Calabi-Yau threefolds, quivers, and quantum algebras

Andrei Neguț

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Chapter 1

Introduction

The goal of these lectures is to define and relate several remarkable algebras that arise from a toric Calabi-Yau threefold (abbreviated CY3) X. The origin of this story is in mathematical physics, where CY3's have long been studied in superstring theory (in particular the vast subject of mirror symmetry, see for instance [4, 5, 37]). I will not survey any of this physics today.

In the first lecture, we will present a connection

$$\Big(\text{toric CY3 } X\Big) \rightsquigarrow \Big(\text{toric diagram}\Big) \rightsquigarrow \\ \rightsquigarrow \Big(\text{brane tiling}\Big) \rightsquigarrow \Big(\text{quiver } Q \text{ drawn on } \mathbb{T}^2\Big) \quad (1.0.1)$$

where $\mathbb{T}^2$ denotes 2d real torus. We will also introduce a notion of **shrubby quivers**, which will turn out to be important in the next lecture.

In the second lecture, we will recall the construction of quiver Yangians ([24], an incarnation of the BPS algebra studied in [19]), or more precisely in the context at hand, quiver quantum toroidal algebras ([12, 13, 31, 32]) associated to Q:

$$\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}^+ \otimes \widetilde{\mathbf{U}}^0 \otimes \widetilde{\mathbf{U}}^-$$

This is an algebra that acts on vector spaces of BPS states for the CY3 X (we will discuss the realizations of BPS states in terms of crystal configurations Λ). For brevity, in these lectures we will only focus on the positive part of this action

$$\widetilde{\mathbf{U}}^+ \curvearrowright \Lambda \quad (1.0.2)$$

Moreover, the algebra $\widetilde{\mathbf{U}}^+$ is expected to be related to the so-called K-theoretic Hall algebra ([23, 35]) that geometric representation theory associates to the quiver Q and potential W of (2.4.2). In other words, one expects a homomorphism

$$\widetilde{\mathbf{U}}^+ \xrightarrow{\Xi} \text{K-HA}(Q, W)_{\text{loc}} \quad (1.0.3)$$

The remarkable thing is that both the action (1.0.2) and the homomorphism (1.0.3) have the same kernel I^+. Therefore, if we define the **reduced** algebra

$$\mathbf{U}^+ = \widetilde{\mathbf{U}}^+ \Big/ I^+$$

then the main result of these lecture notes can be summarized as the following connection between mathematical physics and representation theory.

Theorem 1.0.1. *([29]) We give explicit generators for the ideal I^+, hence an explicit generators-and-relations presentation of $\mathbf{U}^+$. There is a faithful action*

$$\mathbf{U}^+ \curvearrowright \Lambda \tag{1.0.4}$$

(see (3.3.5) *for the definition of Λ) and an isomorphism*

$$\mathbf{U}^+ \xrightarrow{\sim} \text{K-HA}(Q, W)_{\text{loc}} \Big/ (\text{torsion}) \tag{1.0.5}$$

As a by-product, we also explicitly describe the K-theoretic Hall algebra in the right-hand side of (1.0.5) *in terms of explicit **wheel conditions**.*

Perspectives:

1. By comparison with related cases, it is reasonable to expect that the so-called "torsion" above is trivial, so that it may be removed from formula (1.0.5).

2. To remove the localization (i.e. the word "loc") from formula (1.0.5), one must define an appropriate integral form of the algebra $\mathbf{U}^+$, and in the level of generality above it is a very interesting and challenging problem.

3. By analogy with the cohomological case ([23] and subsequent works, such as [21, 22]), one expects there to be a K-theoretic Hall algebra associated to any Calabi-Yau threefold X, which should match K-HA(Q, W) for the quiver Q from (1.0.1) and the potential W of (2.4.2).

Chapter 2

From CY3s to quivers

The main purpose of this lecture is to explain the arrows in (1.0.1), in which we start from a toric Calabi-Yau threefold X and we obtain a quiver Q drawn on the torus (with potential W given by (2.4.2)). Physically, the procedure is that the quiver encodes D-branes on X. Mathematically, the procedure is such that there exists an equivalence of derived categories

$$D^b(\text{Coh}(X)) \cong D^b(J(Q,W)\text{-mod}) \tag{2.0.1}$$

(see [20]). We will define the Jacobi algebra $J(Q,W)$ that features in the RHS in Definition 2.4.2. However, the construction that we will provide for the connection (1.0.1) is the one that originated in quiver gauge theory ([11, 16, 17, 18], see also [15]). Mathematicians have other constructions, which are more algebraic but less combinatorial (see for instance [14, Proposition 3.3.1] and [25]). While we will not even claim the equivalence of all these constructions, in practice they all give the same answers for many toric Calabi-Yau threefolds.

2.1 Toric Calabi-Yau threefolds

In what follows, the word "variety" means normal integral scheme over $\mathbb{C}$ (although you may replace it by "singular complex manifold" if you prefer).

Definition 2.1.1. *A threefold X is a variety of (complex) dimension 3. It is called **Calabi-Yau** if there exists an isomorphism*

$$\mathcal{O}_X \cong \mathcal{K}_X \tag{2.1.1}$$

where $\mathcal{K}_X$ is the line bundle of top (i.e. 3) dimensional differential forms.

While technically speaking, our definition of the right-hand side of (2.1.1) requires X to be smooth, one can get away with allowing X to have certain "mild" singularities (Gorenstein will be enough for these lectures, because in that case, one can define the line bundle $\mathcal{K}_X$ by extension from the smooth locus of X).

Definition 2.1.2. *A threefold X is called* ***toric*** *if it is acted on by the torus*

$$(\mathbb{C}^*)^3 \curvearrowright X \tag{2.1.2}$$

such that there exists a dense open subset $(\mathbb{C}^*)^3 \subset X$ on which $(\mathbb{C}^*)^3$ acts via

$$(t_1, t_2, t_3) \cdot (x_1, x_2, x_3) = (t_1 x_1, t_2 x_2, t_3 x_3) \tag{2.1.3}$$

for all $t_1, t_2, t_3, x_1, x_2, x_3 \in \mathbb{C}^*$.

Examples of toric Calabi-Yau threefolds include affine space $\mathbb{C}^3$ with the standard torus action from formula (2.1.3), the **conifold**

$$\{xy = zt\} \subset \mathbb{C}^4 \tag{2.1.4}$$

(which has a singularity at the origin). The resolution of the conifold is

$$\mathrm{Tot}_{\mathbb{P}^1}\Big(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\Big) \tag{2.1.5}$$

with the torus action induced from $\mathbb{C}^* \curvearrowright \mathbb{P}^1$ and dilating the two $\mathcal{O}_{\mathbb{P}^1}(-1)$. For a more complicated example, we have the **suspended pinch point** (SPP)

$$\{xy = zt^2\} \subset \mathbb{C}^4 \tag{2.1.6}$$

(which has singularities along a line) and its resolutions.

If X is a toric Calabi-Yau threefold, the action $(\mathbb{C}^*)^3 \curvearrowright X$ of (2.1.2) induces actions of $(\mathbb{C}^*)^3$ on the line bundles $\mathcal{O}_X$ and $\mathcal{K}_X$. The identification (2.1.1) does not match these two actions on the nose, but only up to a character $\chi : (\mathbb{C}^*)^3 \twoheadrightarrow \mathbb{C}^*$. The **Calabi-Yau torus** is defined as

$$T = \mathrm{Ker}\ \chi \tag{2.1.7}$$

Tautologically, it has the property that the isomorphism (2.1.1) is T-invariant. To see that the character χ is non-trivial (and so T is a proper, i.e. rank 2, subtorus of $(\mathbb{C}^*)^3$), it is enough to calculate it on the dense open subset (2.1.3). On this dense open, the fibers of $\mathcal{O}$ are spanned by the function 1, while the fibers of $\mathcal{K}$ are spanned by the 3-form $dx_1 \wedge dx_2 \wedge dx_3$. Thus, the character χ is simply the torus character which scales the aforementioned 3-form, so

$$\chi((t_1, t_2, t_3)) = t_1 t_2 t_3$$

Then $T = \{(t_1, t_2, t_3) | t_1 t_2 t_3 = 1\} = \{(u, v, u^{-1} v^{-1}) | u, v \in \mathbb{C}^*\}$ is indeed rank 2.

2.2 Toric diagrams

We will review the main constructions of toric geometry in the context of Calabi-Yau threefolds, and refer to [6] for further details and proofs.

Definition 2.2.1. *A **cone** is a subset of the form*

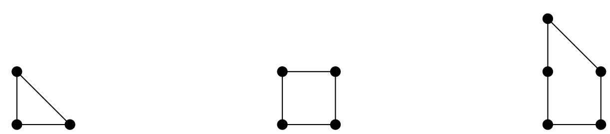
$$\left\{ a_1 v_1 + \cdots + a_n v_n \middle| a_1, \ldots, a_n \in \mathbb{R}_{\geq 0} \right\} \subset \mathbb{R}^3 \tag{2.2.1}$$

*(for various vectors $v_1, \ldots, v_n \in \mathbb{Z}^3$) which does not contain any pair of opposite vectors in $\mathbb{R}^3$. A **face** of a cone C is its intersection with any hyperplane H such that the C lies in one of the two closed half-spaces bounded by H. A **fan** is a collection of cones which is closed under taking faces, and such that the intersection of any two cones in the fan is a face of each of the two cones.*

The fundamental theorem of toric varieties states that there is a one-to-one correspondence between toric threefolds and fans as above (indeed, $\mathbb{R}^3$ in (2.2.1) is simply the real form of the Lie algebra of the torus $(\mathbb{C}^*)^3$ from (2.1.2)).

Proposition 2.2.2. *(see [6, Section 4.3]) A toric threefold X is Calabi-Yau if and only if there exists a plane $P \subset \mathbb{R}^3$ which contains the generating vectors $v \in \mathbb{Z}^3$ of all the one-dimensional cones in the fan corresponding to X.*

Moreover, we can choose a basis of the plane P so that the aforementioned generating vectors v intersect P in lattice points. The two-dimensional cones in the fan will intersect the plane P in a bunch of edges connecting the above lattice points. The collection of these lattice points and edges is referred to as the **toric diagram** of the toric Calabi-Yau threefold X.



Picture 2.2: The toric diagrams of $\mathbb{C}^3$, the conifold and the SPP.

One can read off the singularities of X from its toric diagram. In particular, X is smooth if and only if the edges give a triangulation of the toric diagram. Therefore, the toric diagram of the resolved conifold (2.1.5) is obtained by adding a diagonal to the middle diagram in Picture 2.2.

2.3 Brane tilings

The next step in our procedure arose in quiver gauge theory ([11, 16, 17, 18]). We refer to *loc. cit.* for details on this construction (dubbed the "fast inverse algorithm"), and just sketch it. Start from a toric diagram as in Section 2.2.

1. Draw an outward green arrow perpendicular to each edge of the toric diagram (this leads to the so-called (p, q)-web that is dual to the toric diagram).

2. Consider a square large enough to contain the toric diagram, and interpret it as the 2d torus $\mathbb{T}^2$ by identifying opposite edges. The green arrows in the previous item will partition $\mathbb{T}^2$ into polygonal tiles.

3. These polygonal tiles can be colored in red, white and blue such that no three green arrows intersect, and any intersection of two green arrows is surrounded by colors as indicated below (with respect to the standard orientation of $\mathbb{T}^2$).

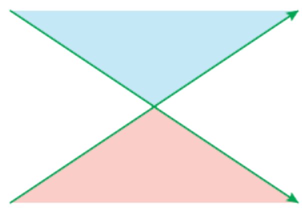


Figure 2.1: The allowed colors near an intersection in a brane tiling

We call the above partition and coloring of $\mathbb{T}^2$ a **brane tiling**. It is not true that any choice of green arrows gives rise to a partition and coloring compatible with Figure 2.1, but there exists a choice (precisely given by branes on X) which does.

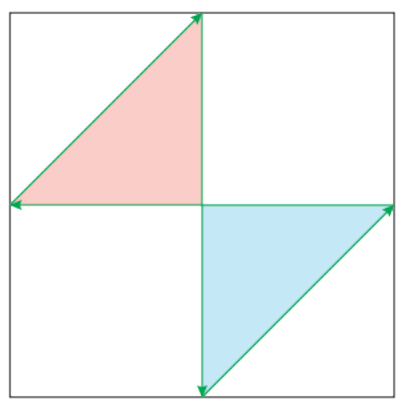


Figure 2.2: The brane tiling associated to $X = \mathbb{C}^3$.

Definition 2.3.1. *A **dimer model** (on the torus) is a bipartite graph drawn on $\mathbb{T}^2$, such that every edge has at least two neighbors, and removing all edges breaks up $\mathbb{T}^2$ into simply connected components.*

The brane tiling procedure leads to a dimer model by regarding the red/blue tiles as vertices, and drawing an edge between a red tile and a blue tile if and only if they are on opposite sides of a green intersection (as in Figure 2.1). The literature on dimer models is vast, see for instance [2, 3, 7] for works that are closer to the topic at hand.

2.4 Quivers with potential

In the previous Section, we associated a so-called brane tiling (on $\mathbb{T}^2$) to the toric diagram of X, and we showed how this leads to a dimer model. The dual of the dimer model is the **quiver** Q associated to X in (1.0.1). This quiver has:

1. A vertex per white region of the brane tiling.

2. An edge connecting any two white regions which are on opposite sides of a green intersection (as in Figure 2.1).

3. A red/blue face surrounding a red/blue polygonal tile of the brane tiling.

Very importantly, any two faces that share an edge must be of different colors. Because of the orientation in Figure 2.1, the edges in the quiver can be oriented to go clockwise around the blue faces and counterclockwise around the red faces.

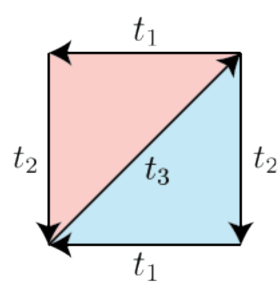


Figure 2.3: The quiver (drawn on $\mathbb{T}^2$) associated to $X = \mathbb{C}^3$.

Note that the quiver In Figure 2.3 has 1 vertex, 3 edges and 2 faces. This is due to Euler's formula, which says that because the quiver is drawn on a torus, the number of vertices plus the number of faces must equal the number of edges.

Definition 2.4.1. *The* ***path algebra*** *of a quiver Q is given by*

$$\mathbb{C}Q = \bigoplus_{\textit{path } e_1 \dots e_k \textit{ in } Q} \mathbb{C} \cdot e_1 \dots e_k$$

(a path p is a sequence of edges $e_1, \dots, e_k$, in which the source of each edge e_i is the target of the next edge e_{i+1}) with multiplication given by concatenation

$$p \cdot p' = \begin{cases} pp' & \textit{if the source of } p = \textit{the target of } p' \\ 0 & \textit{otherwise} \end{cases}$$

Definition 2.4.2. *A* ***potential*** *is an element*

$$W \in \mathbb{C}Q$$

which is a linear combination of cycles (i.e. paths that start and end at the same point). The ***Jacobi algebra*** *associated to* W *is*

$$J(Q, W) = \mathbb{C}Q \Big/ \left(\frac{\partial W}{\partial e} \right)_{\forall \text{ edges } e} \tag{2.4.1}$$

where the derivative of the potential with respect to any edge e *is defined by*

$$\frac{\partial e_1 \dots e_k}{\partial e} = \sum_{i=1}^{k} \begin{cases} e_{i+1} \dots e_k e_1 \dots e_{i-1} & \text{if } e_i = e \\ 0 & \text{if } e_i \neq e \end{cases}$$

Note that the Jacobi algebra is unchanged by cyclically permuting the potential, i.e. replacing $e_1 e_2 \dots e_k$ by $e_2 \dots e_k e_1$. For the quivers that we associated to toric Calabi-Yau threefolds in the previous Sections, we can define the potential

$$W = \sum_{\substack{\text{blue face } F \\ \text{with boundary } e_1 \dots e_k}} e_1 \dots e_k - \sum_{\substack{\text{red face } F \\ \text{with boundary } e_1 \dots e_k}} e_1 \dots e_k \tag{2.4.2}$$

For example, the quiver in Figure 2.3 has potential

$$W = e_1 e_2 e_3 - e_1 e_3 e_2$$

and so its Jacobi algebra is

$$\mathbb{C}\langle e_1, e_2, e_3 \rangle \Big/ (e_1 e_2 - e_2 e_1, e_2 e_3 - e_3 e_2, e_3 e_1 - e_1 e_3) = \mathbb{C}[e_1, e_2, e_3]$$

Exercise 1. Work out the brane tiling and the quiver with potential associated to the conifold, i.e. the toric diagram in the middle of Picture 2.2 (*Hint: see [18, Section 5.2] or [32, Section 4.1]*).

Exercise 2. Work out the brane tiling and the quiver with potential associated to the suspended pinch point, i.e. the toric diagram on the right of Picture 2.2 (*Hint: see [11, Section 4.1], [24, Section 2.1] or [31, Section B.1]*).

2.5 Shrubby quivers

Recall that the plane $\mathbb{R}^2$ is the universal cover of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. This allows us to lift the quiver Q drawn on the torus to a quiver $\widetilde{Q}$ drawn on the plane, which is **periodic**, i.e. invariant under translation by $\mathbb{Z}^2$. For instance, in Figure 2.4 we can visualize the periodic quiver associated to the quiver in Figure 2.3. Clearly, $\widetilde{Q}$ also comes with blue and red faces, such that the edges go clockwise around the former and counterclockwise around the latter.

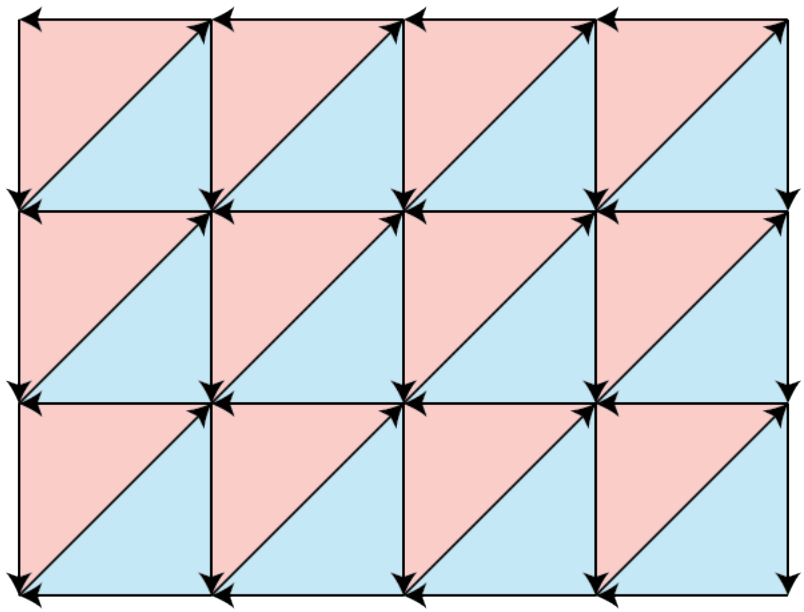


Figure 2.4: The periodic quiver associated to $X = \mathbb{C}^3$.

The following Definitions originated in [29], and they apply more generally to all quivers Q drawn on a torus with blue and red faces (such that the edges go clockwise around the blue faces and counterclockwise around the red faces), not just to the ones that come from toric Calabi-Yau threefolds.

Definition 2.5.1. *A **broken wheel** refers to a path obtained by removing a single edge e from the boundary of any face F (of either Q or $\widetilde{Q}$). The **mirror image** of the aforementioned broken wheel is the path obtained by removing e from the boundary of the other face $F' \neq F$ incident to e. The edge e will be called the **interface** of the broken wheel (and of its mirror image).*

Broken wheels are quite relevant for the study of the potential (2.4.2) and its Jacobi algebra (2.4.1). Specifically, if we consider the derivative of W with respect to the black arrow (the interface) in Figure 2.5, then we obtain precisely the difference between the red and blue paths therein.

$$\frac{\partial W}{\partial e} = p_{\text{red}} - p_{\text{blue}} \tag{2.5.1}$$

Definition 2.5.2. *The quiver Q is called **shrubby** if given any paths $p \neq p'$ in $\widetilde{Q}$ with the same start and end points, at least one of p and p' contains a broken wheel whose interface lies in the closed region between the two paths.*

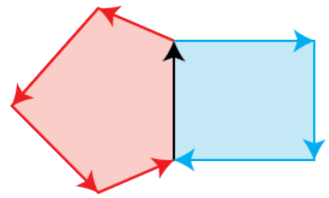


Figure 2.5: A broken wheel (the path in red, denoted by p_{red}) and its mirror image (the path in blue, denoted by p_{blue}). The black arrow is the interface.

The terminology above is a reference to this work, see [29, Claim 3.24]. More relevant to our purposes right now is that shrubbiness is a technical condition on the quiver Q which will allow the contents of the next lecture to run through (indeed, in the absence of this condition, it is easy to cook up examples of quivers for which Theorems 3.5.2 and 3.6.2 fail, and significant adjustments would be necessary in order to fix them). Dmitrii Rachenkov showed me a proof of the following result while he was a graduate student at SISSA.

Theorem 2.5.3. *The quiver Q drawn on $\mathbb{T}^2$ that arises from any toric Calabi-Yau threefold X via the procedure* (1.0.1) *is shrubby.*

The following notions were defined in [29] for any shrubby quiver Q.

Definition 2.5.4. *A* ***pre-shrub*** *S is an subgraph of $\widetilde{Q}$ which does not contain the entire boundary of any face, and moreover has the property that if S contains a broken wheel then it must also contain its mirror image.*

Exercise 3. Prove that a pre-shrub cannot contain any oriented cycles (*Hint: see [29, Proposition 3.13]*).

Definition 2.5.5. *A* ***shrub*** *S is a pre-shrub with a single root, which contains all the vertices contained in the interior of S (i.e. the area encompassed by the unoriented cycles contained in S). See Figure 2.6.*

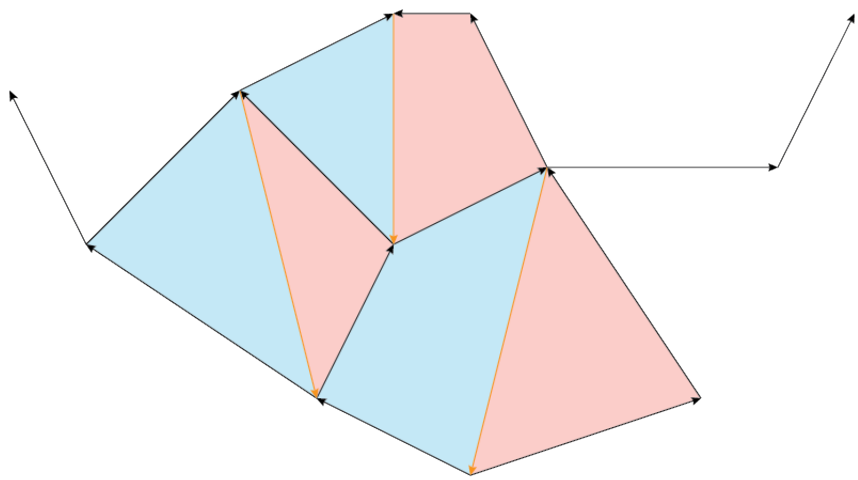


Figure 2.6: The black arrows determine a shrub S (the root is on the bottom). The orange arrows are in the quiver, but not in the shrub S.

Chapter 3

BPS algebras and K-HAs

The main purpose of this lecture is to establish isomorphisms

$$\mathbf{U}^+ \cong \mathcal{S} \cong \text{K-HA}(Q,W)_{\text{loc}} \Big/ (\text{torsion}) \tag{3.0.1}$$

where the three algebras above are the reduced BPS algebra, the shuffle algebra, and the localized K-theoretic Hall algebra associated to a shrubby quiver Q. The key notion is the shuffle algebra $\mathcal{S}$, as it provides the connection between the algebra on the left (which is a quotient, defined in [29], of the quiver quantum toroidal algebra of [12, 13, 24, 31, 32]) and the algebra on the right (see [23, 35] for cohomological and K-theoretic Hall algebras of quivers with potential).

3.1 Edge parameters

We will henceforth fix a quiver Q drawn on $\mathbb{T}^2$, with blue and red faces such that the arrows go clockwise around the former and counterclockwise around the latter; assume that Q is shrubby in the sense of Definition 2.5.2. We will work over a field of $\mathbb{K}$ characteristic zero, which is endowed with **edge parameters**

$$\Big\{ t_e \in \mathbb{K}^* \Big\}_{e \text{ edge of } Q} \tag{3.1.1}$$

These parameters are required to satisfy the following constraint for any face F of Q whose boundary is traced out by edges $e_1, \dots, e_k$

$$t_{e_1} \dots t_{e_k} = 1 \tag{3.1.2}$$

If we think of the parameters t_e as scaling the generators of the path algebra $\mathbb{C}Q$, then the constraint (3.1.2) precisely implies that the potential W of (2.4.2) is invariant. For example, for the quiver in Figure 2.3, we will denote the edge parameters by t_1, t_2, t_3 and they will satisfy the relation $t_1 t_2 t_3 = 1$.

Beside the face constraints (3.1.2), the edge parameters are required to be sufficiently generic, in the following sense. For any path $p = e_1 \dots e_k$ in $\widetilde{Q}$, define

$$t_p = t_{e_1} \dots t_{e_k} \tag{3.1.3}$$

Then the genericity condition that we require is

$$|t_p| \neq |t_{p'}| \tag{3.1.4}$$

for any paths p, p' in $\widetilde{Q}$ with the same start/end point but different end/start points, where $|\cdot|$ denotes absolute value with respect to some embedding $\mathbb{K} \to \mathbb{C}$.

Remark 3.1.1. If the quiver Q comes from a toric Calabi-Yau threefold X as in (1.0.1), then the natural choice of ground field is $\mathbb{K} = \text{Frac}(\text{Rep}_T)$, where T is the rank 2 torus of (2.1.7). The edge parameters $t_e \in \mathbb{K}^*$ are defined to be certain characters of T defined in [24], and beside the face constraint (3.1.2), they will enjoy the property that the product of incoming t_e's equals the product of outgoing t_e's at any vertex.

3.2 Quiver quantum toroidal algebras

Let I denote the vertex set of Q. For all $i, j \in I$, let us define

$$\zeta_{ij}(x) = \frac{\alpha_{ij} x^{s_{ij}}}{(1-x)^{\delta_{ij}}} \prod_{e \text{ arrow from } i \text{ to } j} (1 - x t_e) \tag{3.2.1}$$

for any fixed $\alpha_{ij} \in \mathbb{K}^*$, $s_{ij} \in \mathbb{Z}$. The following construction is the trigonometric version ([12, 13, 31, 32]) of the quiver Yangians defined in [24].

Definition 3.2.1. *The (half)* ***quiver quantum toroidal algebra*** $\widetilde{\mathbf{U}}^+$ *is*

$$\widetilde{\mathbf{U}}^+ = \mathbb{K}\Big\langle e_{i,d} \Big\rangle_{i \in I, d \in \mathbb{Z}} \Big/ \Big(\textit{relation } (3.2.3)\Big) \tag{3.2.2}$$

where if we write

$$e_i(z) = \sum_{d \in \mathbb{Z}} \frac{e_{i,d}}{z^d}$$

then the defining relations are given by the formula

$$e_i(z) e_j(w) \zeta_{ji}\left(\frac{w}{z}\right) = e_j(w) e_i(z) \zeta_{ij}\left(\frac{z}{w}\right) \tag{3.2.3}$$

for all $i, j \in I$. Formula (3.2.3) is interpreted as an infinite collection of relations obtained by equating the coefficients of all $\{z^a w^b\}_{a,b \in \mathbb{Z}}$ in the left and right-hand sides (if $i = j$, one clears the denominators $z - w$ before equating coefficients).

Let $\widetilde{\mathbf{U}}^- = (\widetilde{\mathbf{U}}^+)^{\text{op}}$. The (full) quiver quantum toroidal algebra is defined as

$$\widetilde{\mathbf{U}} = \widetilde{\mathbf{U}}^+ \otimes \mathbb{K}[h_{i,d}, h'_{i,d'}]^{i \in I}_{d,d' \text{ bounded below}} \otimes \widetilde{\mathbf{U}}^- \tag{3.2.4}$$

modulo certain relations that teach us how to commute elements from the three tensor factors above with each other. We will not be concerned with these relations, as we will only study $\widetilde{\mathbf{U}}^+$, but the interested reader can find them in [12, Section 2.2] or [31, Section 4.2]. We note that while the quantities α_{ij} and s_{ij} in (3.2.1) can be left arbitrary for the purpose of defining the half algebra $\widetilde{\mathbf{U}}^+$, they must be chosen carefully when considering the full algebra $\widetilde{\mathbf{U}}$.

Remark 3.2.2. The name "quiver quantum toroidal algebra" reflects the fact that the reduced versions of (3.2.4) (see (3.6.4)) includes examples such as the quantum toroidal algebras of $\mathfrak{gl}_{m|n}$, see [1] (specifically, $X = \mathbb{C}^3$, the conifold and the SPP correspond to (m, n) being $(1, 0)$, $(1, 1)$ and $(2, 1)$, respectively).

3.3 Crystals - reduction I

Consider a quiver Q drawn on a torus with red and blue faces, such that the edges go counterclockwise around the former and clockwise around the latter. Bogomolnyi-Prasad-Sommerfield (BPS for short) states can be realized combinatorially by a procedure called crystal melting, see [33, 34] (as well as [8, 25, 38] in the mathematical literature). To define crystals, fix a vertex i_0 called the "origin" and consider paths in $\widetilde{Q}$

$$\square = \Big\{ i_0 \mapsto i_1 \mapsto \cdots \mapsto i_{k-1} \mapsto i_k \Big\} \tag{3.3.1}$$

An **atom** is an equivalence class of such paths modulo the relation that identifies a broken wheel with its mirror image (see Figure 2.5). It is a fact that the equivalence class of a path is completely determined by the shortest path from i_0 to i_k (which is unique) and an integer which keeps track of how many times we wind around faces of the quiver. Thus, atoms are completely determined by

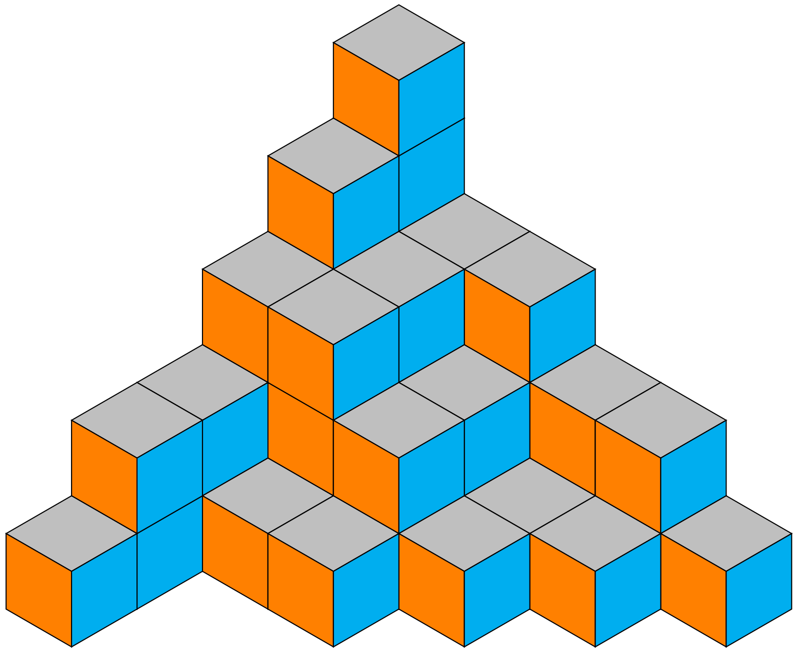
$$(i_k, n) \in \widetilde{Q} \times \mathbb{Z}_{\geq 0} \tag{3.3.2}$$

We abuse notation and denote atoms by $\square$ as well, and we let $\text{col}(\square) \in I$ to be the projection of the endpoint i_k from $\widetilde{Q}$ to Q. It will be called the **color** of $\square$.

Definition 3.3.1. *A **molten crystal configuration** (abbreviated mcc) λ is a finite collection of atoms with the following property: for any $\square$ as in* (3.3.1) *and any edge $e : i_k \mapsto i_{k+1}$, if the concatenation $e\square \in \lambda$, then $\square \in \lambda$.*

The intuition behind molten crystals stems from the case $X = \mathbb{C}^3$, when Q is the quiver drawn in Figure 2.3. In this case, atoms are in one-to-one correspondence with unit boxes in the octant $\mathbb{Z}^3_{\geq 0}$, and the molten crystal condition on λ says that if such a box lies in λ, then so do its immediate neighbors in the directions $(-1, 0, 0)$, $(0, -1, 0)$ and $(0, 0, -1)$. Therefore, in this case a mcc is the same as a 3d partition (a.k.a. "plane partition"), such as the one represented below.

Definition 3.3.2. *Given a molten crystal configuration λ, we call an atom $\square \notin \lambda$ addable to λ if $\lambda \cup \square$ is a molten crystal configuration. Let* $\text{add}(\lambda)$ *denote the set of addable atoms to a mcc λ.*



For the 3d partition depicted above, addable atoms are unit boxes that can be added in the locations . Fix a collection of constants

$$\Gamma_{\lambda,\square} \in \mathbb{K} \tag{3.3.3}$$

for any mcc λ and any atom $\square$, such that $\Gamma_{\lambda,\square} = 0$ unless $\square \in \text{add}(\lambda)$. We assume that the following identities hold

$$\frac{\Gamma_{\lambda,\square}\Gamma_{\lambda\cup\square,\blacksquare}}{\zeta_{\text{col}(\blacksquare)\text{col}\square}\left(\frac{t_\blacksquare}{t_\square}\right)} = \frac{\Gamma_{\lambda,\blacksquare}\Gamma_{\lambda\cup\blacksquare,\square}}{\zeta_{\text{col}(\square)\text{col}\blacksquare}\left(\frac{t_\square}{t_\blacksquare}\right)} \tag{3.3.4}$$

for all mcc λ and all atoms $\square, \blacksquare$. The quantity $t_\square$ is well-defined by (3.1.3), because a broken wheel and its mirror image have the same product of edge parameters due to (3.1.2). One has an action ([12, 13, 31, 32], based on [24])

$$\text{act} : \widetilde{\mathbf{U}}^+ \curvearrowright \Lambda = \bigoplus_{\text{mcc } \lambda} \mathbb{K} \cdot |\lambda\rangle \tag{3.3.5}$$

by the formulas

$$e_{i,d}|\lambda\rangle = \sum_{\square \in \text{add}(\lambda), \text{col}(\square)=i} t_\square^d \Gamma_{\lambda,\square}|\lambda \cup \square\rangle \tag{3.3.6}$$

with $t_\square$ defined in (3.1.3). Formula (3.3.4) is precisely what it takes for relation (3.2.3) to hold between the operators (3.3.6). We may consider the ideal

$$\widetilde{\mathbf{U}}^+ \supset I_1^+ = \text{Ker action (3.3.5)}$$

and define the (half) reduced quiver quantum toroidal algebra as

$$\mathbf{U}_1^+ = \widetilde{\mathbf{U}}^+ \Big/ I_1^+ \tag{3.3.7}$$

Remark 3.3.3. The true power of the action (3.3.5) stems from the fact that it can be extended to the full quiver quantum toroidal algebra (3.2.4)

$$\text{act} : \widetilde{\mathbf{U}} \curvearrowright \Lambda$$

However, to do so, one needs to make some specific choices of the constants that appear in (3.3.3) (see [12, formulas (3.21) and (4.30)] or [31, formula (5.2.1)]).

3.4 Shuffle algebras - reduction II

The following notion is the lynchpin of all our constructions. It originated in [10], see also [9] for a setting closer to ours.

Definition 3.4.1. *The* ***big shuffle algebra*** *is*

$$\mathcal{V} = \bigoplus_{\boldsymbol{n} \in \mathbb{Z}^I_{\geq 0}} \mathcal{V}_{\boldsymbol{n}}, \qquad \textit{where} \qquad \mathcal{V}_{(n_i)_{i\in I}} = \mathbb{K}[z_{i1}, z_{i1}^{-1}, \dots, z_{in_i}, z_{in_i}^{-1}]^{\text{sym}}_{i \in I}$$

endowed with the following multiplication (let $\boldsymbol{n}! = \prod_{i \in I} n_i!$*)*

$$R(z_{i1}, \dots, z_{in_i}) * R'(z_{i1}, \dots, z_{in'_i}) = \tag{3.4.1}$$

$$\text{Sym} \left[\frac{R(z_{i1}, \dots, z_{in_i}) R'(z_{i,n_i+1}, \dots, z_{i,n_i+n'_i})}{\boldsymbol{n}! \boldsymbol{n}'!} \prod_{\substack{1 \leq a \leq n_i \\ n_j < b \leq n_j + n'_j}}^{i,j \in I} \zeta_{ij} \left(\frac{z_{ia}}{z_{jb}} \right) \right]$$

*Above, "*sym*" (resp. "*Sym*") denotes symmetric functions (resp. symmetrization) with respect to the variables* $z_{i1}, z_{i2}, \dots$ *for each* $i \in I$ *separately.*

Exercise 4. Prove that the operation (3.4.1) is well-defined (i.e. takes values in $\mathcal{V}$ despite the apparent poles at $z_{ia} - z_{ib}$) and is associative.

Proposition 3.4.2. *There exists an algebra homomorphism*

$$\widetilde{\mathbf{U}}^+ \xrightarrow{\widetilde{\Upsilon}^+} \mathcal{V} \tag{3.4.2}$$

which takes $e_{i,d}$ *to the function in a single variable* $z_{i1}^d \in \mathcal{V}_{\boldsymbol{\varsigma}^i}$*, where*

$$\boldsymbol{\varsigma}^i = \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{1 \text{ on } i\text{-th position}}$$

We may consider the ideal

$$\widetilde{\mathbf{U}}^+ \supset I_2^+ = \text{Ker } \widetilde{\Upsilon}^+$$

and define the (half) reduced quiver quantum toroidal algebra as

$$\mathbf{U}_2^+ = \widetilde{\mathbf{U}}^+ \Big/ I_2^+ \tag{3.4.3}$$

The fact that this notion agrees with the one of (3.3.7) was observed in [12]. It is a straightforward check, which boils down to the following result.

Proposition 3.4.3. *The action* (3.3.5) *descends to a faithful action*

$$\text{Im } \widetilde{\Upsilon}^+ \curvearrowright \Lambda \tag{3.4.4}$$

Proof. For any mcc λ and any atoms $\square, \blacksquare$, let

$$\Gamma_{\lambda, \square \cup \blacksquare} = \Gamma_{\lambda, \blacksquare \cup \square}$$

denote the quantity in (3.3.4). For any number of atoms $\square_1, \dots, \square_n$, we define

$$\Gamma_{\lambda, \square_1 \cup \dots \cup \square_n} = \frac{\prod_{a=1}^n \Gamma_{\lambda \cup \square_{a+1} \cup \dots \cup \square_n, \square_a}}{\prod_{1 \leq a < b \leq n} \zeta_{\text{col}(\square_a)\text{col}(\square_b)} \left(\frac{t_{\square_a}}{t_{\square_b}} \right)} \tag{3.4.5}$$

Successively applying formula (3.3.4) shows that the expression (3.4.5) is symmetric in $\square_1, \dots, \square_n$. With this in mind, applying formula (3.3.6) repeatedly gives

$$\begin{aligned} e_{i_1,d_1} \dots e_{i_n,d_n} |\lambda\rangle &= \sum_{\substack{\square_1, \dots, \square_n \\ \text{col}(\square_a) = i_a}} t_{\square_1}^{d_1} \dots t_{\square_n}^{d_n} \Gamma_{\lambda, \square_n} \dots \Gamma_{\lambda \cup \square_2 \cup \dots \cup \square_n, \square_1} |\lambda \cup \square_1 \cup \dots \cup \square_n\rangle \\ &= \sum_{\square_1, \dots, \square_n} R(t_{\square_1}, \dots, t_{\square_n}) \Gamma_{\lambda, \square_1 \cup \dots \cup \square_n} |\lambda \cup \square_1 \cup \dots \cup \square_n\rangle \end{aligned}$$

where $R = \widetilde{\Upsilon}^+(e_{i_1,d_1} \dots e_{i_n,d_n})$. The meaning of the evaluation $R(t_{\square_1}, \dots, t_{\square_n})$ is that we set up a one-to-one correspondence between the parameters $t_{\square_a}$ and the variables $z_{i_a \bullet}$ of R (the evaluation is defined to be 0 if this one-to-one correspondence does not exist). From the formula displayed above, it is clear that any element of Ker $\widetilde{\Upsilon}^+$ acts on Λ by 0, so (3.4.4) is a well-defined action.

To show that this action is faithful, one needs to establish the following fact: if a polynomial R has the property that the evaluation $R(t_{\square_1}, \dots, t_{\square_n})$ is 0 for any collection of addable atoms $\square_1, \dots, \square_n$ to any mcc λ, then $R = 0$. To see this, it suffices to fix paths

$$p_a : i_0 \mapsto \dots \mapsto i_a \quad \text{and} \quad r_a : i_a \mapsto \dots \mapsto i_a$$

(note that r_a should not a cycle, but a path in $\widetilde{Q}$ whose start and end points are different lifts of the same $i_a \in Q$). Then for large enough $d_a \in \mathbb{N}$, the numbers

$$\left\{ t_{r_a}^{d_a} t_{p_a} \right\}_{a \in \{1,\dots,n\}}$$

do not all satisfy any given algebraic relation due to (3.1.4). Since we can always find an mcc λ to which we can add the atoms $r_1^{d_1} p_1, \dots, r_n^{d_n} p_n$, the fact that $R(t_{r_1}^{d_1} t_{p_1}, \dots, t_{r_n}^{d_n} t_{p_n}) = 0$ for all large enough $d_1, \dots, d_n$ implies that $R = 0$.

□

3.5 Wheel conditions - reduction III

Beside the homomorphism (3.4.2), we have a pairing

$$\widetilde{\mathbf{U}}^+ \otimes \mathcal{V} \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{K} \tag{3.5.1}$$

given by the following formula for all $i_1, \dots, i_n \in I$, $d_1, \dots, d_n \in \mathbb{Z}$ and $R \in \mathcal{V}$

$$\Big\langle e_{i_1,d_1} \dots e_{i_n,d_n}, R \Big\rangle = \left[\frac{z_1^{d_1} \dots z_n^{d_n} R(z_1, \dots, z_n)}{\prod_{1 \leq a < b \leq n} \zeta_{i_b i_a} \left(\frac{z_b}{z_a} \right)} \right]_{|z_1| \gg \dots \gg |z_n|} \tag{3.5.2}$$

The meaning of the right-hand side is that we set up a one-to-one correspondence between $\{z_1, \dots, z_n\}$ and the variables of R such that each z_a is plugged into a variable of the form $z_{i_a \bullet}$ of R (the pairing is defined to be 0 if this one-to-one correspondence does not exist). Then we expand the right-hand side of (3.5.2) as indicated and take the constant term of the resulting power series.

Exercise 5. Show that the pairing (3.5.2) is well-defined, i.e. that it respects any linear relation between the $e_{i_1,d_1} \dots e_{i_n,d_n}$ that is induced by (3.2.3).

Definition 3.5.1. *Let $\mathcal{S} \subset \mathcal{V}$ denote the subset of polynomials R which satisfy*

$$R \Big|_{z_1 = z_k t_{e_1}, z_2 = z_1 t_{e_2}, \dots, z_k = z_{k-1} t_{e_k}} = 0 \tag{3.5.3}$$

for any face $F = \{i_0, \dots, i_{k-1}, i_k = i_0\}$ of Q, which is bounded by edges $i_{a-1} \xrightarrow{e_a} i_a$. The vanishing (3.5.3) *is called a* ***wheel condition****, inspired by the analogous notion of [10]: its meaning is that while R can have many variables, we choose k distinct ones among these variables with first index $i_1, \dots, i_k$, and then require that R vanishes when we specialize the chosen k variables as indicated in* (3.5.3).

Exercise 6. Prove that $\mathcal{S}$ is a subalgebra with respect to the product (3.4.1).

We call $\mathcal{S}$ the **shuffle algebra**. We may consider the ideal

$$\widetilde{\mathbf{U}}^+ \supset I_3^+ = \left\{ x \in \widetilde{\mathbf{U}}^+ \text{ s.t. } \left\langle x, \mathcal{S} \right\rangle = 0 \right\}$$

and define the (half) reduced quiver quantum toroidal algebra as

$$\mathbf{U}_3^+ = \widetilde{\mathbf{U}}^+ \Big/ I_3^+ \tag{3.5.4}$$

The following is the main result of [29], and together with Proposition 3.4.3, shows that all three notions of reduced quiver quantum toroidal algebras are actually the same. We will henceforth use the notation $\mathbf{U}^+$ for $\mathbf{U}_1^+ = \mathbf{U}_2^+ = \mathbf{U}_3^+$.

Theorem 3.5.2. *If Q is shrubby, we have $I_2^+ = I_3^+$, i.e. we obtain a pairing*

$$\mathbf{U}^+ \otimes \mathcal{S} \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{K} \tag{3.5.5}$$

non-degenerate in both arguments. Moreover, $\widetilde{\Upsilon}^+$ induces an isomorphism

$$\Upsilon^+ : \mathbf{U}^+ \xrightarrow{\sim} \mathcal{S} \tag{3.5.6}$$

Shrubs are the key combinatorial tool which goes into the analysis of the pairing (3.5.5) and allows us to prove Theorem 3.5.2. Indeed, in [29, (3.26)], we give a formula for the pairing (3.5.5) as a sum over shrubberies, which are disjoint union of shrubs. In the absence of the shrubbiness assumption, not only does our proof fail, but the statement of Theorem 3.5.2 also fails (to fix the statement, one would need to add more complicated wheel conditions in Definition 3.5.1, which we do not know how to completely describe). Thus, the analogue of Theorem 3.5.2 for arbitrary quivers is wide open, though it is known in other special cases ([26, 28, 30], see [27] for an overview).

3.6 Generators-and-relations

The gist of Theorem 3.5.2 is that reduced quiver quantum toroidal algebras are not just isomorphic to shuffle algebras, they are dual to them as well. Since every wheel condition (3.5.3) can be thought of as a linear condition that cuts out the linear subspace $\mathcal{S} \subset \mathcal{V}$, its dual with respect to (3.5.5) is an element of

$$I^+ = \mathrm{Ker} \left(\widetilde{\mathbf{U}}^+ \twoheadrightarrow \mathbf{U}^+ \right) \tag{3.6.1}$$

Moreover, because the pairing is explicit, we can work out the aforementioned elements explicitly. Consider the Laurent polynomial $\widetilde{\zeta}_{ij}(x) = \zeta_{ij}(x)(1-x)^{\delta_{ij}}$.

Definition 3.6.1. *For every face $F = \{i_0, i_1, \dots, i_{k-1}, i_k = i_0\}$ of Q, which is bounded by edges $e_a : i_{a-1} \to i_a$, consider the formal series of elements of $\widetilde{\mathbf{U}}^+$*

$$e_F(x_1, \dots, x_k) = \sum_{a=1}^{k} \frac{x_1 t_2 \dots t_{e_a}}{x_a} \cdot \frac{\prod_{b \succ c} \widetilde{\zeta}_{i_c i_b}\left(\frac{x_c}{x_b}\right)\left(-\frac{x_b}{x_c}\right)^{\delta_{i_b i_c} \delta_{b<c}}}{\prod_{b \sim c+1}\left(1 - \frac{x_c t_{e_b}}{x_b}\right)}$$

$$e_{i_a}(x_a) \dots e_{i_1}(x_1) e_{i_k}(x_k) \dots e_{i_{a+1}}(x_{a+1}) \tag{3.6.2}$$

In (3.6.2), the notation $b \succ c$ (respectively $b \sim c+1$) means that b precedes (respectively immediately precedes) c in the sequence $(a, \dots, 1, k, \dots, a+1)$.

Theorem 3.6.2. *If Q is shrubby, then the ideal I^+ of (3.6.1) is generated by one coefficient of the formal series (3.6.2) of each total homogeneous degree (in $x_1, \dots, x_k$) per face F of Q.*

Theorem 3.6.2 then gives us the following explicit generators-and-relations presentation of reduced (half) quiver quantum toroidal algebras:

$$\mathbf{U}^+ = \mathbb{K}\Big\langle e_{i,d} \Big\rangle_{i \in I, d \in \mathbb{Z}} \Big/ \Big(\text{relation (3.2.3) and } e_F(x_1, \dots, x_k) = 0, \forall \text{face } F\Big) \tag{3.6.3}$$

(it is actually enough to factor out by a single coefficient of e_F of every total given homogeneous degree, and all other coefficients will be redundant). Define

$$\mathbf{U} = \mathbf{U}^+ \otimes \mathbb{K}[h_{i,d}, h'_{i,d'}]^{i \in I}_{d,d' \text{ bounded below}} \otimes \mathbf{U}^- \tag{3.6.4}$$

by letting $\mathbf{U}^- = (\mathbf{U}^+)^{\text{op}}$ and using the same relations between the three tensor factors above as between the three tensor factors in (3.2.4).

For the quivers corresponding to $X = \mathbb{C}^3$, the conifold and the SPP, we have

$$\mathbf{U} \cong U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1), U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_{1|1}) \text{ and } U_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_{2|1})$$

respectively. As an example, let us recall the first of the algebras in the right-hand side: its half subalgebra is given by

$$U^+_{t_1,t_2}(\widehat{\widehat{\mathfrak{gl}}}_1) = \mathbb{K}\Big\langle e_d \Big\rangle_{d \in \mathbb{Z}} \Big/ \Big(\text{relation (3.6.5) and (3.6.6)}\Big)$$

(as the quiver in Figure 2.3 has a single vertex, we suppress the index i), where

$$e(z)e(w)(z - wt_1)(z - wt_2)(zt_1t_2 - w) = \\ = e(w)e(z)(zt_1 - w)(zt_2 - w)(z - wt_1t_2) \tag{3.6.5}$$

and

$$[[e_{d-1}, e_{d+1}], e_d] = 0, \qquad \forall d \in \mathbb{Z} \tag{3.6.6}$$

Meanwhile, our reduced quiver quantum toroidal algebra is given by

$$\mathbf{U}^+ = \mathbb{K}\Big\langle e_d \Big\rangle_{d\in\mathbb{Z}} \Big/ \Big(\text{relation (3.6.5) and } e_{F_{\text{red}}} = e_{F_{\text{blue}}} = 0\Big)$$

where

$$\begin{aligned} e_{F_{\text{red}}}(x_1,x_2,x_3) = & \frac{\prod_{i=1}^3 [(x_1 - x_2t_i)(x_1 - x_3t_i)(x_3 - x_2t_i)]}{x_1 x_2^3 x_3^3 (x_1 - x_3t_1)(x_3 - x_2t_3)} e(x_1)e(x_3)e(x_2) \\ & + \frac{t_2 \prod_{i=1}^3 [(x_2 - x_1t_i)(x_1 - x_3t_i)(x_2 - x_3t_i)]}{x_2^3 x_3^4 (x_2 - x_1t_2)(x_1 - x_3t_1)} e(x_2)e(x_1)e(x_3) \\ & + \frac{t_2t_3 \prod_{i=1}^3 [(x_2 - x_1t_i)(x_3 - x_1t_i)(x_3 - x_2t_i)]}{x_1 x_2^2 x_3^4 (x_3 - x_2t_3)(x_2 - x_1t_2)} e(x_3)e(x_2)e(x_1) \end{aligned}$$

and $e_{F_{\text{blue}}}$ is obtained by replacing $\{t_1, t_2, t_3\}$ by $\{t_3, t_2, t_1\}$.

Exercise 7. Show that modulo relations (3.6.5), relation (3.6.6) is equivalent to

$$e_{F_{\text{red}}} = e_{F_{\text{blue}}} = 0$$

The exercise above shows that once we impose the quadratic relations (3.2.3), the higher degree relations can always be modified by adding multiples of the quadratic relations, without changing the overall algebra. So while previously known formulas such as (3.6.6) are simpler than ours, we prefer our more complicated formulas (3.6.2) because they capture the duality between higher degree relations and wheel conditions for general shrubby quivers.

3.7 K-theoretic Hall algebras

Consider any shrubby quiver Q (drawn on a torus with red and blue faces, such that the edges go counterclockwise around the former and clockwise around the latter), together with the potential (2.4.2). For any $\boldsymbol{n} = (n_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}$, we consider $\boldsymbol{n}$-dimensional representations of the Jacobi algebra

$$J(Q,W) \curvearrowright V = (V_i \cong \mathbb{C}^{n_i})_{i\in I}$$

i.e. collections of linear maps $e : V_i \to V_j$ for every arrow $e : i \to j$, which satisfy the equations $\frac{\partial W}{\partial e} = 0$ for every e. For example, for the quiver of Figure 2.3, a representation consists of a single vector space V, endowed with three mutually commuting endomorphisms e_1, e_2, e_3. Then one would like to consider

$$\text{K-HA}(Q,W) \text{``}=\text{''} \bigoplus_{\boldsymbol{n}\in\mathbb{N}^I} K^T(\text{moduli stack of } \boldsymbol{n}\text{-dim reps of } J(Q,W)) \quad (3.7.1)$$

and make it into an algebra via a suitable convolution product that is additive in $\boldsymbol{n}$. However, the stack above (and more importantly the stacks of extensions that give the aforementioned convolution product) are very singular, and so one needs to replace the right-hand side of (3.7.1) by something else. In cohomology, this "something else" is the cohomology of a certain sheaf of vanishing cycles on a smooth space ([23]). In K-theory, one uses instead categories of singularities associated to W ([35]). We will not have time to go into the technical details.

Definition 3.7.1. *In the appropriate replacement of* (3.7.1)*, T is a torus which acts on the moduli stack by dilating the linear maps e via certain characters*

$$t_e : T \to \mathbb{C}^*$$

These characters are assumed to satisfy the analogue of (3.1.2) *(so that the potential W is T-invariant) and to be sufficiently generic so that* (3.1.4) *holds.*

We let $\mathbb{K} = \mathrm{Frac}(\mathrm{Rep}_T)$ and consider the **localized** K-theoretic Hall algebra

$$\text{K-HA}(Q, W)_{\mathrm{loc}} = \text{K-HA}(Q, W) \bigotimes_{\mathrm{Rep}_T} \mathbb{K}$$

Recall the big shuffle algebra in Definition 3.4.1. We have a homomorphism

$$\iota : \text{K-HA}(Q, W)_{\mathrm{loc}} \to \mathcal{V} \tag{3.7.2}$$

by combining [35, Section 9] with [36, Proposition 3.6].

Remark 3.7.2. Different choices of α_{ij} and s_{ij} in (3.2.1) are accounted for by different line bundle twists in the convolution product on the K-HA.

The kernel of the homomorphism ι is called the torsion submodule of the K-HA, and it is precisely what we need to factor out in (1.0.5). Then the connection between BPS algebras, shuffle algebras and localized K-theoretic Hall algebras is made precise by the following.

Lemma 3.7.3. *If Q is shrubby, the image of ι coincides with $\mathcal{S} \overset{(3.5.6)}{\cong} \mathbf{U}^+$.*

Proof. The fact that Im $\iota \supseteq \mathcal{S}$ follows from the fact that $\mathcal{S}$ is generated by $z_{i1}^d \in \mathcal{V}_{\boldsymbol{\varsigma}^i}$ (see Theorem 3.5.2, this is where we need the shrubbiness assumption), and these generators lie in the image of the K-theoretic Hall algebra by a simple inspection of the stack of $\boldsymbol{\varsigma}^i$-dimensional representations of $J(Q, W)$. For the opposite inclusion Im $\iota \subseteq \mathcal{S}$, we need to show that ι(any element of the K-HA) satisfies the wheel conditions (3.5.3). This is explained in [29, Corollary 1.18], and it follows from the fact that any element of the K-HA satisfies the equations $\frac{\partial W}{\partial e} = 0$ for all edges e.

□

Bibliography

[1] Bezerra L., Mukhin E., *Quantum Toroidal Algebra Associated with* $\mathfrak{gl}_{m|n}$, Algebr. Represent. Theor. 24, 541-564 (2021).

[2] Bocklandt R., *Consistency conditions for dimer models*, Glasgow Mathematical Journal. 2012; 54(2): 429-447.

[3] Broomhead N., *Dimer models and Calabi-Yau algebras*, Memoirs of the American Mathematical Society (2012), Volume 215, Number 1011 ISBNs: 978-0-8218-5308-5.

[4] Candelas P., de la Ossa X., Green P., Parkes L., *A pair of Calabi–Yau manifolds as an exactly soluble superconformal field theory* Nuclear Physics B. 359 (1) (1991), 21-74.

[5] Candelas P., Horowitz G., Strominger A., Witten E., *Vacuum configurations for superstrings*, Nuclear Physics B. 258 (1985), 46-74.

[6] Closset C., *Toric geometry and local Calabi-Yau varieties: An introduction to toric geometry (for physicists)*, arχiv:0901.3695.

[7] Davison B., *Consistency conditions for brane tilings*, Journal of Algebra, Volume 338, Issue 1 (2011), Pages 1-23.

[8] Davison B., *BPS Lie algebras and the less perverse filtration on the preprojective CoHA*, arχiv:2007.03289.

[9] Enriquez B., *On correlation functions of Drinfeld currents and shuffle algebras*, Transform. Groups 5 (2000), no. 2, 111–120.

[10] Feigin B., Odesskii A., *Quantized moduli spaces of the bundles on the elliptic curve and their applications*, Integrable structures of exactly solvable two-dimensional models of quantum field theory (Kiev, 2000), 123–137, NATO Sci. Ser. II Math. Phys. Chem., 35, Kluwer Acad. Publ., Dordrecht, 2001.

[11] Franco S., Hanany A., Vegh D., Wecht B., Kennaway K., *Brane dimers and quiver gauge theories*, JHEP 01 (2006).

[12] Galakhov D., Li W., Yamazaki M., *Toroidal and elliptic quiver BPS algebras and beyond*, J. High Energy Phys., 24 (2022).

[13] Galakhov D., Li W., Yamazaki M., *Gauge/Bethe correspondence from quiver BPS algebras*, J. High Energy Phys., 119 (2022).

[14] Ginzburg V., *Calabi-Yau algebras*, arχiv:math/0612139.

[15] Gulotta D., *Properly ordered dimers, R-charges, and an efficient inverse algorithm*, JHEP 0810, 014 (2008).

[16] Hanany A., Herzog C., Vegh D., *Brane Tilings and Exceptional Collections*, J. High Energ. Phys. 27 (2006).

[17] Hanany A., Kennaway K., Dimer models and toric diagrams, arχiv:hep-th/0503149.

[18] Hanany A., Vegh D., *Quivers, Tilings, Branes and Rhombi*, J. High Energ. Phys. 10 (2007).

[19] Harvey J., Moore G., *On the algebras of BPS states*, Commun. Math. Phys. vol. 197, 489-519 (1998).

[20] Ishii A., Ueda K., *Dimer models and the special McKay correspondence*, Geom. Top. 19 (2015) 3405-3466.

[21] Joyce D., Upmeier M., *Orientation data for moduli spaces of coherent sheaves over Calabi-Yau 3-folds*, Adv. Math. vol 381 (2021) 107627.

[22] Kinjo T., Park H., Safronov P., *Cohomological Hall algebras for 3-Calabi-Yau categories*, arχiv:2406.12838.

[23] Kontsevich M., Soibelman Y., *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. 5 (2011), no. 2, 231-352.

[24] Li W., Yamazaki M., *Quiver Yangian from crystal melting*, J. High Energ. Phys. 35 (2020).

[25] Mozgovoy S., Reineke M., *On the noncommutative Donaldson-Thomas invariants arising from brane tilings*, Adv. Math. Volume 223, Issue 5 (2010), Pages 1521-1544.

[26] Neguț A., *Quantum loop groups for symmetric Cartan matrices*, arχiv:2207.05504.

[27] Neguț A., *Quantum loop groups for arbitrary quivers*, arχiv:2209.09089.

[28] Neguț A., *Shuffle algebras for quivers and wheel conditions*, J. Reine Angew. Math. (2023) no. 795, 139-182.

[29] Neguț A., *Reduced quiver quantum toroidal algebras*, J. Inst. Math. Jussieu, doi:10.1017/S1474748024000306 (2024).

[30] Neguț A., Sala F., Schiffmann O., *Shuffle algebras for quivers as quantum groups*, Math. Ann. (2024).

[31] Noshita G., Watanabe A. *A note on quiver quantum toroidal algebra*, J. High Energ. Phys. 2022, 11 (2022).

[32] Noshita G., Watanabe A. *Shifted Quiver Quantum Toroidal Algebra and Subcrystal Representations*, J. High Energ. Phys., 122 (2022).

[33] Okounkov A., Reshetikhin N., Vafa C., *Quantum Calabi-Yau and classical crystals*, Prog. Math. 244, 597 (2006).

[34] Ooguri H., Yamazaki M., *Crystal Melting and Toric Calabi-Yau Manifolds*, Commun. Math. Phys. 292, 179 (2009).

[35] Pădurariu T., *K-theoretic Hall algebras for quivers with potential*, Thesis (Ph.D.)-Massachusetts Institute of Technology, ProQuest LLC, Ann Arbor, MI, 2020.

[36] Pădurariu T., *Categorical and K-theoretic Hall algebras of quivers with potential*, J. Inst. Math. Jussieu. 2023;22(6):2717-2747.

[37] Strominger A., Yau S.-T., Zaslow E., *Mirror symmetry is T-duality*,. Nuclear Physics B. 479 (1) (1996), 243-259.

[38] Szendrői B., *Non-commutative Donaldson-Thomas invariants and the conifold*, Geom. Topol. 12, 1171 (2008).