Integral Models of Quantum Invariants of Three-Manifolds

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Overview

1. Integral Representation of WRT Invariants
2. AEC for $\mathrm{WRT}_k(\Sigma(p_1, ..., p_n))$
3. Integral Representation of GPPV Invariants

The results I will be presenting are based on joint projects with

1. J. E. Andersen and S. Hindson,
2. J.E. Andersen, L. Han, Y. Li, D. Sauzin and S. Sun and
3. Y. Murakami.

Classical Chern-Simons Theory

Let $G = \mathrm{SU}(2)$ with Lie algebra $\mathfrak{g}$. Let M be a 3-manifold and let $\mathcal{A} \cong \Omega^1(M, \mathfrak{g})$ denote the space of G-connections on the trivial principal G-bundle

$$M \times G \to M.$$

Let $\mathcal{G} \cong C^\infty(M, G)$ denote the space of gauge transformations. Consider the Chern-Simons action $\mathrm{CS} : \mathcal{A}/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$ given by

$$\mathrm{CS}([A]) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \mod \mathbb{Z}.$$

For a G-connection A, let $F_A := dA + \frac{1}{2}[A \wedge A]$ be the curvature. Then A solves the Euler-Lagrange equation if and only if A is flat

$$\delta_A \mathrm{CS} = 0 \iff F_A = 0.$$

Set

$$\mathcal{M}_{\mathrm{Flat}}(M, G) = \{A \in \mathcal{A} : F_A = 0\}/\mathcal{G}.$$

The Witten-Reshetikhin-Turaev Invariant $\mathrm{WRT}_{\mathrm{k}}(M, L, \mu)$

Let $k \in \mathbb{Z}_+$. Let W_+ denote the set of dominant integral positive weights of $\mathfrak{g}$, let θ be the longest root (normalized: $\langle\theta,\theta\rangle = 2$) and

$$\Lambda_k := \{\lambda \in W_+ : 0 \leq \langle \lambda, \theta\rangle \leq k\}.$$

For a triple (M, L, μ) of a 3-manifold M with a framed oriented link L and a labelling $\mu \in \Lambda_k^{\pi_0(L)}$, consider the level-$k$ WRT invariant

$$\mathrm{WRT}_{\mathrm{k}}(M, L, \mu) \in \mathbb{C}.$$

This is the mathematical model of Witten's path integral formula for the expectation of the Wilson operator in Chern-Simons theory (where the inclusion $\Lambda_k \hookrightarrow Rep_{f.d}(G)$ is implicit):

$$\int_{A \in \mathcal{A}/\mathcal{G}} e^{2\pi i k \,\mathrm{CS}(A)} \prod_{K \in \pi_0(L)} \mathrm{tr}(\tilde{\mu}_K \circ \mathrm{Hol}_A(K)) \mathcal{D}A.$$

Witten argued that this gives a geometric extension of the colored Jones polynomial to links in general 3-manifolds.

Witten's Asymptotic Expansion Conjecture

Consider the case $L = \emptyset$. Recall: The space of classical solutions $\delta\,\mathrm{CS}_A = 0$ is given by the moduli space of flat G-connections

$$\mathcal{M}_{\mathrm{Flat}}(M) := \{[A] \in \mathcal{A}/\mathcal{G} : F_A = 0\}.$$

This is a compact space, and we define the finite set

$$\mathrm{CS}(M) := \mathrm{CS}(\mathcal{M}_{\mathrm{Flat}}(M)) \subset \mathbb{R}/\mathbb{Z}.$$

Set $r = k + 2$. Motivated by the path integral picture:

Conjecture 1 (The asymptotic expansion conjecture)

For each $\theta \in \mathrm{CS}(M)$ there exists $W_\theta(x) \in \cup_{m=1}^{\infty} \mathbb{C}((x^{1/m}))$ such that the following large k asymptotic expansion holds

$$\mathrm{WRT}_k(M) \sim \sum_{\theta \in \mathrm{CS}(M)} e^{2\pi i r\theta} W_\theta(r^{-1}).$$

The AEC for Three-Manifolds with colored Links (M, L, μ)

Fix $k_0 \in \mathbb{Z}_+$, $\mu_0 \in \Lambda_{k_0}^{\pi_0(L)}$. Let $k = sk_0, s \in \mathbb{Z}_+$, and set $\mu_k = s\mu_0$. Let C_G be the space of conjugacy classes of G. Set $c := \iota_{k_0}(\mu_0) \in C_G^{\pi_0(L)}$ where $\iota_k : \Lambda_k \hookrightarrow C_G^{\pi_0(L)}$ is given by

$$\iota_k(\lambda) = [\exp(\lambda/k)].$$

Let $\mathcal{M}_{\mathrm{Flat}}(M, L, c)$ be the moduli space of flat G-connections on $M \setminus L$ with meridional holonomy around a component $K \in \pi_0(L)$ within the conjugacy class $c_\mu(K)$. Set

$$\mathrm{CS}(M, L, c) := \mathrm{CS}(\mathcal{M}_{\mathrm{Flat}}(M, L, c_\mu)) \subset \mathbb{R}/\mathbb{Z}.$$

Conjecture 2 (The asymptotic expansion conjecure (AEC))

For each $\theta \in \mathrm{CS}(M, L, c)$ there exists $W_\theta(x) \in \cup_{m=1}^{\infty} \mathbb{C}((x^{1/m}))$:

$$\mathrm{WRT_k}(M, L, \mu_k) \sim \sum_{\theta \in \mathrm{CS}(M,L,\mu)} e^{2\pi i r\theta} W_\theta(r^{-1}), \text{ as } k \text{ tends to } +\infty.$$

The AEC is open in general, but proven in some cases including:

- Lens spaces, torus bundles by work of Jeffrey, Garoufalidis, Andersen–Jørgensen.
- Mapping tori of surface diffeomorphisms of finite order in the mapping class group by work of Andersen, Andersen–Jørgensen–Himpel–Martens–McLellan.
- The mapping tori of a surface diffeomorphism $\varphi : \Sigma \to \Sigma$ of two-manifolds for which $\mathcal{M}_{\mathrm{Flat}}(\Sigma)^{\varphi}$ is smooth, by work of Charles, Andersen–M, Ios.
- Surgeries on the figure 8 knot due to work of Charles–Marche, and Andersen–M (unpublished) building on Andersen-Hansen.
- **New: Seifert fibered integral homology spheres** by work of Andersen–Han–Li–M–Sauzin-Sun as explained later today. .

Analytic Continuation and Chern-Simons Theory

- **Analytic Continuation:** Witten and Garoufalidis independently argued that resurgence appears through analytic continuation of the partition function as a function of k

$$k \mapsto \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k\,\mathrm{CS}} \mathcal{D}.$$

- **Quest:** We seek an analytic continuation of WRT invariants of triples (M, L, μ_k) as a function of the level

$$k \mapsto \mathrm{WRT_k}(M, L, \mu_k)$$

which illuminates the asymptotic expansion conjecture and the connection to Chern-Simons with complex gauge group

$$G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C}).$$

- **Approach:** R-matrix formula for WRT invariants and analytic continuation of the R-matrix via Faddeev's quantum dilog.

The Colored Jones Polynomial $J(L, \mu, t_r) \in \mathbb{Z}[t_r^{\pm}]$

Colored Jones: Set $t_r = e^{\pi i/(2r)}$. For a framed oriented link $L \subset S^3$ and $\mu \in \Lambda_k^{\pi_0(L)}$ consider the colored Jones polynomial

$$J(L, \mu, t_r) \in \mathbb{Z}[t_r^{\pm}].$$

Quantum Integers: Identify $\Lambda_k = \{1, ..., r-1\}$. For $n \in \Lambda_k$ define the quantum integer and quantum factorial

$$[n]_r := (t_r^{2n} - t_r^{-2n})/(t_r^2 - t_r^{-2}), \quad [n]_r! := \prod_{j=1}^{n} [j]_r.$$

WRT-prefactor: Let σ_L be the signature of the linking matrix and

$$\alpha_r(L) := \exp\left(\frac{\sigma_L 3\pi i(2-r)}{4r}\right)\left(\sqrt{\frac{2}{r}}\sin\left(\frac{\pi}{r}\right)\right)^{\pi_0(L)+1}.$$

Surgery: There exists a pair of disjoint framed oriented links $L_i \subset S^3, i = 1, 2,\ \nu(L_1) \cong \cup_{j=1}^m (S^1 \times B^2)_j$, with an orientation-preserving diffeomorphism $(M, L) \cong (S^3_{L_1}, L_2)$. Here

$$S^3_{L_1} := \left(S^3 \setminus \nu(L_1) \bigcup_{j=1,\ldots,m,} (B^2 \times S^1)_j \right) / \sim,$$

where, for $j = 1, \ldots, m$, we identify $\partial(S^1 \times B^2)_j = \partial(B^2 \times S^1)_j$.

Definition 1 (Reshetikhin–Turaev)

$$\mathrm{WRT}_\mathrm{k}(M, L, \mu) = \alpha_r(L_1) \sum_{\lambda \in \Lambda_k^{\pi_0(L_1)}} J_{\lambda \cup \mu}(L_1 \cup L_2, t_r) \prod_{K \in \pi_0(L_1)} [\lambda_K]_r.$$

The pair L_1, L_2 is unique up to Kirby equivalence, and Reshetikhin and Turaev proved $\mathrm{WRT}_\mathrm{k}(M, L, \mu)$ is an invariant of the Kirby class.

Let $U = U_{q_r}(\mathfrak{sl}(2, \mathbb{C})), q_r = e^{2\pi i/r}$. Let $R \in U \otimes U$ be the universal R-matrix. For $m \in \Lambda_k$, let $l_m = (m-1)/2$, let $B_m = \{-l_m, \ldots, l_m\}$ and let $V^{(m)} = \mathbb{C}^{B_m}$ with basis $(e_j^{(m)})_{j \in B_m}$. For $n, m \in \Lambda_k$, the R-matrix induces an isomorphism

$$C_{r,n,m} : V^{(n)} \otimes V^{(n)} \to V^{(m)} \otimes V^{(n)}.$$

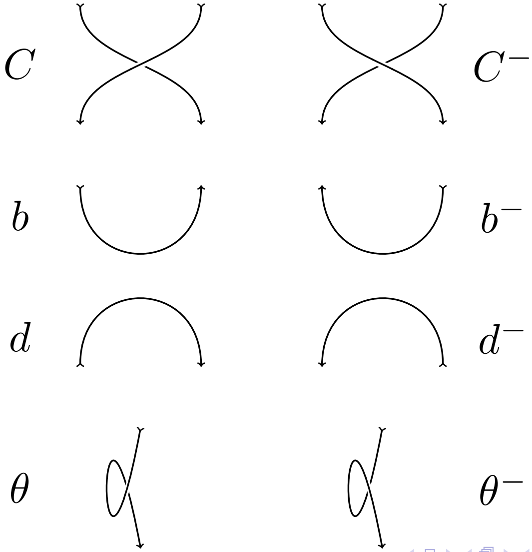
For all $i, w \in B_n$ and $j, v \in B_m$, define $C_r^{\pm}(n, m, i, j, v, w) \in \mathbb{C}$ to be the matrix coefficient of $C_{r,n,m}$ w.r.t. these basis elements, i.e. (second equation is proven by Reshetikhin-Turaev)

$$C_r^{\pm}(n, m, i, j, v, w) := (e_v^{(m)} \otimes e_w^{(n)})^* C_{r,n,m}^{\pm}\left(e_i^{(n)} \otimes e_j^{(m)}\right) =$$

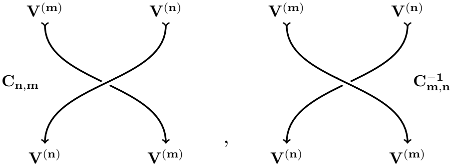
$$\begin{cases} \frac{(\pm(s-\bar{s}))^{\pm(w-i)}}{[\pm(w-i)]!} \frac{[l_n \pm w]!}{[l_n \pm i]!} \frac{[l_m \mp v]!}{[l_m \mp j]!} t_r^{P_{\pm}(i,j,v,w)} \delta_{v+w}^{i+j}, & \text{if } \pm(w-i) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$P_{\pm}(i, j, v, w) = \pm(4ij - 2(w-i)(i-j) - (w-i)(w-i \pm 1)).$$

The R-matrix approach to $J(L, \mu, t_r)$ is combinatorial and depends on a decomposition of a link diagram into elementary tangles



In the R-matrix approach to the colored Jones polynomial $J(L,\mu,t_r)$, one associates R-matrices to crossings:



Let E be the set of edges of the graph of a link diagram D of L. Recall B_m is the basis index of $V^{(m)}$ for $m \in \Lambda$. Set

$$B_\mu := \bigtimes_{e\in E} B_{\mu_e}.$$

Each crossing $c \in C$ has a sign $\epsilon_c \in \{\pm\}$, two colors $\mu_c \in \Lambda_k^2$ and four edges $x_c \in E^4$. The R-matrix approach equals $J(L,\mu,t_r)$ to

$$\sum_{x\in B_\mu}\prod_{c\in C} C_r^{c_\epsilon}(\mu_c,x_c)\prod_{\theta\in \text{Twists}} t_r^{\epsilon_\theta(\lambda_\theta^2-1)}\prod_{\cup\in\bigcup^-} t_r^{-4x_\cup}\prod_{\cap\in\bigcap^-} t_r^{4x_\cap}.$$

Faddeev's quantum dilog: Let $\gamma = \pi/r, \in \mathbb{C}, \mathrm{Re}(r) \geq 2$ and set

$$S_\gamma(z) := \exp\left(\int_{\mathbb{R}(+)} \frac{e^{zy}}{4\sinh(\pi y)\sinh(\gamma y)y} dy\right), |\mathrm{Re}(z)| < \gamma + \pi,$$

$$\tilde{R}_r(z) := \exp\left(\frac{\pi i z(z+1)}{2r}\right) \frac{S_\gamma\left(\pi - (2z+1)\gamma\right)}{S_\gamma\left(\pi - \gamma\right)}.$$

The following equations holds, where (1) holds for $\zeta \in \mathbb{C}$ with $|\mathrm{Re}(\zeta)| < \pi$ and (2) is a consequence and holds for $m \in \Lambda_k$

$$(1 + e^{i\zeta})S_\gamma(\zeta + \gamma) = S_\gamma(\zeta - \gamma), \tag{1}$$

$$[m]_r! = \tilde{R}_r(m)((i2\sin(\gamma)))^{-m}. \tag{2}$$

The first extends S_γ (and consequently $\tilde{R}_r$) to a meromorphic function on all of $\mathbb{C}$. We have

Pole divisor of $z \mapsto \tilde{R}_r(z)$: $\mathcal{P}(r) = \mathbb{Z}_{\leq -1} + r\mathbb{Z}_{\leq 0}$,
Zero divisor of $z \mapsto \tilde{R}_r(z)$: $\mathcal{Z}(r) = \mathbb{Z}_{\geq 0} + r\mathbb{Z}_{\geq 1}$.

Let $y = (y_1, y_2)$, resp. $x = (x_1, x_2, x_3, x_4)$, be coordinates on $\mathbb{C}^2$, resp. $\mathbb{C}^4$. Set $z_j = (y_j - 1)/2$ and $\tilde{x} = x_1 + x_2 - x_3 - x_4$.

$$Q_r^{\pm}(x) := \pm\frac{2x_1x_2 - (x_4 - x_1)(x_1 - x_2)}{r} + \frac{x_3 - x_2 \mp (x_1 - x_4)}{2},$$

$$R_r^{\pm}(y,x) := \frac{\tilde{R}_r(z_1 \pm x_4)\tilde{R}_r(z_2 \mp x_3)\exp\left(\pi i Q_r^{\pm}(x)\right)}{\tilde{R}_r(\pm(x_4 - x_1))\tilde{R}_r(z_1 \pm x_1)\tilde{R}_r(z_2 \mp x_2)} \times \frac{\tilde{R}_r(0)}{\tilde{R}_r(-(\tilde{x})^2)}.$$

Lemma 2 (Andersen–M–Hindson)

For all $n, m \in \Lambda_k, i, w \in B_n, j, v, \in B_m$ it holds that

$$C_r^{\pm}(n,m)_{i,j}^{v,w} = R_r^{\pm}(n,m,i,j,v,w).$$

The right hand side is a meromorphic function of (n, m, i, j, v, w) and depends analytically on r.

Recall the surgery link $(L_1, L_2) : (M, L) \cong (S^3_{L_1}, L_2)$. Let $D = D_1 \cup D_2$ be a diagram of $L_1 \cup L_2$. Set $\mathbb{C}_y^{\pi_0(L_1)} \times \mathbb{C}_x^E = \mathbb{C}^d$. Define $\Omega_r(D, D_2, \mu) \in \mathcal{M}(\wedge^d(\mathbb{C}^d))$ to be equal to

$$\alpha_r(L_1) \prod_{c\in C} R_r^{\epsilon_c}(y_c, x_c) \prod_{k\in K} t_r^{\epsilon_k(y_k^2-1)} \prod_{\cup\in\bigcup^-} t_r^{-4x_\cup} \prod_{\cap\in\bigcap^-} t_r^{4x_\cap}$$

$$\bigwedge_{e\in E} \frac{\cot(\pi(x_e - z_e))dx_e}{2\pi i} \bigwedge_{K\in\pi_0(L_1)} \frac{\cot(\pi y_K)\sin(\gamma y_K)dy_K}{\sin(\gamma)2\pi i}_{|y_2=\mu}.$$

Let $S_r \subset \mathbb{C}^{\pi_0(L_1)}$ be the $|\pi_0(L)|$-torus enclosing $[1, r-1]^{|\pi_0(L_1)|}$. Let $T_r \subset \mathbb{C}^d$ be the fibre bundle over S_r with fibre over $y \in S_r$ given by the $|E|$-torus enclosing $\bigtimes_{e\in E}[-z_e(y), z_e(y)]$.

Theorem 3 (Andersen–M–Hindson)

$$\mathrm{WRT}_k(M, L, \mu) = \int_{T_r} \Omega_r(D, D_2, \mu).$$

The semi-classical approximation of Faddeev's quantum dilog:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u} du,$$

$$S_\gamma(\zeta) \approx \exp\left(\frac{r}{2\pi i}\,\mathrm{Li}_2(-e^{i\zeta})\right).$$

Using this approximation leads to a formal semi-classical approximation of $\mathrm{WRT}_\mathrm{k}(M, L, \mu_k)$ of the form

$$I(r) = \int \exp(2\pi i r W(D, D_2, \mu))\,\mathrm{Vol}$$

where $W(D, D_2, \mu)$ is holomorphic. Let $\Sigma(D, D_2, \mu)$ denote the set of critical values of the phase function $W(D, D_2, \mu)$.

Conjecture 3 (Andersen–M–Hindson)

$$\mathrm{CS}(M, L, G_\mathbb{C}, c) \subset \Sigma(D, D_2, \mu) \mod \mathbb{Z}.$$

This is expected from the resurgence picture and builds on work of Yoon, and on work of Garoufalidis–Thurston–Zickert.

The AEC for Seifert Fibered Homology Spheres

Let $n \geq 3$, let $(p_j,q_j)_{j=1}^n \subset \mathbb{Z}_+ \times (\mathbb{Z} \setminus \{0\})$ with $(p_j,p_l) = (p_j,q_j) = 1$ and $P\sum_{j=1}^n \frac{q_j}{p_j} = 1$, where $P := p_1 \cdots p_n$. Consider the associated Seifert fibered homology sphere

$$X = \Sigma(p_1,\ldots,p_n).$$

Theorem 4 (Andersen–Han–Li–M–Sauzin–Sun)

The AEC holds for X, where $n \geq 3$. Moreover, for each $\theta \in \mathrm{CS}(X) \setminus \{0\}$, we have that $r^{-1/2}W_\theta(r^{-1}) \in \mathbb{C}[r]$.

Our proof builds on work of Andersen-M and Han–Li–Sauzin–Sun, and involves the Gukov–Putrov–Pei–Vafa invariant:

$$\widehat{Z}(X,q) \in q^{\Delta_X}\mathbb{Z}[[q]], \Delta_X \in \mathbb{Q}.$$

In the following, we outline the proof.

The Lawrence–Rozansky formula

Let y be a complex variable, set $g(y) := iy^2/(8\pi P)$ and define

$$F(y) = (e^{y/2} - e^{-y/2})^{2-r} \prod_{j=1}^{r} (e^{y/(2p_j)} - e^{-y/(2p_j)}).$$

Consider the oriented contour $C' := \mathbb{R}e^{\pi i/4} \subset \mathbb{C}$. LR showed

$$\mathrm{WRT}_k(X) = \int_{C'} \frac{F(y)e^{rg(y)}}{2\pi i} dy - \sum_{m=1}^{2P-1} \mathrm{Res}\left(\frac{F(y)e^{rg(y)}}{1 - e^{-ry}}, y = 2\pi i m \right).$$

There exists polynomials $p_m(x) \in \mathbb{C}[x]$ such that

$$-\sum_{m=1}^{2P-1} \mathrm{Res}\left(\frac{F(y)e^{rg(y)}}{1 - e^{-ry}}, y = 2\pi i m \right) = \sum_{m=1}^{2P-1} e^{rg(2\pi i m)} p_m(r).$$

Let W_0 be the Ohtsuki series of X. Let $\mathcal{B}$ be the Borel transform. Set $q = e^{2\pi i\tau}, \tau \in \mathbb{H}$. Set $c = \sqrt{2\pi i P}$. Define $\left(\tilde{\chi}(m)\right)_{m=m_0}^{\infty} \subset \mathbb{Z}$ by

$$G(z) := (z^P - z^{-P})^{2-n} \prod_{j=1}^{n} (z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}}) = (-1)^n \sum_{m=1}^{\infty} \tilde{\chi}(m) z^m.$$

Joint with Andersen we showed in a paper from 2022 the following

$$\mathcal{B}(W_0)(\zeta) = \frac{4c}{\pi i \sqrt{\zeta}} G\left(e^{\frac{c\sqrt{\zeta}}{P}}\right), \quad \mathrm{CS}_{\mathbb{C}}^*(X) = \frac{i}{2\pi} \mathcal{P}(\mathcal{B}(W_0)) \mod \mathbb{Z},$$

$$\widehat{Z}(X, q) = \sum_{m=1}^{\infty} \tilde{\chi}(m) q^{\frac{m^2}{2P}} = v.p. \frac{\lambda}{\sqrt{\tau}} \int_{i\mathbb{R}_+} e^{-\frac{\xi}{\tau}} \mathcal{B}(W_0)(\xi) \,\mathrm{d}\,\xi,$$

$$\mathrm{WRT}_k(X) = \lim_{q \to e^{2\pi i/r}} \widehat{Z}(X, q).$$

This builds on Lawrenze–Rozansky 95, Lawrenze–Zagier 97 and Gukov–Mariño–Putrov 16.

The above theorem shows in particular, that $\widehat{Z}(M, q)$ is essentially of the following form, where $j, N \in \mathbb{N}$ and $g : \mathbb{Z} \to \mathbb{C}$ is N-periodic

$$\Theta(\tau, g, j) := \sum_{m \geq 1} m^j g(m) \exp\left(\frac{2\pi i \tau m^2}{N}\right), \quad \tau \in \mathbb{H}.$$

Resurgence and quantum modularity properties of such series are proven by Han, Li, Sauzin and Sun (2023).

Definition 5 (Zagier 2010)

Let $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ be a subgroup, let $\mathcal{Q} \subset \mathbb{Q}$ be preserved by Γ and let $h \in \frac{1}{2}\mathbb{Z}$. A strong quantum modular form on $(\Gamma, \mathcal{Q}, h)$ is a map

$$f : \mathcal{Q} \to \mathbb{C}[[x]], \quad \alpha \mapsto f_\alpha(x),$$

such that for all $\alpha \in \mathcal{Q}, \gamma \in \Gamma$, there is an analytic function g_γ on $\mathbb{R} \setminus D$, with D being finite, with the following Taylor series at α

$$g_\gamma(x + \alpha) := f_\alpha(x) - (c(\alpha + x) + d)^{-h} f_{\gamma(\alpha)}(\gamma(x + \alpha) - \gamma(\alpha)).$$

For any root of unity ξ, let $W(X, \xi) \in \mathbb{Q}[\xi]$ be Habiro's extension of the WRT invariant, i.e. $W(X, e^{2\pi i/r}) = \mathrm{WRT}_k(X), r = k + 2$.

Theorem 6 (Andersen–Han–Li–M–Sauzin–Sun)

There is a family of explicitly defined resurgent formal series $\widehat{Z}_\alpha(x), \alpha \in \mathbb{Q}$, such that the following asymptotic expanion holds

$$\widehat{Z}(X, \tau) \sim \widehat{Z}_\alpha(\tau - \alpha), \text{ as } \tau \to \alpha.$$

Further, for all $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$ we have that

$$\lim_{\tau \to \alpha} \widehat{Z}(X, \tau) = \mathrm{WRT}_k(X, e^{2\pi i \alpha}),$$

and $\mathbb{Q} \ni \alpha \mapsto \widehat{Z}_\alpha \in \mathbb{C}[[x]]$ is a strong higher depth quantum modular form with congruence subgroup $\Gamma_1(4P)$.

Towards proving the AEC we identify $\mathcal{M}(X) := \mathcal{M}_{\mathrm{Flat}}(X, G)$ with a union of moduli spaces of flat G-connections on the Seifert surface (a compact oriented genus 0 surface with n marked points)

$$\Sigma_{0,n} = X/U(1)$$

with special holonomy around the boundary circles $\partial_j, j = 1, \ldots, n$, of discs centered at the exceptional orbits of $X \to \Sigma_{0,n}$.

Definition 7 (Label set for boundary holonomies)

Let $\mathfrak{R}(p_1, \ldots, p_n)$ be the set of $l = (l_1, \ldots, l_n) \in \bigtimes_{j=1}^{n} \{0, \ldots, p_j\}$:

1. For $j \geq 2$ we have that l_j is even. Let t_l denote the number of indices $j \in \{1, \ldots, n\}$ with $l_j = 0 \mod p_j$. Then $t_l \leq n - 3$.
2. For every $J \subset \{1, \ldots, n\}$ with odd cardinality, we have that

$$\sum_{j \in J} \frac{p_j - l_j}{p_j} + \sum_{j \in \{1, \ldots, n\} \setminus J} \frac{l_j}{p_j} > 1.$$

For $l \in \mathbb{Z}^n, j \in \{1, ..., n\}$, set $C_j^{(l)} := \text{diag}\left(e^{\pi i l_j/p_j}, e^{-\pi i l_j/p_j}\right)$ and

$$\mathcal{M}(\Sigma_{0,n}, C^{(l)}) := \mathcal{M}_{\text{Flat}}(\Sigma_{0,n}) \bigcap_{j\in\{1,...,n\}} \text{hol}_{\partial_j}^{-1}([C_j^{(l)}]).$$

Let T be the trivial connection. Then $\mathcal{M}(X) = \mathcal{M}^{\text{Irr}}(X) \sqcup \{T\}$

Theorem 8 (Andersen–Han–Li–M–Sauzin–Sun)

For all $l \in \mathfrak{R}(p_1, ..., p_n)$ we have a non-empty connected moduli space

$$\mathcal{M}(\Sigma_{0,n}, C^{(l)}) = \mathcal{M}^{\text{Irr}}(\Sigma_{0,n}, C^{(l)}) \neq \emptyset.$$

Pullback with respect to $\Sigma_{0,n} \hookrightarrow X$ induces a homeomorphism

$$\mathcal{M}^{\text{Irr}}(X) \cong \bigsqcup_{l\in\mathfrak{R}(p_1,...,p_n)} \mathcal{M}^{\text{Irr}}(\Sigma_{0,n}, C^{(l)}).$$

This builds on work of Jeffrey.

Let $\mathfrak{R}_{\mathbb{C}}(p_1,\ldots,p_n)$ be the set of $l \in \bigtimes_{j=1}^n \{0,\ldots,p_j\}$ satisfying the first of the two conditions defining $\mathfrak{R}(p_1,\ldots,p_n)$. Let $d = n-3$

Theorem 9 (Andersen–M 22, (building on Kirk–Klassen 90))

We have that $\pi_0(\mathcal{M}_{\mathrm{Flat}}(X, G_{\mathbb{C}})) \cong \mathfrak{R}_{\mathbb{C}}(p_1,\ldots,p_n)$ *and:*

$$\mathrm{CS}(\rho_l) = -\frac{P}{4}\left(\sum_{j=1}^n l_j/p_j\right)^2 \in \mathbb{Q}/\mathbb{Z}, \ \forall l \in \mathfrak{R}_{\mathbb{C}}(p_1,\ldots,p_n).$$

Further, the set $\mathrm{CS}_{\mathbb{C}}(X) \setminus \{0\}$ *is equal to*

$$\left\{-\frac{m^2}{4P} \in \mathbb{Q}/\mathbb{Z} : m \in \mathbb{Z}, \text{ is divisible by at most } d \text{ of the } p_j' s.\right\}$$

and if $p_1,\ldots,p_j$ *are all primes, the Chern-Simons action induces a bijection*

$$\mathfrak{R}_{\mathbb{C}}(p_1,\ldots,p_n) \to \mathrm{CS}_{\mathbb{C}}(X) \setminus \{0\}.$$

Conclusion: Using

- The identification $\pi_0(\mathcal{M}_{\mathrm{Flat}}(X, G)) \cong \mathfrak{R}(p_1, ..., p_n)$,
- the formula $\mathrm{CS}(\rho_l) = -\frac{P}{4}\left(\sum_{j=1}^n l_j/p_j\right)^2$,
- the result $\lim_{q \to q_r} \widehat{Z}(X, q) = \mathrm{WRT}_k(X)$,
- and an analysis of $\widehat{Z}(X, q)$ in the limit $q \to e^{2\pi i/r}$,

we arrive at

Theorem 10 (Andersen–Han–Li–M–Sauzin–Sun)

The asymptotic expansion conjecture holds for X and the WRT invariant admits an asymptotic expansion of the form

$$\mathrm{WRT}_k(X) \sim \sum_{\theta \in \mathrm{CS}(X)} e^{2\pi i r} W_\theta(1/r),$$

where W_0 is the Ohtsuki series and $r^{-1/2} W_\theta(r^{-1}) \in \mathbb{C}[r]$ for all non-zero Chern-Simons invariants θ.

The Gukov–Pei–Putrov–Vafa invariant: of a plumbed 3-manifold M with a plumbing graph (tree) Γ with a negative definite linking matrix B and a choice of $s \in \mathrm{spin}^c(M)$ is a q-series convergent for $|q| < 1$

$$\widehat{Z}_s(M,q) \in q^{\Delta_s}\mathbb{Z}[[q]], \qquad \Delta_s \in \mathbb{Q}.$$

Residue formula: For each vertex v set $F_v(z) = \left(z - \frac{1}{z}\right)^{2-\deg(v)}$. Set $\delta = (\deg(v))_{v \in V}$. Let $a \in (\mathbb{Z}^V + \delta)/2B\mathbb{Z}^V \cong \mathrm{spin}^c(M)$.

$$\widehat{Z}_a(M,q) := q^{\frac{\psi}{4}} \cdot v.p. \oint_{|z_v|=1} \prod_{v \in V} \frac{\mathrm{d}\, z_v}{2\pi i z_v} F_v(z_v) \Theta_a^{-B}(\vec{z}),$$

$$\Theta_a^{-B}(\vec{z}) := \sum_{\vec{l} \in 2B\mathbb{Z}^s + a} q^{-\frac{(\vec{l}, B^{-1}\vec{l}))}{4}} \prod_{v \in V} z_v^{l_v}.$$

Topological invariance: This was proven to be a topological invariant of (M, a) by Gukov-Manolescu.

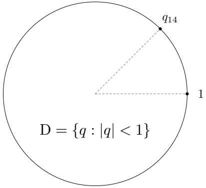
The following theorem was conjectured by GPPV, where

$$\widehat{Z}_r(M;\tau) := \sum_{b\in(\mathbb{Z}^V+\delta)/2B\mathbb{Z}^V} z_r(b)\cdot\widehat{Z}_b(M,e(\tau+1/r))),$$

$$z_r(b) := \frac{1}{2(q_{2r}-q_{2r}^{-2})\sqrt{\det(B)}}\sum_{a\in\mathbb{Z}^V/B\mathbb{Z}^V} e(-ra^tB^{-1}a - a^tB^{-1}b).$$

Theorem 11 (Murakami, 24)

$$\lim_{\tau\to 0}\widehat{Z}_r(M;\tau) = \mathrm{WRT}_k(M).$$



WRT: $\mathrm{WRT}(M):\mathbb{N}\to\mathbb{C}$

GPPV: $\widehat{Z}_b(M):\mathrm{D}\to\mathbb{C}$

Analytic Continuation:

$\sum_b z_r(b)\widehat{Z}_b(M,q_r) = \mathrm{WRT}_k(M)$

Non-semisimple Quantum Invariants: Let M be a negative definite plumbed 3-manifold. Consider the non-semisimple invariant defined by Costantino, Geer, Patureau & Mirand ($r \in \mathbb{Z}_+ \setminus 4\mathbb{Z}_+$)

$$N_r(M,\omega) \in \mathbb{C}, \quad \omega \in H^1(M,\mathbb{Q}/2\mathbb{Z}) \setminus H^1(M,\mathbb{Z}/2\mathbb{Z}).$$

For each $s \in \mathrm{Spin}^c(M)$ let $z_r(\omega,s) \in \mathbb{C}$ be a certain constant defined in Constantino–Gukov–Putrov 2023. Set

$$\widehat{Z}_r(M,\omega;\tau) := \sum_{s\in\mathrm{Spin}^c(M)} z_r(\omega,s)\widehat{Z}_s(M,e(\tau+1/r)).$$

The following was conjectured by Costantino–Gukov–Putrov and proven by them for the smaller class of Y-shaped plumbing graphs

Theorem 12 (M–Murakami)

For any sector $S \subset \mathbb{H}$ the following limit holds as $\tau \in S$ tends to 0

$$\lim_{\tau\to 0} \widehat{Z}_r(M,\omega;\tau) = N_r(M,\omega).$$

Generating Function. For $v \in V$ let $m_v \in H_1(M)$ be the meridian of the corresponding component of the surgery link. Define $\tilde{\omega} \in (\mathbb{R}/2\mathbb{Z})^V$ by $\tilde{\omega}_v = \omega(m_v), \forall v \in V$. We can assume that

$$\tilde{\omega}_v \notin \mathbb{Z}/2\mathbb{Z}, \ \forall v \in V.$$

Let Q denote the quadratic form associated with $-B$ (where B is the linking matrix). Recall $F_v(x) = (x - 1/x)^{2-\deg(v)}, \forall v \in V$, set $\tilde{e} = (1, \dots, 1) \in \mathbb{Z}^V$ and $e(x) = e^{2\pi i x}$. Define

$$G_{\omega,r}(x) := \sum_{\alpha \in \frac{1}{2}(\tilde{\omega} + r\tilde{e}) + \mathbb{Z}^V / r\mathbb{Z}^V} e\left(\frac{-Q(\alpha)}{r}\right) \prod_{v \in V} F_v\left(e\left(\frac{\alpha_v}{r} + \frac{x_v}{2\pi i}\right)\right).$$

Then

$$G_{\omega,r}(0) = N_r(M,\omega).$$

The Pole Divisor: of $x \mapsto G_{\omega,r}(ix)$ is

$$\mathcal{P}_{\omega,r} = \bigcup_{\substack{v \in V : \deg(v) \geq 3, \\ \alpha \in \frac{1}{2}(\tilde{\omega} + re) + \mathbb{Z}^m / r\mathbb{Z}^m}} \left\{ x \in \mathbb{C}^V : \frac{2\pi i \alpha_v}{r} + i x_v \in \pi i \mathbb{Z} \right\} \subset \mathbb{R}^V.$$

Gaussian Reciprocity (version due to Deloup and Turaev)

Let L be a lattice of finite rank n equipped with a non-degenerate symmetric $\mathbb{Z}$-valued bilinear form $\langle \cdot, \cdot \rangle$. Consider the dual lattice

$$L' := \{ y \in L \otimes \mathbb{R} \mid \langle x, y \rangle \in \mathbb{Z} \text{ for all } x \in L \}$$

Let $0 < m \in |L'/L| \mathbb{Z}, u \in \frac{1}{m} L$, and $h : L \otimes \mathbb{R} \to L \otimes \mathbb{R}$ be a self-adjoint automorphism such that

$$h(L') \subset L', \quad \text{and} \quad \tfrac{m}{2} \langle y, h(y) \rangle \in \mathbb{Z}, \ \forall y \in L'.$$

Let σ be the signature of $x \mapsto \langle x, h(x) \rangle$. Recall $e(x) := e^{2\pi i x}$. Then the following holds

$$\sum_{x \in L/mL} e\left(\frac{1}{2m} \langle x, h(x) \rangle + \langle x, u \rangle \right) =$$

$$\frac{e(\sigma/8) m^{n/2}}{\sqrt{|L'/L| \, |\det h|}} \sum_{y \in L'/h(L')} e\left(-\frac{m}{2} \langle y + u, h^{-1}(y + u) \rangle \right).$$

In CGP23 the conjecture is proven for Γ being Y-shaped and assuming an open condition on B. The key step is an application of Gaussian reciprocity. Following this and using ideas from Murakami 2024, we prove the following, where, for each $\nu \in \{\pm 1\}^V$, we define the linear map $I_\nu : \mathbb{C}^V \to \mathbb{C}^V$, the quadratic form $Q_\nu^{-1} : \mathbb{C}^V \to \mathbb{C}$ and the sequence $\{G^\nu_{\omega,r,\ell}\} \subset \mathbb{C}$ as follows

$$I_\nu((x_v)_{v\in V}) := (\nu_v x_v)_{v\in V}, \quad Q_\nu^{-1}(x) := -x^t I_\nu^t B^{-1} I_\nu x,$$
$$G_{\omega,r}(x) = \sum_{\ell \in (\deg(v))_{v\in V} + 2\mathbb{Z}^V_{\geq -1}} G^\nu_{\omega,r,\ell} \exp\left(I_\nu(\ell)^t x\right).$$

Lemma 13 (M-Murakami)

$$\widehat{Z}_r(M,\omega;\tau) = 2^{-|V|} \sum_{\nu\in\mu_2^V} \sum_{\ell \in (\deg(v))_{v\in V} + 2\mathbb{Z}^V_{\geq -1}} G^\nu_{\omega,r,\ell}\, e\left(\frac{\tau Q_\nu^{-1}(\ell)}{4}\right).$$

Gaussian integration: Let $B' \in M_{m\times m}(\mathbb{C})$ be a symmetric and non-degenerate $m \times m$ matrix with positive definite imaginary part, and let $w \in^m$. Then

$$\int_{\mathbb{R}^m} \exp\left(\frac{i}{2} x B' x + iwx\right) d^m x = \sqrt{\frac{(2\pi i)^m}{\det(B')}} \exp\left(-\frac{i}{2} w (B')^{-1} w\right).$$

Stationary phase: Let $B' \in M_{m\times m}(\mathbb{C})$ be a symmetric non-degenerate $m \times m$ matrix with semi-positive definite real part. Let $G \in C_0^\infty(\mathbb{R}^m; \mathbb{C})$. For each $j \in \{1, \ldots, m\}$ define $D_j := -i\frac{\partial}{\partial x_j}$, and define $D_B := \sum_{i,j} (B^{-1})_{i,j} D_i D_j$. For every sector $S \subset \mathbb{H}$ the following Poincare asymptotic expansion holds as $\tau \in S$ tends to 0 and $\rho = 2\pi i \tau$

$$\left(\det\left(\frac{-B'}{\pi\rho}\right)\right)^{1/2} \int_{\mathbb{R}^m} \exp\left(\frac{x^t B' x}{\rho}\right) G(x) dx \sim \sum_{l=0}^{\infty} \rho^l \frac{D_B^l(G)}{4^l l!}(0).$$

Let $\varepsilon > 0$ be a small positive parameter. For each $\nu \in \mu_2^V = \{-1, 1\}^V$, define $\Gamma_\nu := i\varepsilon\nu + \mathbb{R}^V$. An application of Gaussian integration to Lemma 14 gives the result below, where $\widehat{Z}_{\omega,r,l} := \frac{1}{4^l l!}\left(\sum_{v,w\in V} B^{-1}_{v,w}\frac{\partial}{\partial x_v}\frac{\partial}{\partial x_w}\right)^l (G_{\omega,r})(0), \quad \forall l \in \mathbb{Z}_{\geq 0}$.

Theorem 14 (M–Murakami)

The integral representation (3) *holds, and for any sector $S \subset \mathbb{H}$ an application of the method of steepest descent to the right hand side of* (3) *gives the asymptotic expansion* (4) *as $\tau \in S$ tends to 0*

$$\widehat{Z}_r(M,\omega;\tau) = \sum_{\nu\in\mu_2^V}\left(\frac{\det(B)}{(8\pi^2 i\tau)^{|V|}}\right)^{\frac{1}{2}}\int_{\Gamma_\nu}\exp\left(\frac{Q(x)}{2\pi i\tau}\right)G_{\omega,r}(ix)dx, \tag{3}$$

$$\widehat{Z}_r(M,\omega;\tau) \sim N_r(M,\omega) + \sum_{l=1}^{\infty}\widehat{Z}_{\omega,r,l}(2\pi i\tau)^l. \tag{4}$$

Generalization to WRT Invariants

Consider the WRT invariant $\mathrm{WRT}_k(M)$ and $\widehat{Z}_r(M;\tau)$. Let

$$G_r(x) := \sum_{\alpha\in\mathbb{Z}^V/(2r\mathbb{Z})^V} e\left(\frac{-Q(\alpha)}{4r}\right)\prod_{v\in V} F_v(t_r^{2\alpha_v}e^{x_v}).$$

In comparision with $G_{\omega,r}$, it is now non-trivial to show that the prinicpal part at $x=0$ vanish, and that $G_r(0)=\mathrm{WRT}_k(M)$. This is done by Murakami 2024. Our method then gives

Theorem 15 (M–Murakami)

$$\widehat{Z}_r(M;\tau) = \sum_{\nu\in\mu_2^V}\left(\frac{\det(B)}{(8\pi^2 i\tau)^{|V|}}\right)^{\frac{1}{2}}\int_{\Gamma_\nu}\exp\left(\frac{Q(x)}{2\pi i\tau}\right)G_r(ix)dx,$$

$$\widehat{Z}_r(M;\tau)\sim \mathrm{WRT}_k(M)+\sum_{l=1}^{\infty}\widehat{Z}_{r,l}(2\pi i\tau)^l.$$