REFINED GROMOV-WITTEN INVARIANTS (joint w/ Schuler)

Take-home msg:

$$\boxed{\begin{array}{ccc} \text{Refined GW inv'ts} & \subset & \text{equiv. inv'ts of} \\ \text{of CY3} & & \text{CY5} \end{array}}$$

(cf N-O, "The index of M-theory")

Plan:

I) Overview

II) The main Conjecture

III) Results: A)-model / B)-model

I) X smooth q-proj. $/_{\mathbb{C}}$, $\dim_{\mathbb{C}} X = 3$, $\omega_X = \mathcal{O}_X$

$\rightsquigarrow$ count of (param'd) curves in X, fixed genus / degree

$\rightsquigarrow \overline{\mathcal{M}}_g(X,\beta) = \overline{\{ \phi : C \to X \mid \phi_*[C] = \beta,\ h^{1,0}(C) = g \}}$

$$\rightsquigarrow GW(X,\varepsilon,Q) := \sum_{\beta \in H_2(X,\mathbb{Z})} Q^\beta \sum_{g \geq 0} (-\varepsilon^2)^{g-1} \int_{[\overline{\mathcal{M}}_g(X,\beta)]^{vir}} 1$$

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$$GV(X,\varepsilon_{\pm},Q) := \sum_{\beta \in H_2(X,\mathbb{Z})} Q^\beta \frac{\Omega^X_\beta(q_-, q_+)}{[q_-][q_+ q_-]}$$

$q = e^{\varepsilon}$. $[q] := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$

$$\mathbb{Z}[q_\pm^{\pm 1}] \ni \Omega^X_\beta(q_-, q_+) := \sum_{j_-, j_+ \in \frac{\mathbb{Z}}{2}} N^\beta_{j_-, j_+}(X)\,(-1)^{2(j_+ + j_-)} \sum_{d_+ = -j_+}^{j_+} q_+^{2d_+} \cdot \sum_{d_- = -j_-}^{j_-} q_-^{2d_-}$$

$N^\beta_{j_-, j_+} = \#$ BPS M2   charge $= \beta$, Lorentz $= (j_-, j_+)$

$$\boxed{\exp(GW) = \mathrm{Exp}(GV)}$$

[GV]

$X \circlearrowleft \gamma^\times$ rigid CY3

(eg.: $X =$ toric)

$\rightsquigarrow N^\beta_{j_-, j_+}$ refined BPS inv'ts

Maths consequences of refinement

$\dim \mathrm{Gr}_i H^{i,j}(M^{(s)}_{\beta,\chi}, \mathbb{Z})$

- GV : $\Omega^X_\beta(q_+, q_-) = \sum_{i,j\in\mathbb{Z}} (-q_+)^i (-q_-)^j h^i(\mathrm{Chow}_\beta(X), \mathcal{P}^j_i)$

  $(X = K_S)$

  [MT]

- PT : $\mathbb{C}^\times \curvearrowright X$ $\qquad q_+^2$: char on ω_X^{-1}

  [NO]

  $P := P_n(X,\beta)$ proper

  $$PT(q,Q) := \sum_{n\in\mathbb{Z}} (-q_-)^n \sum_{\beta\in\mathbb{Z}} Q^\beta \boxed{\int_{[P]^{vir}} 1}$$

  $\mathbb{Z}$

  $$\chi_{\mathbb{C}^\times}(P, \hat{\mathcal{O}}_P^{vir}) \in \mathbb{Z}[q_+^{\pm}]$$

- gauge theory / instanton count $\rightsquigarrow$ Ω-deformation
- TV $\rightsquigarrow$ refined TV (IKV)
- CS $\rightsquigarrow$ refined CS (AS)
- 2D CFT / MM $\rightsquigarrow$ β-deformation [DV]

Q : GW $\rightsquigarrow$ refined GW ?

$$\left(\text{s.t.} \quad \exp GW^{\mathrm{ref}} = \mathrm{Exp}\left(GV^{\mathrm{ref}}\right) = PT^{\mathrm{ref}}\right)$$

II) Setup: X smooth, q-proj var. $/_{\mathbb{C}}$, $\dim X = 3$, $\omega_X \simeq \mathcal{O}_X$

$Z := X \times \mathbb{A}^2_{\mathbb{C}}$ other CY2's would work

$(\mathbb{C}^*)^2 \simeq T$ acts

- linearly on $\mathbb{A}^2_{\mathbb{C}}$, char's q_1, q_2
- $\boxed{\text{char } q_t^{-2} \text{ on } \omega_X, \quad q_t^2 = q_1 q_2}$
- $X^T \simeq Z^T$ proper

Example (*): $X = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1)^{\oplus 2}) \rightsquigarrow Z = \mathrm{Tot}_{\mathbb{P}^1}(\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}^{\oplus 2})$

T: acts fibrewise w/ weights $(1,0), (0,1), (-1,0), (0,-1)$

Def: $H_2^+(X,\mathbb{Z}) \ni \beta$ is rigid if $\overline{M}_g(X,\beta)$ proper $\forall g$.

Today: restrict to β rigid $\forall \beta \neq 0$ ("equi-rigid target")

Example: X = res. cfd., local del Pezzo

Defn: $$GW_{ref}(X, \varepsilon_1, \varepsilon_2, Q) := \sum_{\beta \in H_2(X,\mathbb{Z})} Q^\beta \sum_{g \geq 0} \langle 1 \rangle_{g,\beta}^{Z,T}$$

$$\langle 1 \rangle_{g,\beta}^{Z,T} := \int^T_{[\overline{M}_g^T(Z,\beta)]_T^{vir}} \frac{1}{e(N_T^{vir})} \in \mathbb{Q}[\varepsilon_1, \varepsilon_2)^{(2g-2)}$$

$$\varepsilon_\cdot = c_1^T(q_\cdot)$$

$$\rightsquigarrow GW^{ref} \in \frac{1}{\varepsilon_1\varepsilon_2}\mathbb{Q}[[\varepsilon_+^2, \varepsilon_1\cdot\varepsilon_2, Q]]$$

$$= \sum_{k,g} GW_{k,g}(Q)\, \varepsilon_+^{2k} (\varepsilon_1\cdot\varepsilon_2)^{g-1}$$

Conj.: $\exp GW^{ref} = \mathrm{Exp}\, GV^{ref} = PT^{ref}$

This any good ???

0) unrefined limit $q_+ \to 1$ / $\varepsilon_+ \to 0$ (CY action on X)

A) match w/ refined GV/PT ?

B) " " " BCOV ?

0) Thm: ✓ . Pf $\int_{[\overline{M}_g(Z,\beta)]^{vir}_T}^{T} (\quad) = \int_{[\overline{M}_g(X,\beta)]^{vir}_T}^{T} (\quad) \Lambda(\varepsilon_1)\Lambda(\varepsilon_2)$

$$\Lambda(\varepsilon) = \sum_{k=0}^{g} (-1)^k \lambda_k \varepsilon^{g-k-1} \qquad \lambda_k = c_k(\mathbb{E}_g)$$

$$\varepsilon_t = 0 : \quad \Lambda(\varepsilon) \cdot \Lambda(-\varepsilon) = (-\varepsilon^2)^{g-1} \quad (\text{Mumford})$$

A) Thm A1 : let X = res. cfd. . Then

$$\exp(GW^{ref}) = \exp\left(-\sum_{d=1}^{\infty} \frac{Q^d}{d[q_1^d][q_2^d]}\right) = \mathrm{Exp}(GV)^{ref}$$
$$= PT^{ref}$$

$$\left(N^{d[\mathbb{P}^1]}_{\vec{\partial}_+ | \vec{\partial}_-} = \delta_{\vec{\partial}_+, 0}\, \delta_{\vec{\partial}_-, 0}\, \delta_{d,1} \right)$$

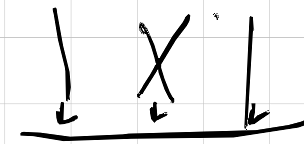
<u>Thm A2</u>: (NS limit for local surfaces). Let S a del Pezzo surface, $D \in |-K_S|$ smooth, $X := \mathrm{Tot}_S(\mathcal{O}(-D))$

i) $\displaystyle \lim_{\varepsilon_2 \to 0} \varepsilon_2\, GW^{ref} = GW^{rel}(S|D)$

ii) $S = \mathbb{P}^2$, $D = E$

$$\lim_{\varepsilon_2 \to 0} \varepsilon_2\, GW^{ref} = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \mathrm{Exp}(GV^{ref})\Big|_{q = e^{\varepsilon}}$$

Proof for both by degeneration



B) B-model $GV^{\mathrm{ref}}_{(k,g)}$ det'd recursively from

$$(k,g) = (0,0)\,,\quad (1,0)\,,\quad (0,1)$$

from "refined BCOV axioms") Huang-Klemm

P1: refined HAE

P2: quasi-modularity / polynomiality

P3: orbifold regularity

refined

P4: refined CGC: $GV^{\mathrm{ref}}_{cf} \sim \log \Gamma_2(\epsilon_1, \epsilon_2)\, t_{cf}^{2-2g}$

Thm B (refined MS for $K_{\mathbb{P}^2}$) : P1-4 hold for $GW^{ref}(K_{\mathbb{P}^2})$

Pf: refinement is a quantised symplectic transformation