z = \varepsilon$ . Show that the length of the regularised geodesics is

$$\ell = 2 \log \left( \frac{|\sigma_1 - \sigma_2|}{\varepsilon} \right) . \quad (3.16)$$

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<sup>2</sup>In the lecture, Julian used the parameterization  $ds^2 = \cosh^2(\rho)d\tau^2 + d\rho^2 + \sinh^2(\rho)d\theta^2$ . The two can be connected via  $r = \sinh(\rho)$ .

- (e) Suppose that  $\sigma_{1,2}$  are the locations of the insertions of two operators. We can compute the correlation function between them in the *geodesic approximation* through

$$\langle \mathcal{O}(\sigma_1) \mathcal{O}(\sigma_2) \rangle \sim e^{-m\ell} , \quad (3.17)$$

where  $m$  is the mass of the dual field in the bulk. What result do you get from (3.17)? Is it the one you would expect from a CFT? What is the dictionary between the mass  $m$  of the dual field and the scaling dimension  $\Delta$  of the operator  $\mathcal{O}$ , in this approximation?

*Bonus* : For later purposes, and for your own knowledge, you can also have fun showing that

$$X_0 = \frac{r}{r_+} \cosh(r_+\theta) , \quad X_1 = \sqrt{\frac{r^2}{r_+^2} - 1} \cos(r_+\tau) , \quad (3.18)$$

$$X_2 = \frac{r}{r_+} \sinh(r_+\theta) , \quad X_3 = \sqrt{\frac{r^2}{r_+^2} - 1} \sin(r_+\tau) , \quad (3.19)$$

which gives the (non-rotating) BTZ black hole metric, namely

$$ds^2 = (r^2 - r_+^2) d\tau^2 + \frac{dr^2}{r^2 - r_+^2} + r^2 d\theta^2 . \quad (3.20)$$

In turn, this means that the BTZ black hole is locally equivalent to  $\text{AdS}_3$ . This is funny, isn't it?

## 4 MALDACENA-MAOZ WORMHOLES

The Maldacena-Maoz construction of two-boundary Wormholes in  $\text{AdS}_{d+1}$  considers generic metrics of the form

$$ds^2 = d\rho^2 + \cosh^2(\rho) d\Sigma_d^2 , \quad \text{with} \quad \rho \in (-\infty, +\infty) . \quad (4.1)$$

The geometry (4.1) has a constant negative curvature if and only if  $d\Sigma_d^2$  has a constant negative curvature. We call two-boundary wormhole a geometry with the topology  $[0, 1] \times \Sigma_d$ . Not all metrics of the form (4.1) have this topology.

- (a) Let's focus on three-dimensional wormholes, and consider

$$d\Sigma_2^2 = \frac{dz d\bar{z}}{(\text{Im } z)^2} , \quad \text{with} \quad \text{Im } z > 0 , \quad (4.2)$$

namely  $\Sigma_2 = \mathbb{H}_2$ , the hyperbolic upper-half plane. Do you remember that this is just a different parametrisation of the Poincaré patch of  $\text{AdS}_3$ ?

The idea of the construction is to use the simplicity of the hyperbolic upper-half plane to find geometries with topology  $[0, 1] \times \Sigma_2$ . In order to do that, we quotient the upper-half plane  $\mathbb{H}_2$  with a group  $\Gamma$ , such that  $\Sigma_2 = \mathbb{H}_2/\Gamma$ . We take  $\Gamma$  to be a discrete subgroup of  $\text{PSL}(2, \mathbb{R})$ , namely the group of matrices  $\gamma$  with  $\det(\gamma) = 1$ . Such matrices  $\gamma$  act on  $z$ , the complex coordinate of  $\mathbb{H}_2$  as

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{such that} \quad \gamma z \equiv \frac{az + b}{cz + d}. \quad (4.3)$$

(b) A well known example is

$$\gamma = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad \text{with} \quad q \in \mathbb{R}_+. \quad (4.4)$$

What is  $\mathbb{H}_2/\Gamma$  in this case? Looking at the various parametrisation of Exercise 3, can you see that this construction is the BTZ black hole?

The idea of the last part of this exercise is to show that it is possible to construct wormholes such that  $\Sigma_2$  is a two-dimensional sphere with three conical defects. These geometries give informations about the statistics of the OPE structure constants  $C_{ijk}$ . We focus on matrices with  $|\text{Tr}(\gamma)| < 2$ , which are called *elliptic*.

(c) Consider

$$\gamma_1 = \begin{pmatrix} \cos(\phi_1) & -\sin(\phi_1) \\ \sin(\phi_1) & \cos(\phi_1) \end{pmatrix}. \quad (4.5)$$

Show that the fixed point of this map is  $z = i$ .

(d) What is  $\mathbb{H}_2/\Gamma_1$ , with  $\Gamma_1$  the discrete group generated by  $\gamma_1$ ? You should find it generates a conical defect.

(e) Consider

$$\gamma_2 = \begin{pmatrix} \cos(\phi_2) & e^{-\alpha} \sin(\phi_2) \\ -e^{\alpha} \sin(\phi_2) & \cos(\phi_2) \end{pmatrix}, \quad (4.6)$$

with  $\alpha \in \mathbb{R}_+$ . Show that the fixed point of this map is  $z = e^{-\alpha}i$ .

The group  $\Gamma$  that gives a sphere with three defects is the discrete group generated by  $\gamma_1, \gamma_2$  and  $\gamma_3 = \gamma_2\gamma_1$ . These quotients are also called Schwarz triangles<sup>3</sup>.

## 5 UNIVERSAL ASYMPTOTICS FOR 2D CFTs

Let's consider a two-dimensional CFT on a circle of length  $2\pi$ , with Hamiltonian

$$H = L_0 + \bar{L}_0 - \frac{c}{12}. \quad (5.1)$$

<sup>3</sup>[https://en.wikipedia.org/wiki/Schwarz\\_triangle](https://en.wikipedia.org/wiki/Schwarz_triangle)

The two Virasoro generators commute with each other, so that the eigenvalues of the Hamiltonian are  $|h, \bar{h}\rangle$ , thus labeled by the eigenvalues of  $L_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle$  and  $\bar{L}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle$ , with energy

$$E_{h, \bar{h}} = h + \bar{h} - \frac{c}{12} . \quad (5.2)$$

- (a) A holographic CFT has a spectral gap between the vacuum and thermal states (black holes in AdS<sub>3</sub>). For example, there are no eigenvalues between the vacuum and the first excited state which has  $h = c/24$  (and similarly for  $\bar{h}$ ). Show that, in the low temperature regime ( $\beta \rightarrow \infty$ ), the thermal partition function is

$$\lim_{\beta \rightarrow \infty} Z(\beta) = e^{\beta c/12} . \quad (5.3)$$

- (b) Any thermal observable of the system is defined through a torus of the form  $\mathbb{S}_{2\pi}^1 \times \mathbb{S}_\beta^1$ . Moreover, the conformal invariance of the system demands that observables should depend only on the parameter  $t = \beta/2\pi$ . However, a different identification of the thermal and spatial circle implies invariance under the *modular transformation*  $t \rightarrow 1/t$ . Thus, show that the minimal modular completion of the partition function (5.3) is

$$Z(\beta) = e^{\beta c/12} + e^{\pi^2 c/3\beta} . \quad (5.4)$$

This is rather striking, as the low-energy states severely constrain the distribution of the high energy states.

Let's consider the high-energy spectrum, looking at the high-temperature limit  $\beta \rightarrow 0$  and approximating the partition function to  $Z(\beta) = e^{\pi^2 c/3\beta}$ .

- (c) Compute the thermal energy  $E_\beta$  and thermal entropy  $S_\beta$  of the system.  
 (d) Consider the euclidean BTZ black hole,

$$ds^2 = (r^2 - r_+^2) d\tau^2 + \frac{dr^2}{r^2 - r_+^2} + r^2 d\theta^2 , \quad (5.5)$$

which is the geometry dual to the CFT thermal state.

- Look at the near-horizon ( $r = r_+ + \varepsilon$ ) limit, and show that the geometry does not have a conical singularity unless  $\tau$  is periodic with period  $\beta = 2\pi/r_+$ .
- With this information, show that

$$S_\beta = \frac{c\pi r_+}{3} = \frac{cA}{6} , \quad (5.6)$$

where  $A$  is the area of the BTZ black hole. Notice that this is the famous  $S_\beta = A/4G_N$  Bekenstein-Hawking entropy, provided  $c = 3/2G_N$  in AdS-length units.



- (e) Taking the inverse Laplace transform of (5.4), it is possible to show that the high-energy density of states is given by the *Cardy formula*

$$\rho(E) \sim e^{2\pi\sqrt{\frac{c}{3}E}} \quad (5.7)$$

Do that, or show (in a saddle-point approximation) that with the ansatz (5.7), you get the high-temperature partition function.

We can actually do something similar to the OPE structure constants. Consider the four point function (expanded in different channels)

$$\begin{aligned} \langle \mathcal{O}(0)\mathcal{O}(x)\mathcal{O}(1)\mathcal{O}(\infty) \rangle &= \sum_{h_s, \bar{h}_s} |C_{\mathcal{O}\mathcal{O}\mathcal{O}_s}|^2 x^{h_s-2h_{\mathcal{O}}} \bar{x}^{\bar{h}_s-2\bar{h}_{\mathcal{O}}} \\ &= \sum_{h_t, \bar{h}_t} |C_{\mathcal{O}\mathcal{O}\mathcal{O}_t}|^2 (1-x)^{h_t-2h_{\mathcal{O}}} (1-\bar{x})^{\bar{h}_t-2\bar{h}_{\mathcal{O}}} \end{aligned} \quad (5.8)$$

The contribution of the identity in the  $t$ -channel ( $h_t = 0$ ) has a pole at  $x = \bar{x} = 1$ , which has to be reproduced by the terms of the  $s$ -channel with  $h_s$  and  $\bar{h}_s$  large. In particular, we can expand

$$\frac{1}{(1-x)^{2h_{\mathcal{O}}}} = \sum_{n=0}^{\infty} \binom{2h_{\mathcal{O}}+n-1}{n} x^n. \quad (5.9)$$

Remembering also the fact that  $C_{\mathcal{O}\mathcal{O}\mathbb{1}} = 1$ , we have that the scaling of the square of the structure constants is

$$|C_{\mathcal{O}\mathcal{O}\mathcal{O}_s}|^2 \sim \binom{h_s-1}{h_s-2h_{\mathcal{O}}} \binom{\bar{h}_s-1}{\bar{h}_s-2\bar{h}_{\mathcal{O}}} \sim \frac{h_s^{2h_{\mathcal{O}}-1}}{\Gamma(2h_{\mathcal{O}})} \frac{\bar{h}_s^{2h_{\mathcal{O}}-1}}{\Gamma(2\bar{h}_{\mathcal{O}})}, \quad \text{for } h_s, \bar{h}_s \rightarrow \infty. \quad (5.10)$$

This information is captured by the Maldacena-Maoz wormhole we studied in the previous exercise.