Energy Trans-series of Hofstadter Butterfly from Supersymmetric Field Theory

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Resurgence, Wall-Crossing and Geometry, Les Diablerets, Jan 17 2025

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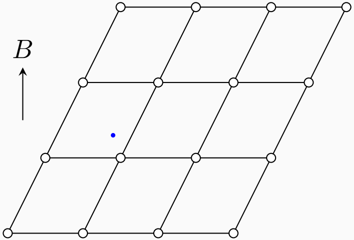
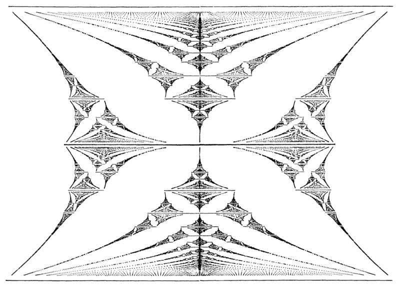
Hofstadter butterfly

- In 1979, D. Hofstadter considered an interesting 2d electron model in a magnetic field. [Hofstadter'79]





- How to explain? Relation with supersymmetric field theory! [Hatsuda,Katsura,Tachikawa'16]

Hofstadter butterfly

2d electron in lattice with magnetic field

- 2d electron in a square lattice with spaceing a: by tight binding approximation

$$\mathrm{H} = 2\cos\frac{\mathrm{p}_x a}{\hbar} + 2\cos\frac{\mathrm{p}_y a}{\hbar}$$

- Adding a uniform and perpendicular magnetic field B

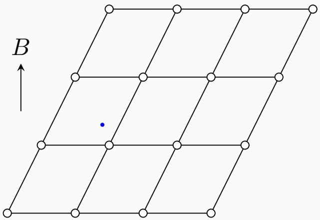
$$\vec{\mathrm{p}} \to \vec{\Pi} = \vec{\mathrm{p}} + e\vec{A}$$

which satisfy commutation relation

$$[\vec{\Pi}_x, \vec{\Pi}_y] = -\mathrm{i}\hbar e(\partial_x A_y - \partial_y A_x) = -\mathrm{i}\hbar eB$$

Hamiltonian of electron becomes

$$\mathrm{H} = \mathrm{e}^{\frac{\mathrm{i}a}{\hbar}\Pi_x} + \mathrm{e}^{-\frac{\mathrm{i}a}{\hbar}\Pi_x} + \mathrm{e}^{\frac{\mathrm{i}a}{\hbar}\Pi_y} + \mathrm{e}^{-\frac{\mathrm{i}a}{\hbar}\Pi_y}$$



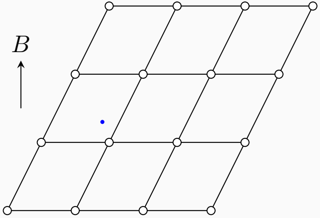
Harper's equation

- Replacing $(a/\hbar)\Pi_{x,y}$ by operators x, y

$$\mathrm{H} = \mathrm{e}^{\mathrm{ix}} + \mathrm{e}^{-\mathrm{ix}} + \mathrm{e}^{\mathrm{iy}} + \mathrm{e}^{-\mathrm{iy}}$$

with the commutation relation

$$[\mathrm{x}, \mathrm{y}] = \frac{\mathrm{i}a^2 eB}{\hbar} =: \mathrm{i}\phi \quad \text{magnetic flux through a plaquette.}$$



- Equivalent to a **1d relativistic QM model** where ϕ is $\hbar$.
- Harper's equationn

$$\psi(x+\phi) + \psi(x-\phi) + 2\cos(x)\psi(x) = E\psi(x)$$

Introducing $x = n\phi + \delta$ and $\psi_n(\delta) = \psi(n\phi + \delta)$

$$\psi_{n+1} + \psi_{n-1} + 2\cos(n\phi + \delta)\psi_n = E\psi_n$$

Energy spectrum at rational flux

- The model simplifies when flux is **rational** [Hofstadter'79]

$$\phi = 2\pi\alpha = 2\pi\frac{P}{Q}, \quad P, Q \in \mathbb{N}, (P, Q) = 1.$$

- Harper's equation is periodic $n \to n + Q$, so that Bloch wavefunciton can be defined,

$$\psi_n(\delta) = \mathrm{e}^{\mathrm{i}kn} u_n(\delta, k) \quad \text{w/ } u_{n+Q}(\delta, k) = u_n(\delta, k).$$

- Energy spectrum is computed by the **polynomial characteristic (secular) equation**

$$F_{P/Q}(E, \delta, k) := \det(H_Q - E\mathbf{1}_Q) = 0$$

with

$$H_Q(\delta, k) = \begin{pmatrix} 2\cos\delta & \mathrm{e}^{\mathrm{i}k} & & & \mathrm{e}^{-\mathrm{i}k} \\ \mathrm{e}^{-\mathrm{i}k} & 2\cos(\delta + 2\pi\frac{P}{Q}) & \mathrm{e}^{\mathrm{i}k} & & \\ & \mathrm{e}^{-\mathrm{i}k} & 2\cos(\delta + 4\pi\frac{P}{Q}) & \mathrm{e}^{\mathrm{i}k} & \\ & & \ddots & \ddots & \ddots \\ \mathrm{e}^{\mathrm{i}k} & & & \mathrm{e}^{-\mathrm{i}k} & 2\cos(\delta + 2\pi(Q-1)\frac{P}{Q}) \end{pmatrix}$$

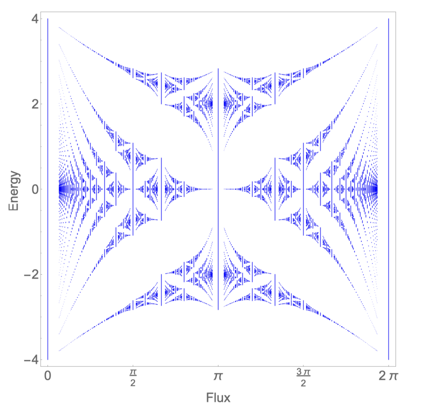
Energy spectrum at rational flux, two Bloch angles

- It can be shown [Hasegawa,Hatsugai,Kohmoto,Montambaux'90]

$$F_{P/Q}(E,0,0) = 2(\cos Qk + \cos Q\delta) =: 2(\cos\theta_x + \cos\theta_y)$$

  where θ_x, θ_y are actually Bloch angles: equation periodic by $\theta_{x,y} \to \theta_{x,y} + 2\pi$.
- Varying $\cos\theta_x + \cos\theta_y \in [-2,2]$, degree Q polynomial $F_{P/Q}(E,0,0)$ yields Q energy bands.

Hofstadter butterfly



- Features of the energy spectrum
  - Fractal structure
- Problems of the energy spectrum
  - How to understand this picture? What is E as a function of ϕ? **Highly non-perturbative**!
- Results:
  - Energy trans-series for $\phi = 2\pi/Q$ that includes **full non-pert. corrections**.

Semi-classical and basic resurgence analysis

Semi-classical analsys of energy series

- Hamiltonian for the Hofstadter model

$$\mathrm{H} = \mathrm{e}^{\mathrm{i}\mathrm{x}} + \mathrm{e}^{-\mathrm{i}\mathrm{x}} + \mathrm{e}^{\mathrm{i}\mathrm{y}} + \mathrm{e}^{-\mathrm{i}\mathrm{y}}, \quad [\mathrm{x}, \mathrm{y}] = \mathrm{i}\phi.$$

- The perturbative energy series can be efficiently calculated by `BenderWu` package with Landau level $N = 0, 1, 2, \ldots$ [Bender,Wu'73; Sulejmanpasic,Unsal'16; **JG**,Sulejmanpasic'17]

$$E^{(0)}(N,\phi) = 4 - (1+2N)\phi + \frac{1}{8}(1+2N+2N^2)\phi^2 + -\frac{1}{192}(1+2N)(1+N+N^2)\phi^3 + \ldots$$

- It is independent of $\theta_{x,y}$ and divergent

$$E^{(0)}(N,\phi) = \sum_{k\geq 0} a_k(N)\phi^k, \quad a_k(N) \sim N!$$

so that non-perturbative corrections are needed.

Instanton corrections

- By path integral analysis of twisted thermal partition function, one finds that for $\phi = 2\pi/Q$, there are instanton and anti-instanton in both x- and y-directions

  [Duan,**JG**,Hatsuda,Sulejmanpasic'18]

$$E^{(1)}_{(\theta_x,\theta_y)}(N=0,\phi) = 8(\cos\theta_x + \cos\theta_y)\left(\frac{\phi}{2\pi}\right)^{1/2} \mathrm{e}^{-A_c/\phi}(1+\ldots), \quad A_c = 8C.$$

- General trans-series

$$E(N,\phi) = E^{(0)}(N,\phi) + \sum_{n\geq 1} E^{(n)}_{\theta_x,\theta_y}(N,\phi), \quad E^{(n)}_{\theta_x,\theta_y}(N,\phi) \sim \mathrm{e}^{-nA_c/\phi},$$

  such that

$$E_{\mathrm{ex}}(N,\phi) = \mathscr{S} E(N,\phi)$$

Resurgence analysis

- Dominant Borel singularity of $E^{(0)}(N,\phi)$ at

$$\mathcal{A}, \quad \mathcal{A} = 2A_c = 16C$$

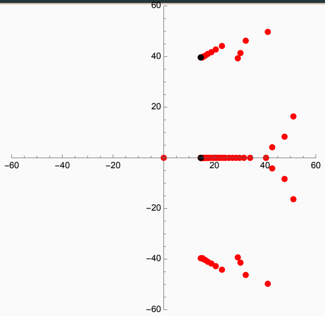
- Lateral Borel resummation is needed, which is ambiguous.
  [Veronica, Murad, Amir's talks]
- Ambiguity (partially) canceled by 2-instanton ($\mathcal{I}\bar{\mathcal{I}}$) corrections [Duan,**JG**,Hatsuda,Sulejmanpasic'18]

$$\mathscr{S}^{(+)}E^{(0)}(N,\phi) = \mathscr{S}^{(-)}E^{(0)}(N,\phi) + S_{\mathcal{A}}\mathscr{S}^{(-)}E^{(\mathcal{A})}(N,\phi) + \ldots$$

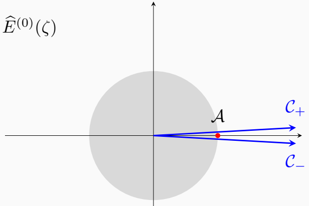
  with

$$E^{(\mathcal{A})}(N,\phi) = E^{(\mathcal{I}\bar{\mathcal{I}})}(N,\phi) = \mathrm{e}^{-\mathcal{A}/\phi}\phi^{b_N}\sum_{k=0}^{\infty} a_k^{(\mathcal{I}\bar{\mathcal{I}})}(N)\phi^k.$$

- Using $E^{(0)}(N,\phi)$, b_N and first few $a_k^{(\mathcal{I}\bar{\mathcal{I}})}$ can be extracted.



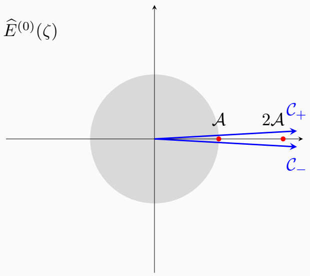
Borel singularities of $E^{(0)}(N=0,\phi)$



Resurgence analysis

- In general, there could be multi-instanton contributions to the ambiguity [Ecalle'81]

$$\mathscr{S}^{(+)}E^{(0)}(N,\phi) = \mathscr{S}^{(-)}\exp\left(\sum_{k=1}\dot{\Delta}_{k\mathcal{A}}\right)E^{(0)}(N,\phi)$$
$$= \mathscr{S}^{(-)}\left(E^{(0)}(N,\phi) + \dot{\Delta}_{\mathcal{A}}E^{(0)}(N,\phi) + \ldots\right)$$



with

$$\dot{\Delta}_{\mathcal{A}}E^{(0)}(N,\phi) = S_{\mathcal{A}}E^{(\mathcal{A})}(N,\phi),$$
$$\dot{\Delta}_{2\mathcal{A}}E^{(0)}(N,\phi) = S_{2\mathcal{A}}E^{(2\mathcal{A})}(N,\phi),$$
$$\ldots$$

so that $\dot{\Delta}_{k\mathcal{A}}$ encodes contributions from $k\mathcal{A}$ sector.

5d SYM and its resurgent properties

5d SYM and TS on local $\mathbb{P}^1 \times \mathbb{P}^1$

- Harper's equation

$$\left(e^{ix} + e^{-ix} + e^{iy} + e^{-iy}\right)\psi = E\psi. \quad [x, y] = i\phi,$$

- It is related to relativistic Toda lattice [Hatsuda,Katsura,Tachikawa'16]

$$\left(e^{x} + e^{-x} + e^{y} + e^{-y}\right)\psi = E\psi, \quad [x, y] = i\hbar,$$

by a double Wick rotation

$$(x, y, \hbar) \mapsto (ix, iy, -\phi)$$

- The latter is quantum Seiberg-Witten curve of 5d $\mathcal{N} = 1$ on $\mathbb{R}^4 \times S^1$ SYM with $G = SU(2)$ or quantum mirror curve of topological string on massless local $\mathbb{P}^1 \times \mathbb{P}^1$.
- The eigen-energy is NS Wilson loop vev in the magnetic frame [Nekrasov,Shatashvili'09; Huang,Lee,Wang'22; Wang'23]

$$E(N, \hbar) = W_{\mathbf{r}}(t_c, \hbar)$$

$\mathbf{r}$ is fundamental representation, and $t_c = \hbar\nu, \nu = N + 1/2$.

Non-perturbative corrections to NS Wilson loop

- **Borel singularities** located at $\pm\mathcal{A}_{(p,q,m)}$ with [**JG**,Marino'22]

$$\mathcal{A}_\gamma = p t_{c,D} + q t_c + 4\pi^2 \mathrm{i} m, \quad \gamma = (p, q, m),$$

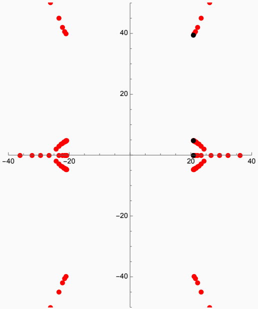
  which are conjecturally central charges of stable D-brane bound states of charge γ with $p \neq 0$.

- **Non-pert. amplitude** [**JG**,Marino'22]

$$\dot{\Delta}_{\mathcal{A}_\gamma} W = S^{\mathrm{BPS}}_{\mathcal{A}_\gamma} W^{(\mathcal{A}_\gamma)}, \quad W^{(\mathcal{A}_\gamma)} = -\frac{\hbar}{2\pi \mathrm{i}} D W \mathrm{e}^{-DF^{\#}_{\mathrm{NS}}/\hbar}$$

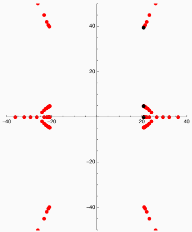
  with $D = p\partial_{t_c}$ $D = p\partial_{t_c}$ and $DF^{\#}_{\mathrm{NS},0} = \mathcal{A}_\gamma$.

- **Stokes constant** $S^{\mathrm{BPS}}_{\mathcal{A}_\gamma}$ is conjecturally counting of stable D-brane bound states $\Omega(\gamma)$ with $\langle \gamma, \gamma_c \rangle \neq 0$, where $\gamma_c = (0, 1, 0)$ is ass'd with the magnetic frame of evaluation.

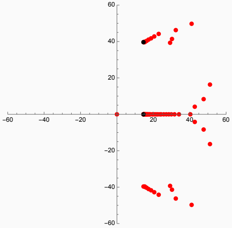


Borel singularities of $W_{\mathbf{r}}(\hbar)$ for $z = E^{-2} = 1/32$ at $\pm\mathcal{A}_{(2,0,0)}$, $\pm\mathcal{A}_{(2,\pm 1,0)}$, $\pm\mathcal{A}_{(2,0,\pm 1)}$

Double scaling limit: Borel singularities



$W_{\mathbf{r}}(\hbar)$ at $\pm\mathcal{A}_{(2,0,0)}$, $\pm\mathcal{A}_{(2,\pm1,0)}$, $\pm\mathcal{A}_{(2,0,\pm1)}$



$E^{(0)}(N=0,\phi)$ at $\mathcal{A}_{(2,0)}$, $\mathcal{A}_{(2,\pm1)}$

- In the double scaling limit

$$\hbar = -\phi, \quad t_c = -\phi\nu, \quad \phi \to 0$$

  Borel singularities become

$$\mathcal{A}_{(p,q,m)} \mapsto \mathcal{A}_{(p,m)} = pA_c + 4\pi^2 \mathrm{i} m, \quad t_{c,D} \mapsto A_c = 8C$$

- The leading singularity is $\mathcal{A}_{(2,0)} = 2A_c$.
- The set of $\mathcal{A}_{(p,q,m)}$ w/ different q collapses to a single $\mathcal{A}_{(p,m)}$.

Double scaling limit: non-pert. amplitudes

- Puzzle: Why does only one of a pair of singularities survive?
- The DS limit of non-perturbative amplitude [Codesido,Marino'17; Marino,Schwick'24]

$$\mathcal{E}^{(1)} = \frac{\hbar}{2\pi\mathrm{i}} \frac{\partial W^{(0)}(t_c,\hbar)}{\partial t_c} \exp\left(\frac{2}{\hbar}\frac{\partial}{\partial t_c} F^{\#}_{\mathrm{NS,c}}(t_c,\hbar)\right)\Bigg|_{\substack{t_c\to-\phi\nu \\ \hbar\to-\phi}}$$

$$= \frac{1}{2\pi\mathrm{i}} \frac{\partial E^{(0)}(\nu,\hbar)}{\partial \nu} \underbrace{\left(\frac{\sqrt{2\pi}}{\Gamma(\nu+1/2)}\right)^2 \left(\frac{16}{\hbar}\right)^{2\nu}}_{F^{\mathrm{sing}}} \underbrace{\mathrm{e}^{2A_c/\hbar} \exp\left(\frac{2}{\hbar}\frac{\partial}{\partial t_c} F^{\mathrm{reg}}_{\mathrm{NS,c}}(t_c,\hbar)\right)}_{F^{\mathrm{reg}}}\Bigg|_{\substack{t_c\to-\phi\nu \\ \hbar\to-\phi}}$$

- The amplitude of the other one of a pair of singularities is obtained by [Marino,Schwick'24]

$$(\phi,\nu)\to(-\phi,-\nu)$$

and it vanishes as [Marino,Schwick'24; **JG**,Xu'24]

$$\mathcal{E}^{(0|1)} \propto \left(\frac{\sqrt{2\pi}}{\Gamma(-\nu+1/2)}\right)^2 \mapsto 0$$

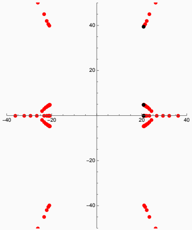
Double scaling limit: Stokes constants

- Stokes constants [Marino,Schwick'24; **JG**,Xu'24]

$$S_{(p,r)} = p \sum_{q} S^{\mathrm{BPS}}_{(p,q,r)} \mathrm{e}^{-2\pi \mathrm{i} q \nu}.$$

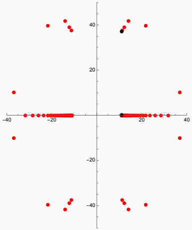
- Puzzle: But what $S^{\mathrm{BPS}}_{(p,q,r)}$? As $\phi \to 0$, $t_c = -\phi\nu \to 0$, we approach the conifold singularity, where **wall-crossing** happens.

Weak coupling regime ($z < 1/16$)



$S^{\mathrm{Weak}}_{(2,0)} = 16$

Strong coupling regime ($z > 1/16$)



$S^{\mathrm{Strong}}_{(2,0)} = 4$

Which side do you take?

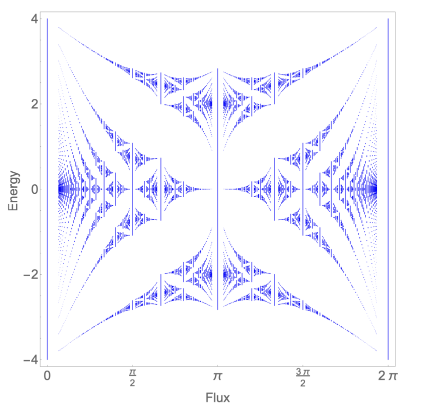
- In the case of Hofstadter butterfly, when $\phi > 0$

$$E < 4 \ \Rightarrow \ z = 1/E^2 > 1/16$$

  we approach the conifold point from the strong coupling regime.

- Numerically, by large order analysis of $E^{(0)}(N, \phi)$, we also find

$$S_{\mathcal{A}_{(2,0)}} = S^{\text{Strong}}_{(2,0)} = 4.$$



Non-perturbative corrections from supersymmetric field theory

- The non-pert. corrections at 2-instanton level from Stokes phenomenon is

$$\dot{\Delta}_{\mathcal{A}_{(2,0)}} E^{(0)}(\nu,\phi) = S_{\mathcal{A}_{(2,0)}} E^{(\mathcal{I}\overline{\mathcal{I}})}(\nu,\phi)$$

- The 2-instanton amplitude is

$$E^{(\mathcal{I}\overline{\mathcal{I}})}(\nu,\phi) = \mathcal{E}^{(1)}(\nu,\phi) = \frac{1}{2\pi \mathrm{i}} \frac{\partial E^{(0)}(\nu,\phi)}{\partial \nu} \exp\left(\frac{2}{\hbar}\frac{\partial}{\partial t_c} F_{\mathrm{NS,c}}(t_c,\hbar)\right)\Big|_{\substack{t_c \to -\phi\nu \\ \hbar \to -\phi}}$$

- The Stokes constant is

$$S_{\mathcal{A}_{(2,0)}} = S^{\mathrm{Strong}}_{(2,0)} = 4$$

Trans-series for Hofstadter butterfly

Exact WKB method

- To solve a 1d non-relativistic QM model

$$-\frac{\hbar^2}{2}\partial_x^2\psi(x) + V(x)\psi(x) = E\psi(x)$$

  one can apply the exact WKB method.

- WKB ansatz

$$\psi(x) = \exp\left(\frac{\mathrm{i}}{\hbar}\int_*^x P(x',\hbar)\mathrm{d}x'\right)$$

  where $P(x,\hbar)$ is a formal power series

$$P(x,\hbar) = \sum_{n\geq 0} P_n(x)\hbar^n$$

- $P_n(x)$ for $n \geq 1$ can be solved recursively with the initial condition $P_0(x)$, which is the classical momentum subject to

$$P_0(x)^2 = 2(E - V(x))$$

Quantum periods

- Perturbative quantum period [Voros'83]

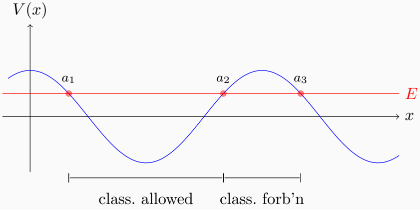
$$t(E,\hbar) = \frac{1}{\pi}\int_{a_1}^{a_2} P(x,\hbar)\mathrm{d}x$$

It is a formal power series which is *not* Borel summable for $\hbar > 0$.

- Non-perturbative quantum period

$$t_D(E,\hbar) = -2\mathrm{i}\int_{a_2}^{a_3} P(x,\hbar)\mathrm{d}x$$

It is a formal power series which *is* Borel summable for $\hbar > 0$.



All-orders Bohr-Sommerfeld QC

- The naive Bohr-Sommerfeld QC

$$t(E) = \mathrm{Vol}(E) = \frac{1}{\pi}\int_{a_1}^{a_2} P_0(x)\mathrm{d}x = \hbar\nu, \ \nu \in \mathbb{N} + 1/2,$$

- The all-orders Bohr-Sommerfeld QC

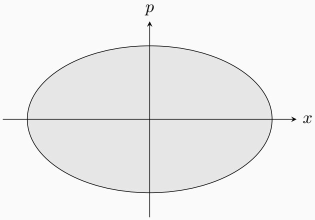
[Dunham,Bender,Robnik,Romanovski,. . .]

$$t(E, \hbar) = \mathrm{Vol}(E, \hbar) = \hbar\nu.$$

is equivalent to

$$\mathrm{e}^{2\pi\mathrm{i}t(E,\hbar)/\hbar} + 1 = 0.$$

from which the perturbative energy series $E^{(0)}(\nu, \hbar)$ can be calculated.



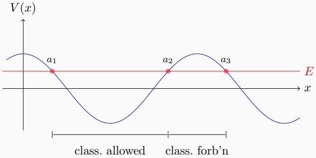
Exact quantisation conditions

- By using appropriate boundary conditions and connection formulas, the Exact Quantisation Conditions (EQCs) for many 1d QM models have been written down.
  [Delabaere,Zinn-Justin,Jentschura,Alvarez,Dunne,Ünsal,. . .]
- The EQCs usually take the form of

$$1+\mathcal{V}_A = f(\mathcal{V}_A,\mathcal{V}_B) \xrightarrow{\hbar\to 0} 0$$

with Voros symbols

$$\mathcal{V}_A = \mathrm{e}^{2\pi\mathrm{i}\frac{t(E,\hbar)}{\hbar}}, \quad \mathcal{V}_B = \mathrm{e}^{-\frac{t_D(E,\hbar)}{\hbar}}.$$



Exact quantisation conditions

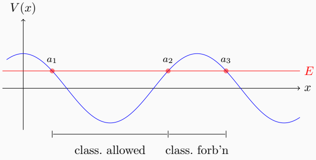
- As an example, the **Cosine Model** with Schrödinger equation

$$-\frac{\hbar^2}{2}\partial_x^2\psi(x) + \cos(x)\psi(x) = E\psi(x)$$

has the EQCs

$$1 + \mathcal{V}_A^{\mp 1}(1 + \mathcal{V}_B) - 2\sqrt{\mathcal{V}_A^{\mp 1}\mathcal{V}_B}\cos\theta = 0$$

for respectively $\mathrm{Im}\hbar > 0$ and $\mathrm{Im}\hbar < 0$.



- The two EQCs are equivalent, connected by the Delabaere-Dillinger-Pham formula [Delabaere,Dillinger,Pham'93]

$$\mathscr{S}^{(+)}\mathcal{V}_A = \mathscr{S}^{(-)}\mathcal{V}_A(1 + \mathcal{V}_B)^2$$

Structure of full energy trans-series

- The EQCs imply a **universal structure** of the full trans-series [van Spaendonck,Vonk'23]

$$E_{\theta_{x,y},\epsilon}(\nu,\hbar) = E^{(0)}(\nu,\hbar) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\theta_{x,y},\epsilon)E^{(n,m)}(\nu,\hbar)$$

- The *n*-**instanton amplitudes** have a model-independent expression

$$E^{(n,m)} = \left(\frac{\partial}{\partial\nu}\right)^m \left(\frac{\partial E^{(0)}(\nu,\hbar)}{\partial\nu} \mathrm{e}^{-n\, t_D(\nu,\hbar)/\hbar}\right), \quad t_D(\nu,\hbar) = A_c + \mathcal{O}(\hbar)$$

- The **trans-series coefficients** are model-dependent and are solved from the specific EQCs. They also depend on Stokes sign $\epsilon = \pm 1$ such that

$$E^{\text{ex}}_{\theta_{x,y}}(N,\hbar) = \mathscr{S}^{(\pm)} E_{\theta_{x,y},\pm 1}(N,\hbar)$$

Full trans-series for Hofstadter butterfly

- **Conjecture**: the same universal structure of full trans-series also applies for relativistic 1d QM models such as Hofstadter butterfly. [**JG**,Xu'24]

$$E_{\theta_{x,y},\epsilon}(\nu,\hbar) = E^{(0)}(\nu,\hbar) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\theta_{x,y},\epsilon)E^{(n,m)}(\nu,\hbar)$$

- The n-instanton amplitudes are

$$E^{(n,m)} = \left(\frac{\partial}{\partial\nu}\right)^m \left(\frac{\partial E^{(0)}(\nu,\phi)}{\partial\nu}\mathrm{e}^{-n\,t_D(\nu,\hbar)/\phi}\right),$$

- In the case of Hofstadter butterfly

$$E^{(0)}(\nu,\phi) = W(t_c,\hbar)\Big|_{\substack{t_c\to-\phi\nu,\\ \hbar\to-\phi}}$$

$$t_D(\nu,\phi) = \partial_{t_c}F_{\mathrm{NS,c}}(t_c,\hbar)\Big|_{\substack{t_c\to-\phi\nu\\ \hbar\to-\phi}}$$

The trans-series coefficiensts $u_{n,m}$ are

$n\backslash m$	0	1	2
1	$\frac{\Theta}{\pi}$		
2	$\frac{\mathrm{i}\epsilon}{\pi}$	$\frac{\Theta^2}{2\pi^2}$	
3	$-\frac{\Theta}{\pi}+\frac{\Theta^3}{6\pi}$	$\frac{\mathrm{i}\epsilon\Theta}{\pi^2}$	$\frac{\Theta^3}{6\pi^3}$

$$\Theta = (-1)^{N+1}(\cos\theta_x + \cos\theta_y)$$

Decomposition of trans-series

- The full trans-series can be decomposed in terms of two trans-series [van Spaendonck,Vonk'23]

$$\text{full} = \text{minimal} \otimes \text{medium}$$

Such that

medium trans-series

$$E_{\theta_{x,y},\epsilon}(\nu,\phi) = E^{(0)}_{\text{min}}(\nu,\phi;\epsilon S_{\mathcal{A}}/2) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} w_{n,m}(\theta_{x,y})\, E^{(n,m)}_{\text{min}}(\nu,\phi;\epsilon S_{\mathcal{A}}/2)$$

minimal trans-series   minimal trans-series

- Minimal trans-series (Stokes data)

$$E^{(0)}_{\text{min}}(\nu,\phi;\sigma) = E^{(0)}(\nu,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} v_{n,m}(\sigma)E^{(2n,m)}(\nu,\phi)$$

- Medium trans-series (symmetry data)

$$E^{(0)}_{\text{med}}(\nu,\phi;\theta_{x,y}) = E^{(0)}(\nu,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} w_{n,m}(\theta_{x,y})E^{(n,m)}(\nu,\phi)$$

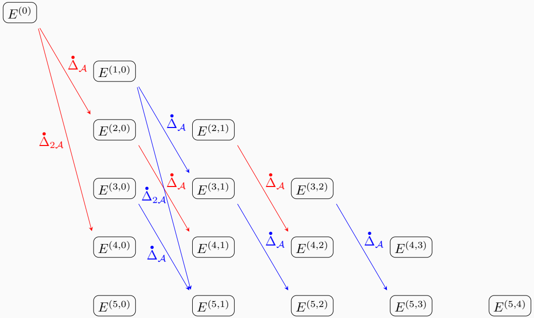
Minimal resurgent structure

- Suppose the Voros symbols satisfy the DDP formula for positive $\hbar$

$$\mathscr{S}^{(+)}\mathcal{V}_A = \mathscr{S}^{(-)}\mathcal{V}_A(1+\mathcal{V}_B)^{S_\mathcal{A}}$$

- The minimal resurgent structure along $\mathbb{R}_+$ is [van Spaendonck, Vonk'23]

$$\dot{\Delta}_{\ell\mathcal{A}} E^{(0)}(\nu,\phi) = \frac{S_\mathcal{A}}{2\pi \mathrm{i}}\frac{(-1)^{\ell-1}}{\ell} E^{(2\ell,0)}(\nu,\phi),$$
$$\dot{\Delta}_{\ell\mathcal{A}} E^{(n,m)}(\nu,\phi) = \frac{S_\mathcal{A}}{2\pi \mathrm{i}}\frac{(-1)^{\ell-1}}{\ell} E^{(n+2\ell,m+1)}(\nu,\phi).$$



Minimal trans-series

- The minimal trans-series is defined by

$$E^{(0)}_{\text{min}}(\nu,\phi;\sigma) = E^{(0)}(\nu,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \sigma^{m+1} v_{n,m} E^{(2n,m)}(\nu,\phi)$$

with

$$v_{n,m} = \frac{1}{n!} B_{n,m+1}(1!s_1, 2!s_2, \ldots),\ s_k = \frac{(-1)^{k-1}}{k\cdot 2\pi\mathrm{i}}.$$

- The minimal trans-series encodes entirely the **Stokes data**, and it has the property that

$$\mathscr{S}^{(+)} E^{(0)}_{\text{min}}(\sigma) = \mathscr{S}^{(-)} E^{(0)}_{\text{min}}(\sigma + S_{\mathcal{A}})$$

- For any $\phi > 0$, the Borel resummation of $E^{(0)}_{\text{min}}$ is **ambiguity-free** and **real**

$$\mathscr{S}^{(+)} E^{(0)}_{\text{min}}(-S_{\mathcal{A}}/2) = \mathscr{S}^{(-)} E^{(0)}_{\text{min}}(+S_{\mathcal{A}}/2)\ \in \mathbb{R}_+$$

In the case of Hofstadter butterfly, $S_{\mathcal{A}} = 4$.

Medium and full trans-series

- The medium trans-series is defined by

$$E^{(0)}_{\text{med}}(\nu,\phi;\theta_{x,y}) = E^{(0)}(\nu,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} w_{n,m}(\theta_{x,y}) E^{(n,m)}(\nu,\phi)$$

so that the full trans-series as

$$E_{\theta_{x,y},\sigma}(\nu,\phi) = E^{(0)}_{\text{min}}(\nu,\phi;\sigma) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} w_{n,m}(\theta_{x,y}) E^{(n,m)}_{\text{min}}(\nu,\phi;\sigma)$$

satisfies

$$E^{\text{ext}}_{\theta_{x,y}}(\nu,\phi) = \mathscr{S}^{(\pm)} E_{\theta_{x,y},\mp s_{\mathcal{A}}/2}(\nu,\phi)$$

- The medium trans-series encodes dependence on additional parameters (**symmetry data**) which arise due to symmetries of the model.
- They can be solved from the EQCs of the *Voros* type

$$1 + \mathcal{V}_A = f(\mathcal{V}_A, \mathcal{V}_B).$$

Full trans-series

- In the case of Hofstadter butterfly, an EQC of the Voros type is *unknown*.
- By comparing with numerical spectrum, we computed $w_{n,m}(\theta_{x,y})$ up to 6-instanton order for $\phi = 2\pi/Q$.
- Conjecturally [**JG**,Xu'24]

$$w_{n,m} = \frac{1}{n!} B_{n,m+1}(1! r_1, \ldots)$$

with the generating series

$$\sum_{j\geq 1} r_j \lambda^j = \frac{1}{\pi} \arcsin \frac{\Theta}{\lambda + \lambda^{-1}}.$$

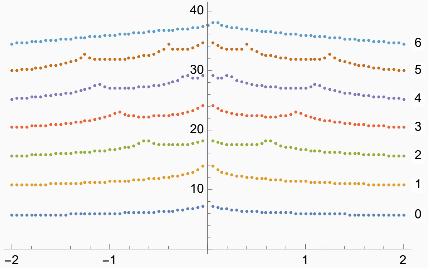
$w_{n,m}$ for $\phi = 2\pi/Q$

m	0	1	2
$w_{1,m}$	$\frac{\Theta}{\pi}$		
$w_{2,m}$	0	$\frac{\Theta^2}{2\pi^2}$	
$w_{3,m}$	$-\frac{\Theta}{\pi} + \frac{\Theta^3}{6\pi}$	0	$\frac{\Theta^3}{6\pi^3}$

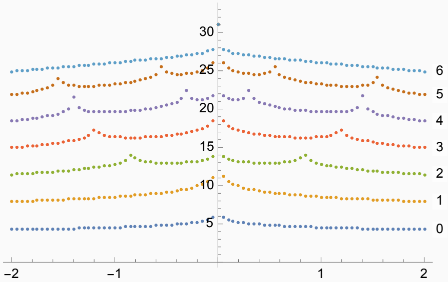
$\Theta = (-1)^{N+1}(\cos\theta_x + \cos\theta_y)$

Full trans-series

- Number of matching digits between $E^{\text{ext}}_{\theta_{x,y}}(N,\phi)$ and $\mathscr{S}^{(\pm)}E_{\theta_{x,y},\mp 2}(N,\phi)$ as a function of Θ with increasing instanton orders.



$\phi = 2\pi/13, N = 0$



$\phi = 2\pi/13, N = 1$

Conclusion and discussion

- We have found the **full energy trans-series** for Hofstadter model when $\phi = 2\pi/Q$

$$E_{\theta_{x,y},\mp S_{\mathcal{A}}/2}(N,\phi) = E^{(0)}_{\min}(\mp S_{\mathcal{A}}/2) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} w_{n,m}(\theta_{x,y}) E^{(n,m)}_{\min}(\mp S_{\mathcal{A}}/2)$$

  with $S_{\mathcal{A}} = 4$ and explicit formulas for $E^{(*)}_{\min}$ and $w_{n,m}$, by exploiting relationship between Hofstader with 5d $\mathcal{N} = 1$ SYM.
- Conversely, it implies the EQCs of the Voros type [**JG**,Xu'24]

$$1 + \mathcal{V}_A^{\mp 1}(1+\mathcal{V}_B)^2 - 2\sqrt{\mathcal{V}_A^{\mp 1}\mathcal{V}_B}(\cos\theta_x + \cos\theta_y) = 0.$$

  for respectively $\mathrm{Im}\phi > 0$ and $\mathrm{Im}\phi < 0$.
- The two EQCs are equivalent, due to the DDP formula

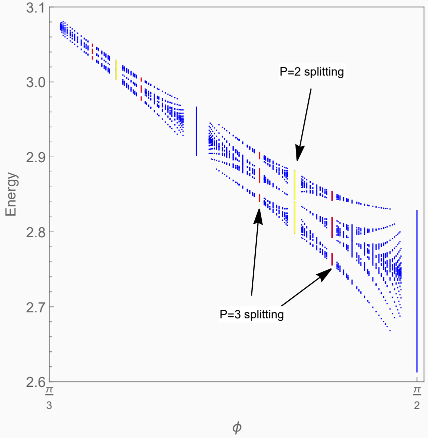
$$\mathscr{S}^{(+)}\mathcal{V}_A = \mathscr{S}^{(-)}\mathcal{V}_B(1+\mathcal{V}_B)^4.$$

Discussion

- Unfortunately, the medium trans-series coefficients $w_{n,m}$ do not work for generic $\phi = 2\pi P/Q$, let alone irrational ϕ.
- For $\phi = 2\pi P/Q$ with $P > 1$, one primary energy band splits to P sub-bands [JG,Xu'24]

$$\cos\theta_x + \cos\theta_y = \frac{1}{2} F_{Q/P}(2\pi w_{1,0}, 0, 0)$$

- This indicates an S-symmetry $\phi \to 4\pi^2/\phi$.



Discussion

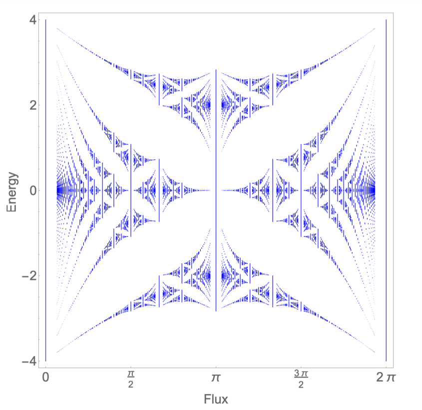
- To study the fractal structure, it is beneficial to perform perturbative expansion at $\phi = 2\pi P/Q$.
- Consider expansion at $\frac{\phi}{2\pi} = 1/n_1$, say $1/n_1 = 1/3$, and using $\tilde{\phi}$ defined by

$$\frac{\phi}{2\pi} = \frac{1}{3 - \frac{\tilde{\phi}}{2\pi}}$$

as expansion parameter [JG,Xu'24]

$$\mathrm{bw}_N(\tilde{\phi}) \sim e^{-8C/\tilde{\phi}}$$

- This also indicates the S-symmetry $\phi \to 4\pi^2/\phi$.



Outlook

- Coefficients of medium trans-series at generic $\phi/(2\pi) = P/Q$.
- Perturbative expansion at $\phi/(2\pi) = P/Q$ and its non-perbative corrections.
- First principle derivation of the EQCs of the Voros type. [Pietro's talk]
- Other models such as triangular lattice or honeycomb lattice.
- Universal energy trans-series structure for other relativistic QM models.

Thank you for your attention!