A guided tour of BPS sectors in 5d, 3d and 3d-5d systems from the viewpoint of exponential networks

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Resurgence, Wall-Crossing and Geometry

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This talk is based on joint works touching on three subjects

- Generalized DT invariants (with S. Banerjee, M. Romo)
- Exact WKB analysis of q-difference equations (with F. Del Monte)
- Aspects of open Gromov-Witten theory (with K. Gupta)

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Understanding these connections leads to

- Clarifying how different BPS sectors interact with each other
- New computational tools, and exact results, for enumerative invariants
- Predictions of new properties & structures from physical arguments

Outline

1. Overview of relevant BPS sectors
2. Exponential networks
3. DT invariants & 5d BPS states
4. Stokes data of qDEs & 3d-5d BPS states
5. Structures in open Gromov-Witten invariants & 3d BPS vortices

1. Overview of relevant BPS sectors

M-theory on toric Calabi-Yau threefolds

Geometric engineering

M-theory : $X \times S^1 \times \mathbb{R}^4$ $\qquad$ $T_{5d}[X]$: $S^1 \times \mathbb{R}^4$

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5d BPS states: compact C_4, C_2

$$\text{M5}: \quad C_4 \times S^1 \times \mathbb{R} \qquad\qquad \text{monopole string}: \quad S^1 \times \mathbb{R}$$

$$\text{M2}: \quad C_2 \times \text{pt} \times \mathbb{R} \qquad\qquad \text{instanton particle}: \quad \mathbb{R}$$

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Counting: KK modes

D4, D2, D0

generalized DT

'BPS index'

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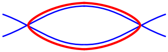
Mirror / Seiberg-Witten description

IIB string theory: $Y_\Sigma \times \mathbb{R}^4$ $\qquad$ mirror curve Σ

D3: sLag $\times \mathbb{R}$ $\qquad$ calibrated 1-cycles



3d-5d systems

If L is a noncompact special Lagrangian with $b_1(L) = 1$, and we introduce

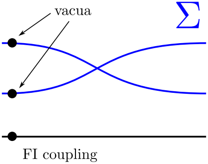
$$\begin{array}{rlcrl} \text{M-theory :} & X \times S^1 \times \mathbb{R}^4 & \qquad & T_{5d}[X] : & S^1 \times \mathbb{R}^4 \\ & \cup & & & \cup \\ \text{M5 :} & L \times S^1 \times \mathbb{R}^2 & & T_{3d}[L] : & S^1 \times \mathbb{R}^2 \end{array}$$

new BPS sectors emerge, from $T_{3d}[L]$ and its interaction with $T_{5d}[X]$

$$\mathcal{H}_{\mathrm{BPS}} = \mathcal{H}_{5d} \oplus \mathcal{H}_{3d} \oplus \mathcal{H}_{3d-5d}$$

3d BPS states: vortices of $T_{3d}[L]$

$T_{3d}[L]$ admits a description as an Abelian 3d $\mathcal{N} = 2$ gauge theory.
On $S^1 \times \mathbb{R}^2$, vacua are parameterized by a complex curve



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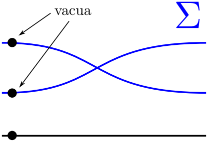

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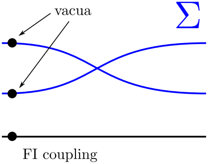

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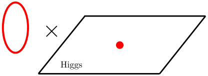

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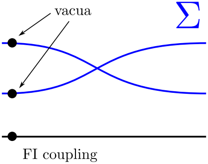

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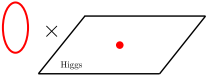

Their counts are related to open GW/LMOV invariants: given $(C, \partial C) \subset (X, L)$

M2 : $C \times S^1 \times \mathrm{pt}$ vortex : $S^1 \times \mathrm{pt}$

3d-5d BPS states: kinky vortices

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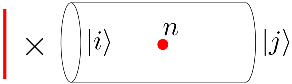

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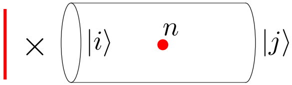

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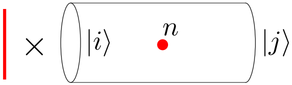

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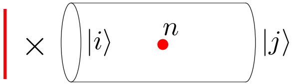

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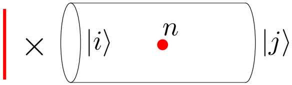

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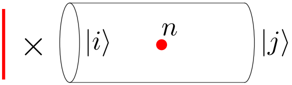

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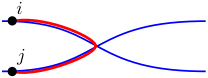

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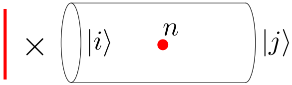

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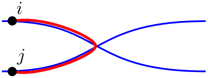

$\Rightarrow$ 3d-5d boundstates $\supset$ 5d BPS states by closed concatenations.

BPS sectors recap

M-theory on $X \times S^1 \times \mathbb{R}^4$ with an M5 brane on $L \times S^1 \times \mathbb{R}^2$ includes a novel 3d-5d BPS sector of 'kinky vortices'. These play a central role by encoding both

- 3d BPS vortices in the $i = j$ sector, in the limit $R \to \infty$.
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While 3d and 5d BPS states have clear mathematical counterparts in open GW and generalized DT theory, there is no obvious counterpart for 3d-5d BPS states.

- Is the 3d-5d CFIV index some kind of enumerative invariant?
- How, exactly, are DT and open-GW invariants related to CFIV indices?
- What new properties/structures does embedding 3d and 5d into 3d-5d predict?

2. Exponential networks

Exponential networks

$T_{3d}[L]$ gives an algebraic curve in $\mathbb{C}^* \times \mathbb{C}^*$

$$\Sigma : \quad F(x, y) = 0$$

with a natural presentation as ramified covering over $\mathbb{C}^*_x$ with sheets $y_i(x)$.

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The study of 3d-5d BPS states motivates a definition of exponential networks.

Mainly two pieces of data:

- Geometric: a web of trajectories on $\mathbb{C}^*_x$ shaped by Σ and ϑ.
- Combinatorial: topological information attached to each trajectory.

Geometric data

An (ij, n) trajectory is labeled by a pair of sheets (i, j) and by an integer $n \in \mathbb{Z}$, and has a shape $x(t)$ governed by

$$(\log y_j - \log y_i + 2\pi i\, n) \frac{d \log x}{dt} = e^{i\vartheta}\,.$$

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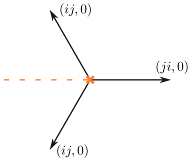

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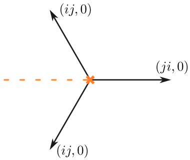
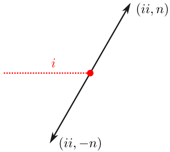


Punctures of Σ, where $y_i(x) \sim (x - x_*)^k$ source (ii, km) trajectories, with $m \in \mathbb{Z}$



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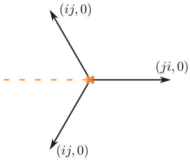

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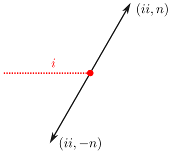
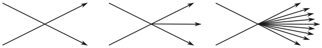


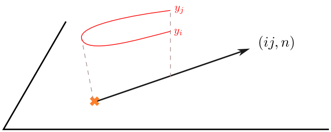
New trajectories can be generated at intersections of (ij, n) and (kl, m)



Combinatorial data

Each trajectory carries:

- paths a on Σ from $y_i(x)$ to $y_j(x)$ winding n times around $\mathbb{C}^*_y$
- a weight $\mu \in \mathbb{Q}$ associated to each path



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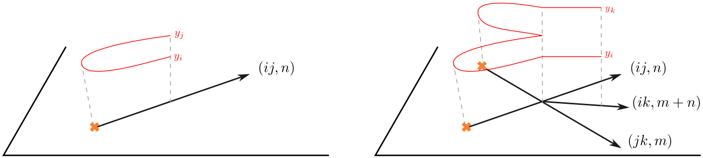

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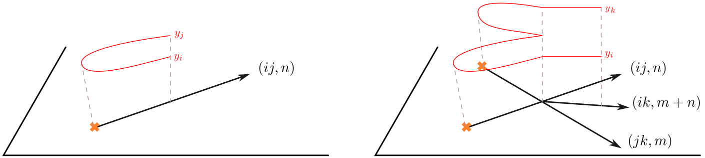

This data is determined by the topology of the underlying network, according to a set of rules motivated by physics.

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The construction is inspired by a 3d uplift of tt^* geometry. It recovers counts of 2d $(2, 2)$ soliton kinks in the limit $R \to 0$. Consistency checks will follow.

3. DT invariants & 5d BPS states

Critical phases

Boundstates of BPS states from conjugate sectors (ij, n) and $(ji, -n)$ carry only flavour charges of $T_{3d}[L]$, corresponding to quantum numbers of $T_{5d}[X]$.

$\Rightarrow$ Boundstates of 3d-5d states probe the space of stable 5d BPS states.

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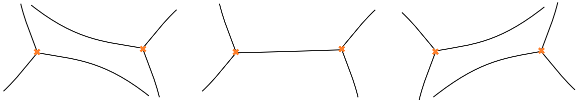

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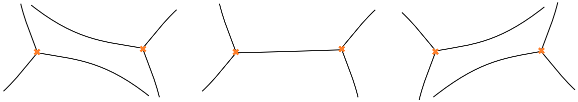
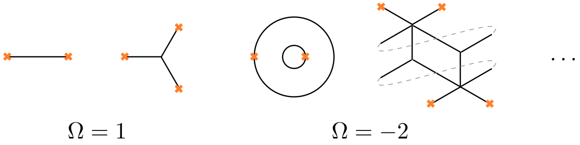
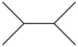


The BPS index $\Omega(\gamma)$ for each saddle is determined by combinatorics of concatenations



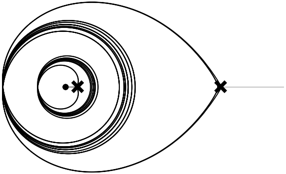
$\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ [Eager Selmani Walcher] [Banerjee L Romo]

X : 

$$\Sigma : \quad 1 + y + xy + Qxy^2 = 0$$

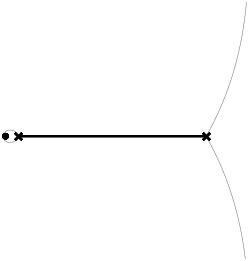
Several critical phases:

$$\vartheta_{\mathsf{cr}} = \arg Z_{\mathsf{D0}}$$



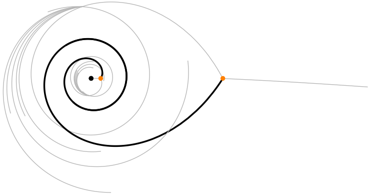
$$Z_{\mathsf{D0}} = \frac{2\pi}{R}\,, \qquad \Omega(k\mathsf{D0}) = -2 = -\chi(X)$$

$\vartheta_{\mathsf{cr}} = \arg Z_{\mathsf{D2}}$



$$Z_{\mathsf{D2}} = \frac{i}{R} \log Q\,, \qquad \Omega(\mathsf{D2}) = 1$$

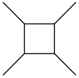
$\vartheta_{\mathsf{cr}} = \arg Z_{\text{D2-D0}}$



$$Z_{\text{D2-D0}} = \frac{2\pi}{R} + \frac{i}{R}\log Q\,, \qquad \Omega(\text{D2-D0}) = 1$$

As well as a whole tower (peacock pattern) of saddles with $\Omega(\text{D2-}k\text{D0}) = 1$ for $k \in \mathbb{Z}$.

$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-2)$

X :  $\Sigma: \quad Q_f(y+y^{-1}) + Q_b(x+x^{-1}) - 1 = 0$

Much richer example, involving wall-crossing, and 'wild' spectrum [Banerjee L Romo]

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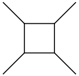
X : 

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However, there is a 'degenerate' chamber of moduli space where the BPS spectrum can be computed exactly [L] [Del Monte L] [Closset Del Zotto]

$$\Omega(\pm\gamma_1 + k(\gamma_1+\gamma_2)) = 1 \qquad \Omega(\pm(\gamma_1+\gamma_2) + k\gamma_{\mathrm{D0}}) = -2$$
$$\Omega(\pm\gamma_3 + k(\gamma_3+\gamma_4)) = 1 \qquad \Omega(k\gamma_{\mathrm{D0}}) = -4$$

with $k \in \mathbb{N}$, and $\langle\gamma_i, \gamma_{i+1}\rangle = -2$

$$\gamma_1 : \ \mathrm{D4} \qquad \gamma_2 : \ \mathrm{D2}_f\overline{\mathrm{D4}} \qquad \gamma_3 : \ \mathrm{D0}\,\mathrm{D2}_b\overline{\mathrm{D2}}_f\overline{\mathrm{D4}} \qquad \gamma_4 : \ \overline{\mathrm{D2}}_b\mathrm{D4}$$

Counting A-branes

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Closely related to an earlier proposal of enumerative invariants of sLags [Joyce].

4. Stokes data of qDEs & 3d-5d BPS states

Stokes graphs

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$$F(x,y) = 0 \quad \rightsquigarrow \quad \hat{F}(\hat{x},\hat{y})\,\psi(x,q) = 0\,.$$

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$$\hat{F} = 1 - \hat{y} - \hat{x}\,, \qquad \psi(x,q) = \exp\left(\sum_{n\geq 0} \frac{B_n}{n!}\hbar^{n-1}\mathrm{Li}_{2-n}(x)\right)$$

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Open questions:

- generalization beyond first-order
- are Stokes constants related to BPS data?

Second order qDEs [Del Monte L]

Any 2nd order qDE can be presented in 'involutive' form

$$\hat{F} = \hat{y} + \hat{y}^{-1} - 2T(\hat{x}, q) \quad \Rightarrow \quad \psi(qx) + \psi(q^{-1}x) = 2T(x,q)\,\psi(x)\,.$$

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The standard WKB ansatz

$$\psi(x) = \exp\left(\int^x S(x, \hbar)\, \frac{dx}{x}\right) \quad \text{with} \quad S(x, \hbar) = \sum_{k=-1}^{\infty} S_k(x)\, \hbar^k$$

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However the WKB ansatz is difficult to work with, because this is a $\hbar$-difference equation, not a differential one.

q-Riccati form

Introducing $R(x) := \psi(qx)/\psi(x)$, the involutive 2nd-order qDE takes the form

$$R(x)R(q^{-1}x) - 2T(x)R(q^{-1}x) + 1 = 0$$

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This admits two solutions $R_{\pm}(x,\hbar) = \sum_{k=0}^{\infty} R_{k,\pm}(x)\hbar^k$

$$R_{n,\pm}(x) = \pm\frac{1}{2\sqrt{T_0^2-1}}\sum_{m=1}^{n-1}\sum_{l=0}^{m}\frac{1}{l!}R_{m-l,\pm}\partial^l_{\log x}\left(R_{n-m,\pm} - 2T_{n-m}\right)$$
$$\pm\frac{1}{2\sqrt{T_0^2-1}}\sum_{l=1}^{n}R_{n-l,\pm}\left(\partial^l_{\log x}\left(R_{0,\pm} - 2T_0\right) - 2T_l\right).$$

The full formal series is known recursively in closed form.

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Over the field of q-periodic functions the space of solutions is just 2-dimensional.

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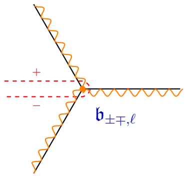

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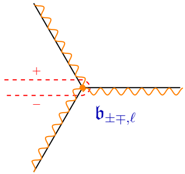

In a basis of suitably normalized vanishing solutions, Stokes matrices given by

$$S^{(\ell)} = \begin{pmatrix} -\xi^{\ell} & i \\ i & 0 \end{pmatrix}, \qquad \xi := \left(\frac{x}{x_0}\right)^{\frac{2\pi i}{\hbar}}.$$

generalizes Voros' single-valuedness condition to include log-monodromy ($\ell \neq 0$)

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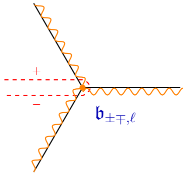

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$\Rightarrow$ The Stokes coefficient $(\mu = 1) \times \xi^{\ell}$ coincides with the combinatorial data encoded by the exponential network (CFIV index of kinky vortices).

Voros data captures changes in normalization between branch points.

$$T_{\mathfrak{b}\mathfrak{b}'} = \begin{pmatrix} 0 & iY_{\mathfrak{b}\mathfrak{b}'} \\ iY_{\mathfrak{b}\mathfrak{b}'}^{-1} & 0 \end{pmatrix} \qquad T_{\mathfrak{b}\mathfrak{b}'} = \begin{pmatrix} Y_{\mathfrak{b}\mathfrak{b}'} & 0 \\ 0 & Y_{\mathfrak{b}\mathfrak{b}'}^{-1} \end{pmatrix}$$

The two types of transport matrices correspond to relative signatures of BP's.

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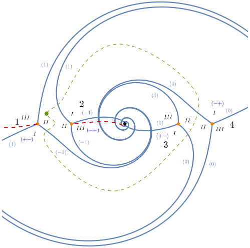
Monodromies can be readily computed by composition of Stokes and Voros data.

Example: the qMathieu equation quantizes Σ of local $\mathbb{P}^1 \times \mathbb{P}^1$ [Del Monte L]



$$\mathrm{Tr}\, M = X_{\frac{1}{2}(\gamma_3+\gamma_4)} + X_{\frac{1}{2}(-\gamma_3-\gamma_4)} + X_{\frac{1}{2}(-\gamma_1+\gamma_2+\gamma_{D0})} + X_{\frac{1}{2}(\gamma_1-\gamma_2-\gamma_{D0})+(\gamma_2+\gamma_4)}$$

5. Structures in open Gromov-Witten invariants & 3d BPS vortices

Field-theoretic properties of kinky vortices

3d-5d BPS kinky vortices plays a central role, but have not been studied as BPS states in QFT. Important to test our heuristic QFT picture of these.

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$T_{3d}[L]$	$S^1 \times \mathbb{R}^2$	vacua $y_i(x) \in \Sigma$
standard vortices	$S^1 \times \mathrm{pt}$	single Higgs vacuum at $S^1_\infty = \partial\mathbb{R}^2$
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A basic check: in the limit $R \to \infty$ with the same Higgs vacuum at both $S^1_{\pm\infty}$, kinky vortices should reduce to standard ones.

From geometric engineering of $T_{3d}[L]$, vortices arise from open strings/M2 in (X, L). Their free energy is encoded by $(g = 0)$ LMOV invariants

$$W_{\text{vortex}} = -\sum_{k\geq 1}\sum_{\beta} \mathfrak{n}_{k,\beta}\, \mathrm{Li}_2(Q^\beta x^k)\,, \qquad (\mathfrak{n}_{k,\beta} \in \mathbb{Z})$$

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$$R^2_{\text{core}} \propto (e^2\zeta)^{-1}\,.$$

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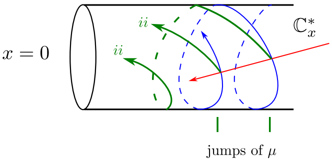
This limit takes $x \propto e^{-\zeta} \to 0$. This is crucial to compute μ, due to wall-crossing.



Let $\mu^*_{n,\beta} := \lim_{x\to 0} \mu_{n,\beta}$.

Conjecture [Gupta L] After an infinite sequence of wall-crossings, the generating function of kinky (ii, n) vortices in the Higgs vacuum $|i\rangle$ stabilizes to

$$\sum_{n\geq 1} \sum_{\beta} \mu^*_{n,\beta}\, x^n Q^\beta = -\sum_{k\geq 1} \sum_{\beta} \mathfrak{n}_{k,\beta}\, \log(1 - x^n Q^\beta)\,.$$

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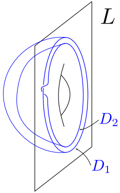
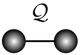
- For other framings the network is much more involved. A 'warping' trick allows for systematic computations. Results are fully consistent with the conjecture.

Knots-quivers correspondence

In some cases, the LMOV spectrum can organized by a stronger underlying structure known as 'knots-quivers' correspondence [Kucharski Reineke Stosic Sulkowski].

M2 branes wrapping holomorphic $(C, \partial C) \subset (X, L)$ are generated by finitely many disks through linking interactions [Ekholm Kucharski L].





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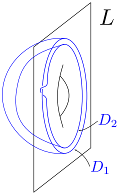

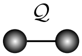

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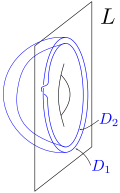

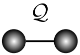

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- Vertices are disks with $[\partial C] = 1$, i.e. single-vortex states.

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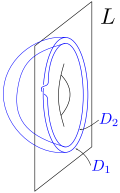

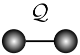

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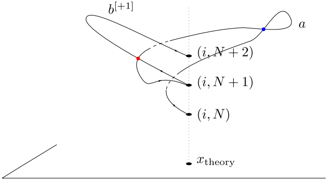
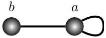
- Links are mixed Chern-Simons couplings of a (dual) QFT description of $T[L]$.

Quivers from Σ [Gupta L]

The relation between standard and kinky vortices (LMOV-CFIV) implies

- Vertices of $\mathcal{Q}$ are 1:1 with $(ii, 1)$ kinky vortices near $x = 0$
- Mixed CS-couplings govern orbital spin of 2-vortex boundstates, which is captured by intersections of paths on Σ [Seiberg Witten] [Galakhov L Moore]





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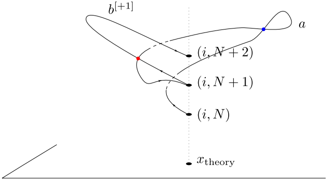

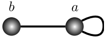

Tests: the proposal has been verified by direct computation for

- toric Lagrangians in $\mathbb{C}^3$ and resolved conifold, in various framings (1 & 2 vertices)
- knot conormal Lagrangian of the trefoil knot, in various framings (3 vertices)
- knot conormal Lagrangian of the figure-eight knot, in various framings (5 vertices)

6. Conclusions

Summary and outlook

Geometric engineering of M-theory on CY3 with a sLag L has been studied extensively, with much attention devoted to

$$\begin{aligned} \text{5d BPS states} \quad &\leftrightarrow \quad \text{(generalized) DT theory} \\ \text{3d BPS states} \quad &\leftrightarrow \quad \text{open GW theory} \end{aligned}$$

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