Summability for state integrals of hyperbolic knots

Veronica Fantini

Laboratoire Mathématique d'Orsay — Université Paris-Saclay

Based on **arXiv:2410.20973** joint with C. Wheeler

Workshop on Resurgence, wall-crossing and geometry
Les Diablerets, Switzerland

Goal

Resum the formal series $\tilde{\Phi}_K$

- We consider the formal integral

$$\tilde{\Phi}_K(\tau) := \int \tilde{\Psi}(z,\tau)^B \, \mathbf{e}\Big(-\frac{A}{2} z^2 \tau \Big) \, dz \,,$$

defined by formal Gaussian integration, where $\mathbf{e}(z) = \exp(2\pi \mathrm{i} z), B > A > 0$ depend on an hyperbolic knot K, and $\tilde{\Psi}(z,\tau) = \sum_{n=0}^{\infty} a_n(z)\, \tau^{-n}$ formal series in τ

- To **resum** a formal series means building an analytic function

$$\tilde{\Phi}_K \xrightarrow{\text{resum}} \hat{\Phi}_K \text{ analytic}$$

- $\tilde{\Phi}_K$ is **divergent**!!!

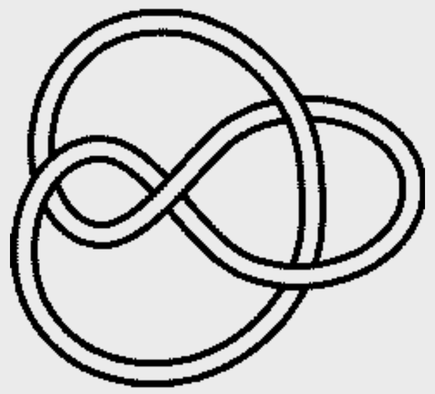
The formal series $\tilde{\Phi}_K$ — part I

Topological invariant of the knot K [Garoufalidis—Strozer—Wheeler]

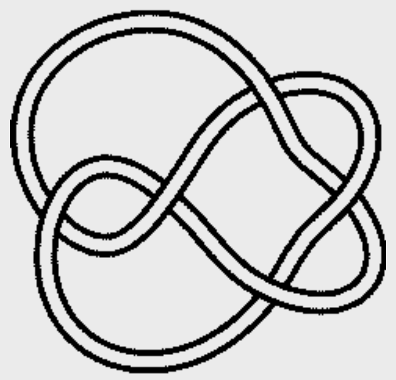
$$\tilde{\Phi}_K(\tau) := \int \tilde{\Psi}(z,\tau)^B \, \mathbf{e}\Big(-\frac{A}{2} z^2 \tau\Big)\, dz\,,$$

- The formal series $\tilde{\Phi}_K$ is conjectured to agree to all orders with the asymptotic series of the Kashaev invariant of the knot K [Dimofte—Garoufalidis]

- The integers $B > A > 0$ are the Neumann—Zagier data, computed from ideal triangulations of $S^3 \backslash K$



$4_1 : (A = 1, B = 2)$



$5_2 : (A = 2, B = 3)$

- The asymptotic expansion of the **complex Chern—Simons** (CS) partition function on the knot complement $S^3 \backslash K$ recovers the formal series $\tilde{\Phi}_K$

- **Theorem** [Garoufalidis—Strozer—Wheeler]: for every hyperbolic knot K, the formal series $\tilde{\Phi}_K$ is a topological invariant

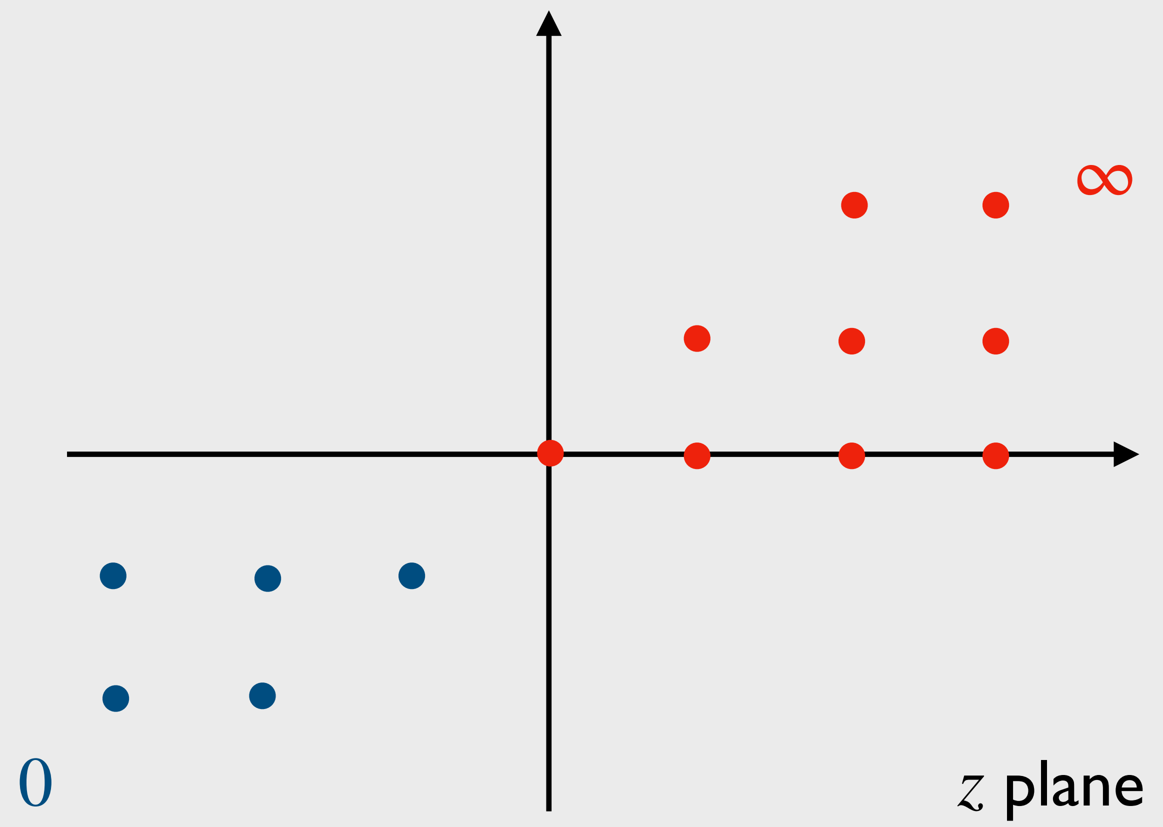
The formal series $\tilde{\Psi}$

The Faddeev's quantum dilogarithm [Faddeev]

- The Faddeev's quantum dilogarithm $\Phi(z;\tau)$

$$\Phi(z;\tau) = \exp\left(\int_{\mathrm{i}\sqrt{\tau}\mathbb{R}+\varepsilon\sqrt{\tau}} \frac{\mathbf{e}((z+1+\tau)w/\tau)}{(\mathbf{e}(w)-1)(\mathbf{e}(w/\tau)-1)} \frac{dw}{w} \right),$$

is a meromorphic function of $\tau \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $z \in \mathbb{C}$



- Its **asymptotics** as $|\tau| \to \infty$ is the **divergent** series

$$\tilde{\Psi}(z,\tau) = \mu_8\, \mathbf{e}\left(-\frac{\tau}{24} - \frac{1}{24\tau} - \sum_{k=0}^{\infty} (2\pi\mathrm{i})^{k-2} \frac{B_k}{k!} \mathrm{Li}_{2-k}(\mathbf{e}(z)) \tau^{1-k} \right),$$

where μ_8 is an eight root of unity

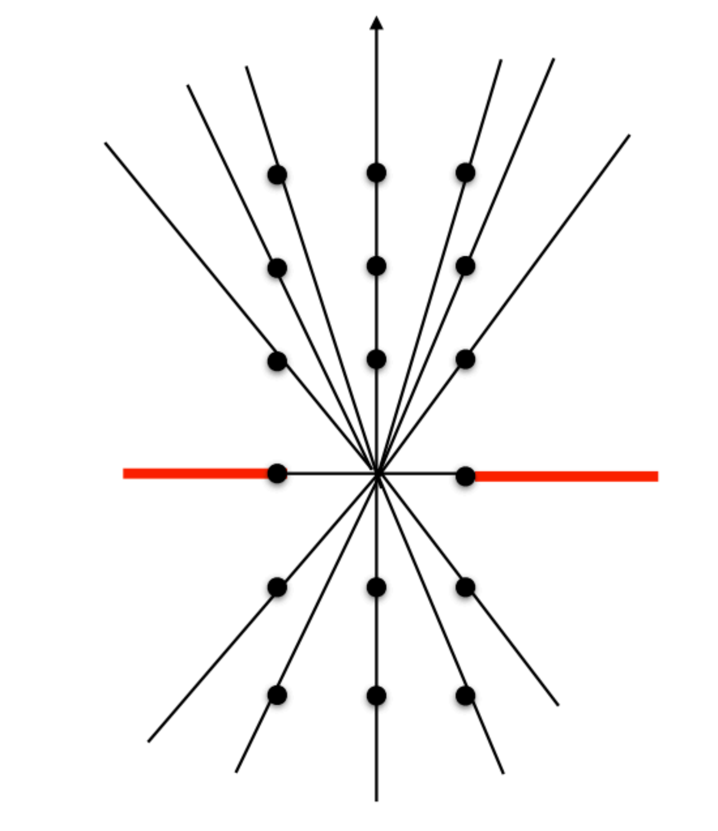
- **Theorem** [Kashaev—Garoufalidis]: $\tilde{\Psi}(z,\tau)$ is resurgent and its Borel—Laplace sum coincides with $\Phi(z;\tau)$

$$\tilde{\Psi}(z,\tau) \xrightarrow{\text{BL sum}} \hat{\Psi}(z,\tau) \equiv \Phi(z;\tau)$$

The formal series $\tilde{\Phi}_K$ — part II

The resurgent structure [Garoufalidis—Gu—Mariño]

- The singularities in the Borel plane are organized in a **peacock pattern**
- and they are located at the critical values of the Chern—Simons functional
- The **Stokes constants** can be computed by solving a q-difference equation



$\tilde{\Phi}_K$ for $K = 4_1$

- The Borel—Laplace resummations of $\tilde{\Phi}_K$ in a given sector can be expressed in terms of certain analytic functions related to the Andersen—Kashaev **state integral**

$\tilde{\Phi}_K$ --- BL sum ---> $\hat{\Phi}_K$ is a linear combination of state integrals

- State integrals should correspond to the partition function of complex Chern-Simons theory on the knot complement $S^3 \backslash K$
- The previous statements are **conjectures**, in general, and **numerically checked** for $K = 4_1$ and 5_2

Main result

Borel—Laplace summation of $\tilde{\Phi}_K$ [F. —Wheeler]

- A step further in the study of the **analytic properties** of the formal series $\tilde{\Phi}_K$

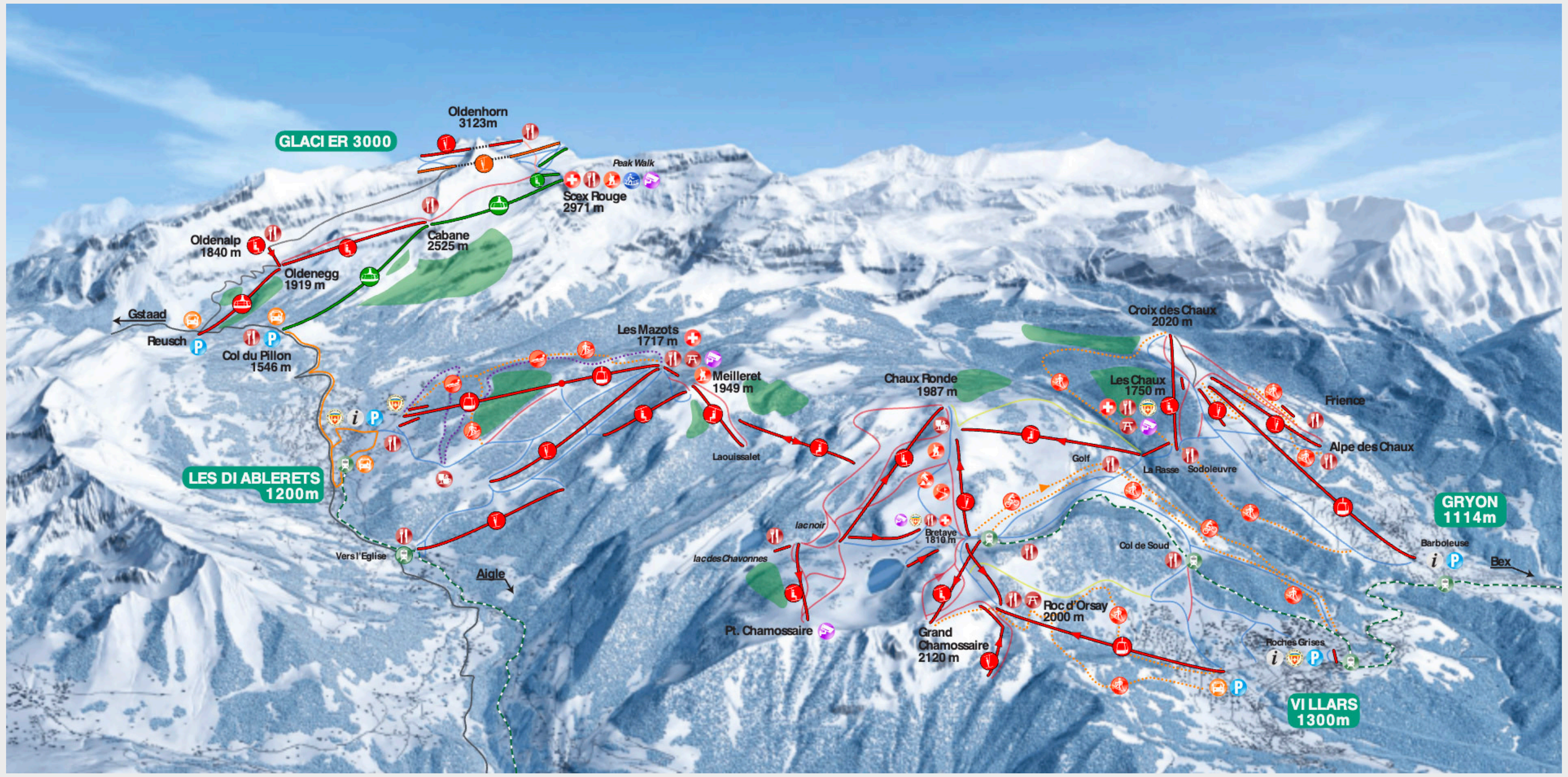
Theorem [F. —Wheeler]: the series $\tilde{\Phi}_K$ is Borel—Laplace summable for $K = 4_1$ and 5_2 knots

- We also give an **algorithm** to compute the correct combinations of state integrals that give the resummation $\hat{\Phi}_K$
- We verify GGM's numerical computations of the first Stokes constants for $K = 4_1$

Plan

- Borel—Laplace summation and Resurgence
- Warm-up: thimble integrals 🔵
- State integrals for the 4_1 and 5_2 knots 🔴
- Conclusion

Watch out!



🔵 Pistes bleues 🔴 Pistes rouges ⚫ Pistes noires

Borel-Laplace summation and Resurgence

Borel-Laplace summation and Resurgence

From formal to analytic

- Summation methods associate to a divergent series $\tilde{\Phi}$ an analytic function $\hat{\Phi}$

$$\tilde{\Phi} = \sum_{n=0}^{\infty} n!\, \tau^{-n-1} \in \mathbb{C}[\![\tau^{-1}]\!]_1 \overset{???}{\dashrightarrow} \hat{\Phi} \text{ analytic}$$

Borel-Laplace summation and Resurgence

Borel-Laplace summation works in three steps

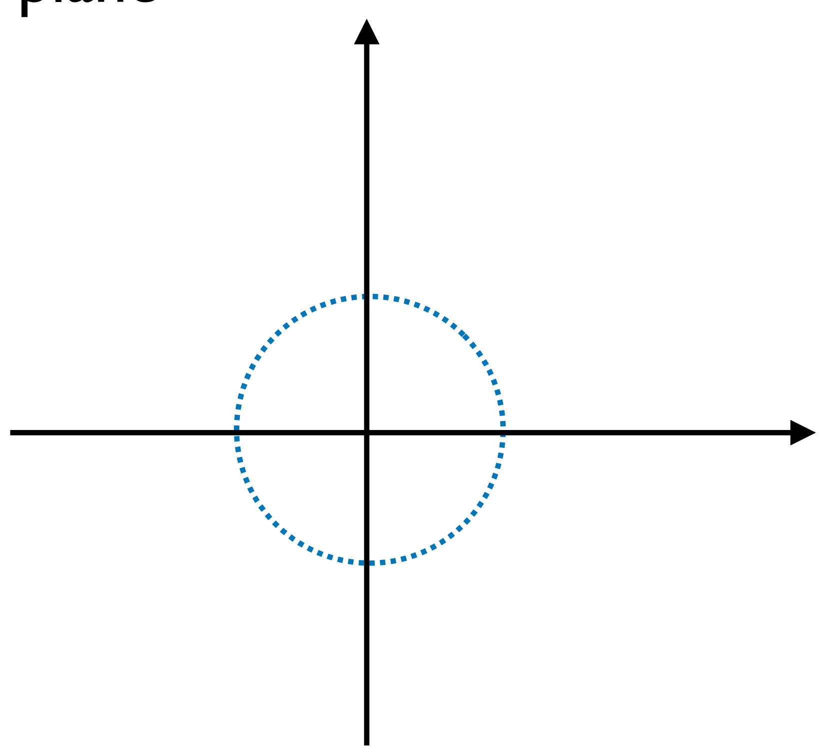
1. The formal **Borel transform** $\mathscr{B}: \mathbb{C}[[\tau^{-1}]]_1 \to \mathbb{C}\{\zeta\}$

$$\tilde{\Phi} = \sum_{n=0}^{\infty} n!\, \tau^{-n-1} \in \mathbb{C}[[\tau^{-1}]]_1 \dashrightarrow \hat{\Phi} \text{ analytic}$$

$$\downarrow \mathscr{B}$$

$$\sum_{n=0}^{\infty} n!\, \frac{\zeta^n}{n!} \in \mathbb{C}\{\zeta\}$$

Borel plane



Borel-Laplace summation and Resurgence

Borel-Laplace summation works in three steps

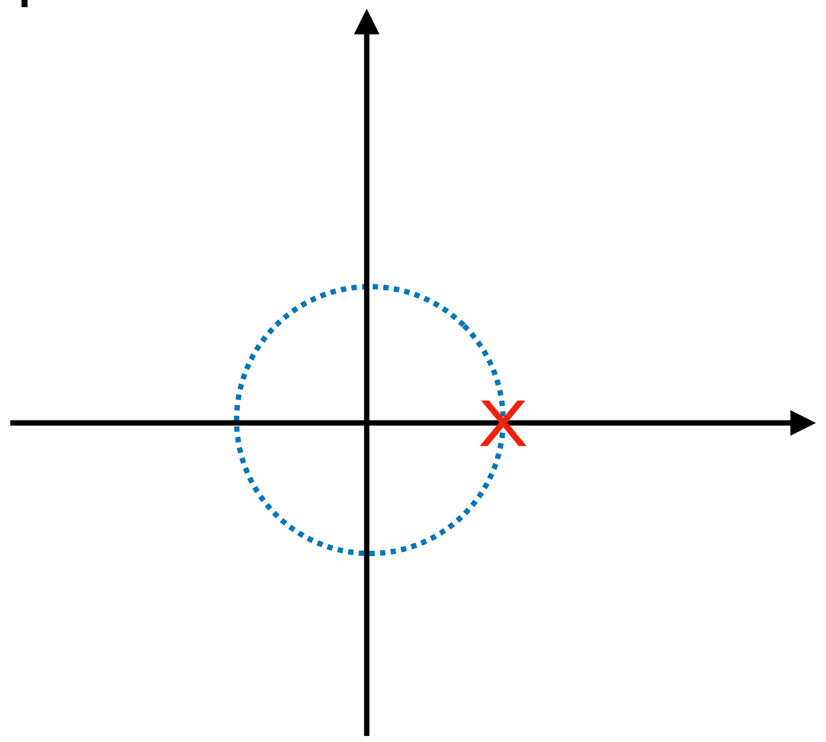
2. **Analytic continuation** in Borel plane

$$\tilde{\Phi} = \sum_{n=0}^{\infty} n!\,\tau^{-n-1} \in \mathbb{C}[\![\tau^{-1}]\!]_1 \dashrightarrow \hat{\Phi} \text{ analytic}$$

$$\Big\downarrow \mathscr{B}$$

$$\sum_{n=0}^{\infty} \zeta^n \in \mathbb{C}\{\zeta\} \xrightarrow[\text{sum}]{} \frac{1}{1-\zeta}$$

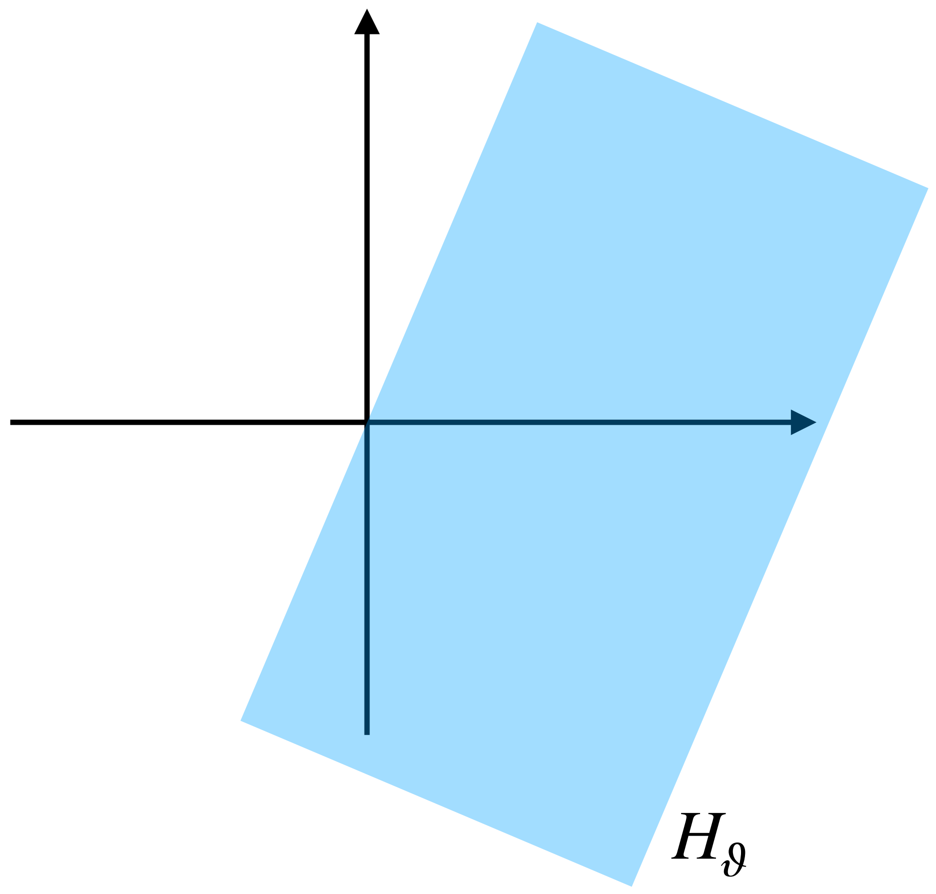
Borel plane



Borel-Laplace summation and Resurgence

Borel-Laplace summation works in three steps

3. The **Laplace transform** $\mathscr{L}^{\vartheta}$ defined along a ray in the direction ϑ that avoids the singularities

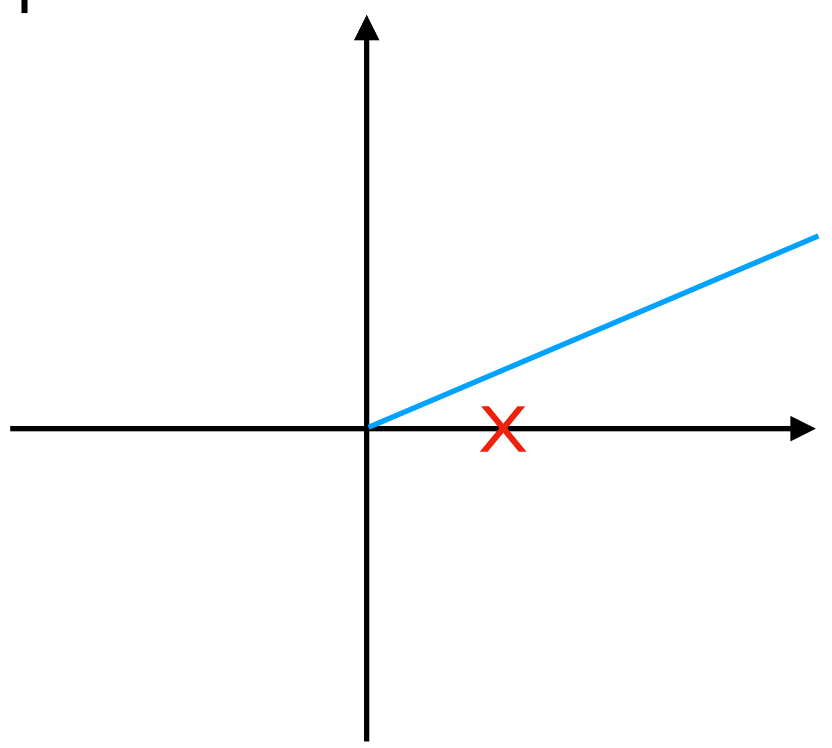


$$\tilde{\Phi} = \sum_{n=0}^{\infty} n!\,\tau^{-n-1} \in \mathbb{C}[\![\tau^{-1}]\!]_1 \xrightarrow{\text{Borel-Laplace sum}} \hat{\Phi} \text{ analytic}$$

$$\downarrow \mathscr{B} \qquad\qquad\qquad\qquad \uparrow \mathscr{L}^{\vartheta}$$

$$\sum_{n=0}^{\infty} \zeta^n \in \mathbb{C}\{\zeta\} \xrightarrow[\text{sum}]{} \frac{1}{1-\zeta}$$

Borel plane



The Borel transform is the **formal inverse** of the Laplace transform

Borel-Laplace summation and Resurgence

Borel-Laplace summation works in three steps

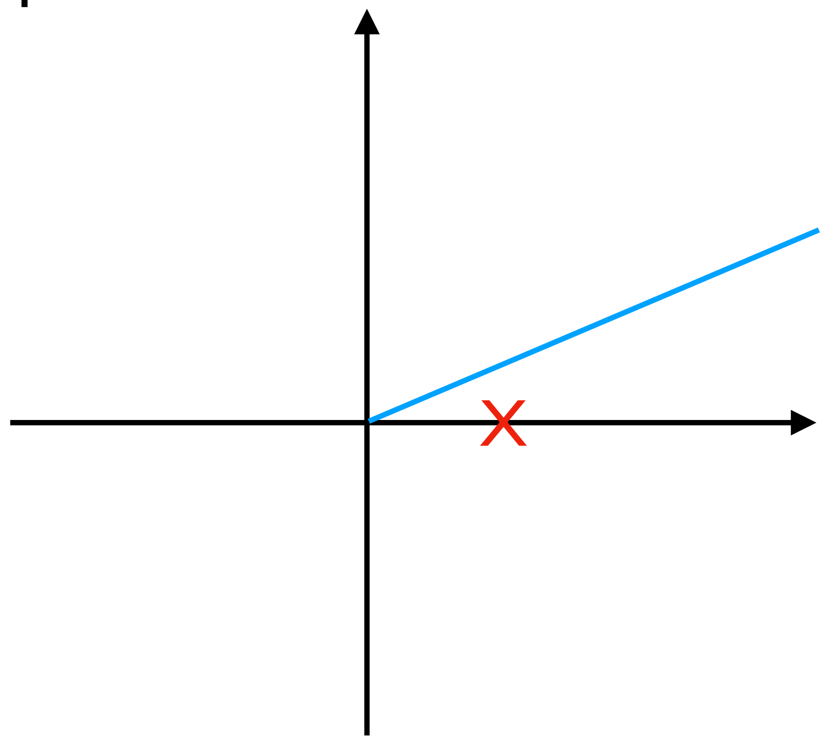
3. The Laplace transform $\mathscr{L}^\vartheta$ defined along a ray in the direction ϑ that avoids the singularities

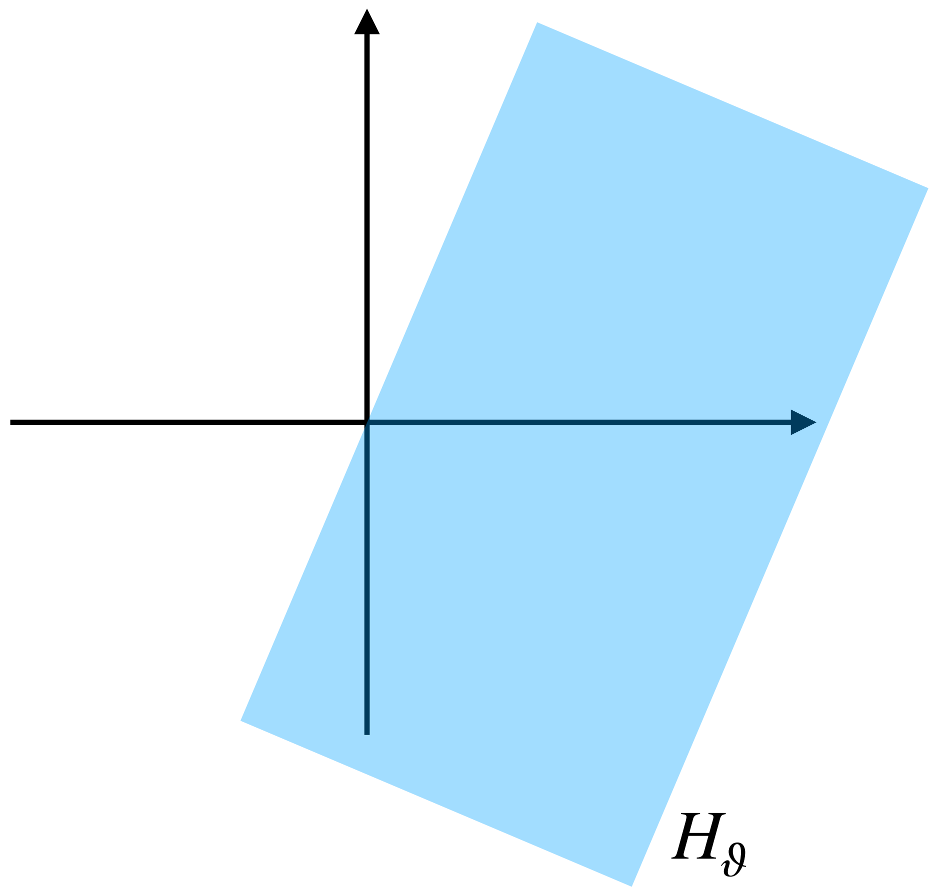
$$\tilde{\Phi} = \sum_{n=0}^{\infty} n!\,\tau^{-n-1} \in \mathbb{C}[\![\tau^{-1}]\!]_1 \quad \xrightarrow[\text{asymptotics}]{\text{Borel-Laplace sum}} \quad \hat{\Phi} \text{ analytic}$$

$$\downarrow \mathscr{B} \qquad\qquad\qquad\qquad \uparrow \mathscr{L}^\vartheta$$

$$\sum_{n=0}^{\infty} \zeta^n \in \mathbb{C}\{\zeta\} \quad \xrightarrow[\text{sum}]{} \quad \frac{1}{1-\zeta}$$

Borel plane



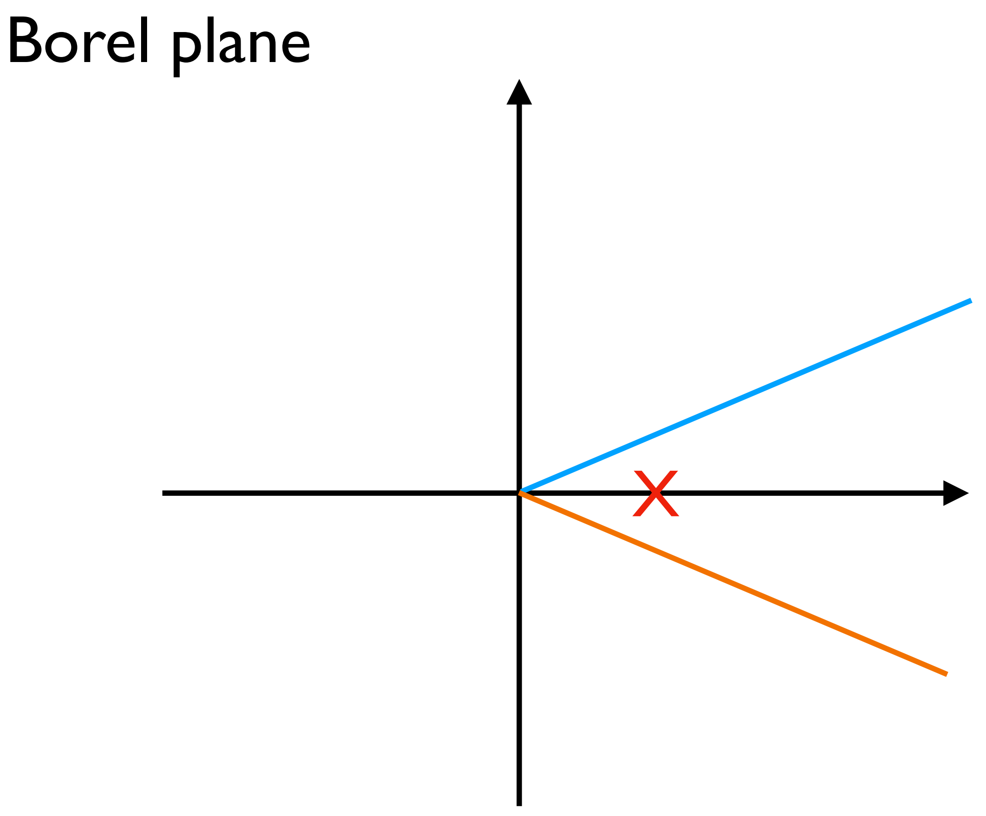


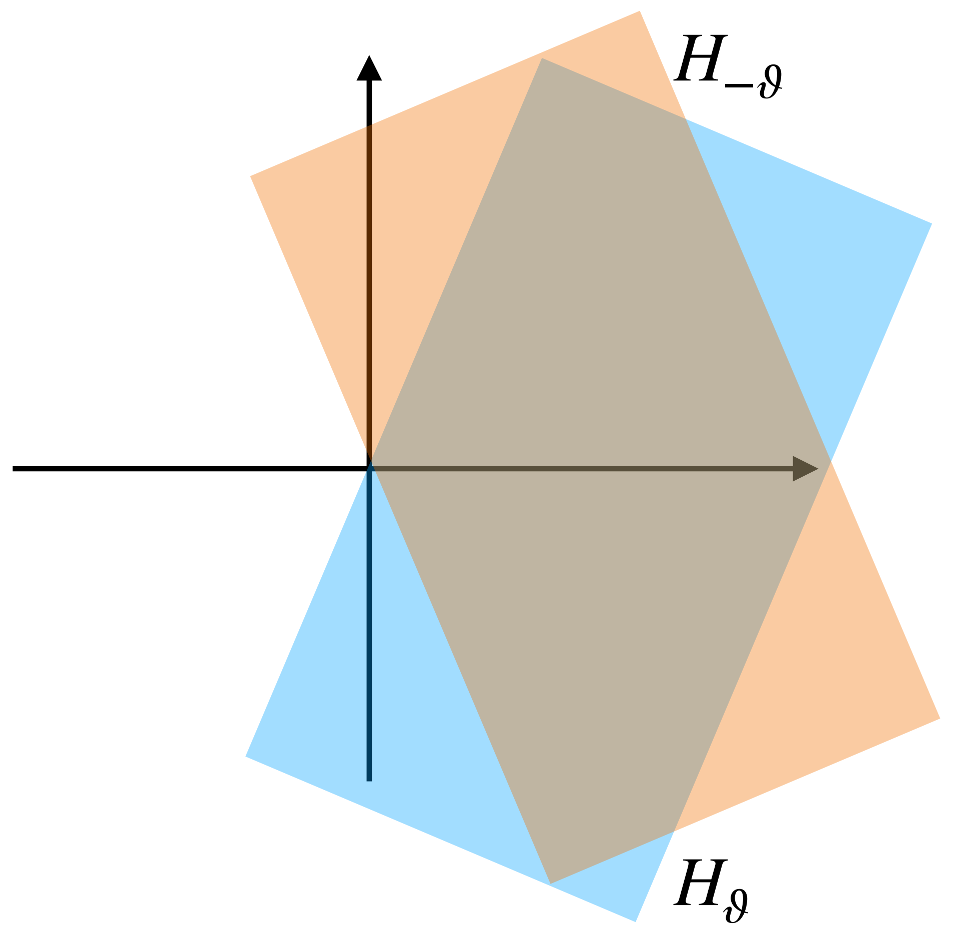
Remark: the Borel-Laplace sum $\hat{\Phi}$ is uniform (Gevrey) asymptotics to $\tilde{\Phi}$

Borel-Laplace summation and Resurgence

The singularities in Borel plane know about exponentially small terms

- What is the effect of the singularity? If the exponentially small terms can be reconstruct from its asymptotics by studying the analytic continuation of the Borel transform.





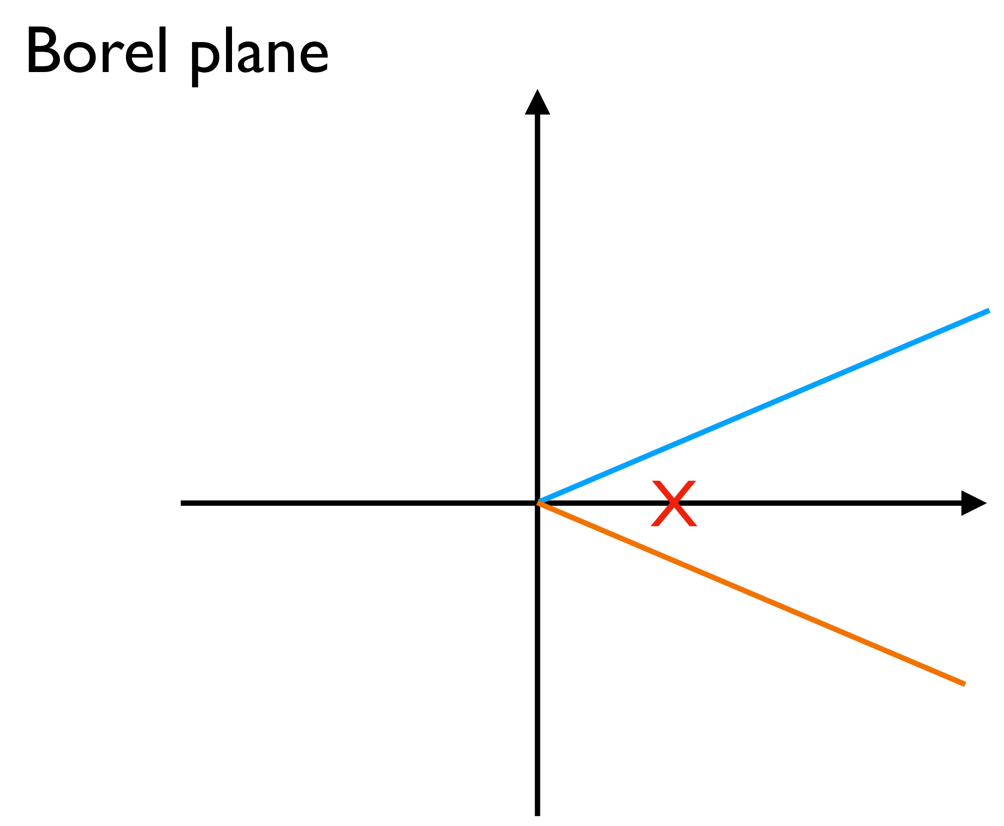
$$\left[\mathscr{L}^{\vartheta} - \mathscr{L}^{-\vartheta}\right]\frac{1}{1-\zeta} = \int_{\mathscr{C}_1} e^{-\tau\zeta}\frac{1}{1-\zeta}\,d\zeta = -2\pi \mathrm{i}\; e^{-\tau}$$

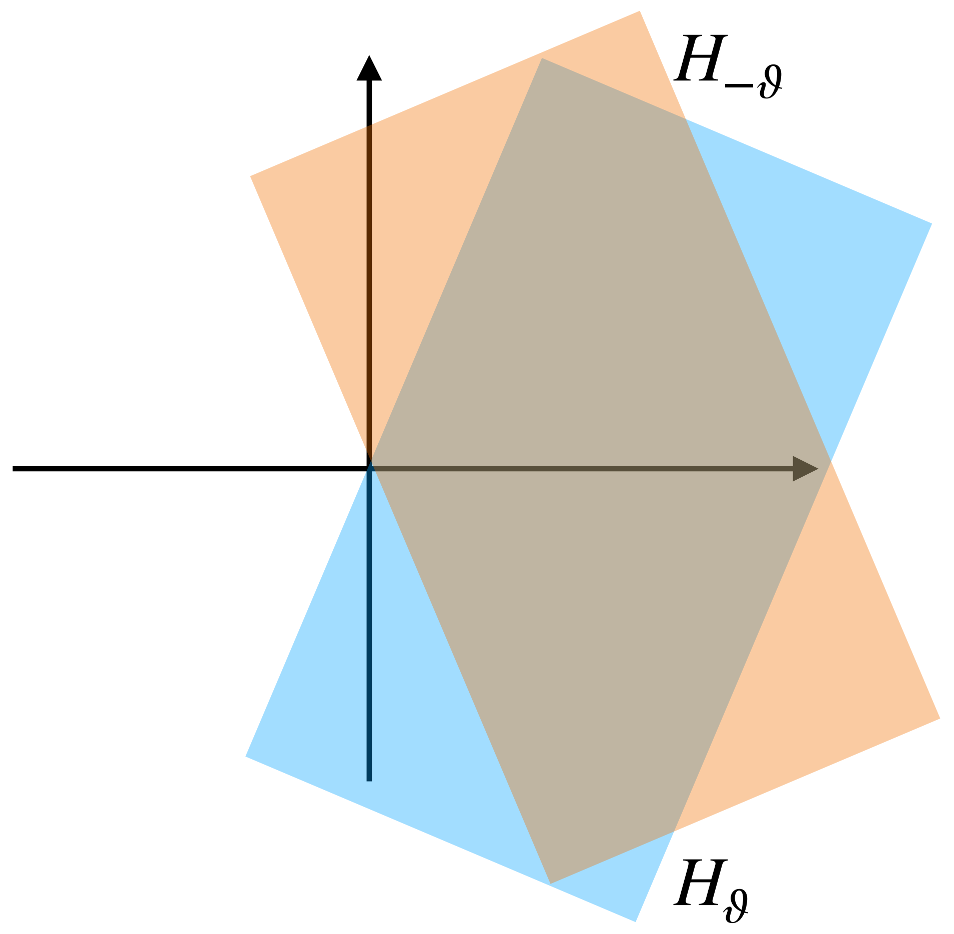
- The analytic continuation of $\hat{\Phi}$ jumps when crossing a singularity and the jump is given by exponentially small corrections

Borel-Laplace summation and Resurgence

Resurgence [Écalle 80s]

- A divergent series $\tilde{\Phi}(\tau) \in \mathbb{C}[\![\tau^{-1}]\!]_1$ is **resurgent** if the exponentially small terms can be reconstruct from $\tilde{\Phi}$ itself by studying the analytic continuation of its Borel transform





$$\left[\mathscr{L}^{\vartheta} - \mathscr{L}^{-\vartheta}\right]\frac{1}{1-\zeta} = \int_{\mathscr{C}_1} e^{-\tau\zeta}\frac{1}{1-\zeta}\,d\zeta = \underbrace{-2\pi\mathrm{i}}_{\text{Residue}}\;\underbrace{e^{-1\cdot\tau}}_{\text{Singularity}}$$

- The constant $-2\pi\mathrm{i}$ is the so-called **Stokes constant**, which constitutes part of the information encoded at the singularity

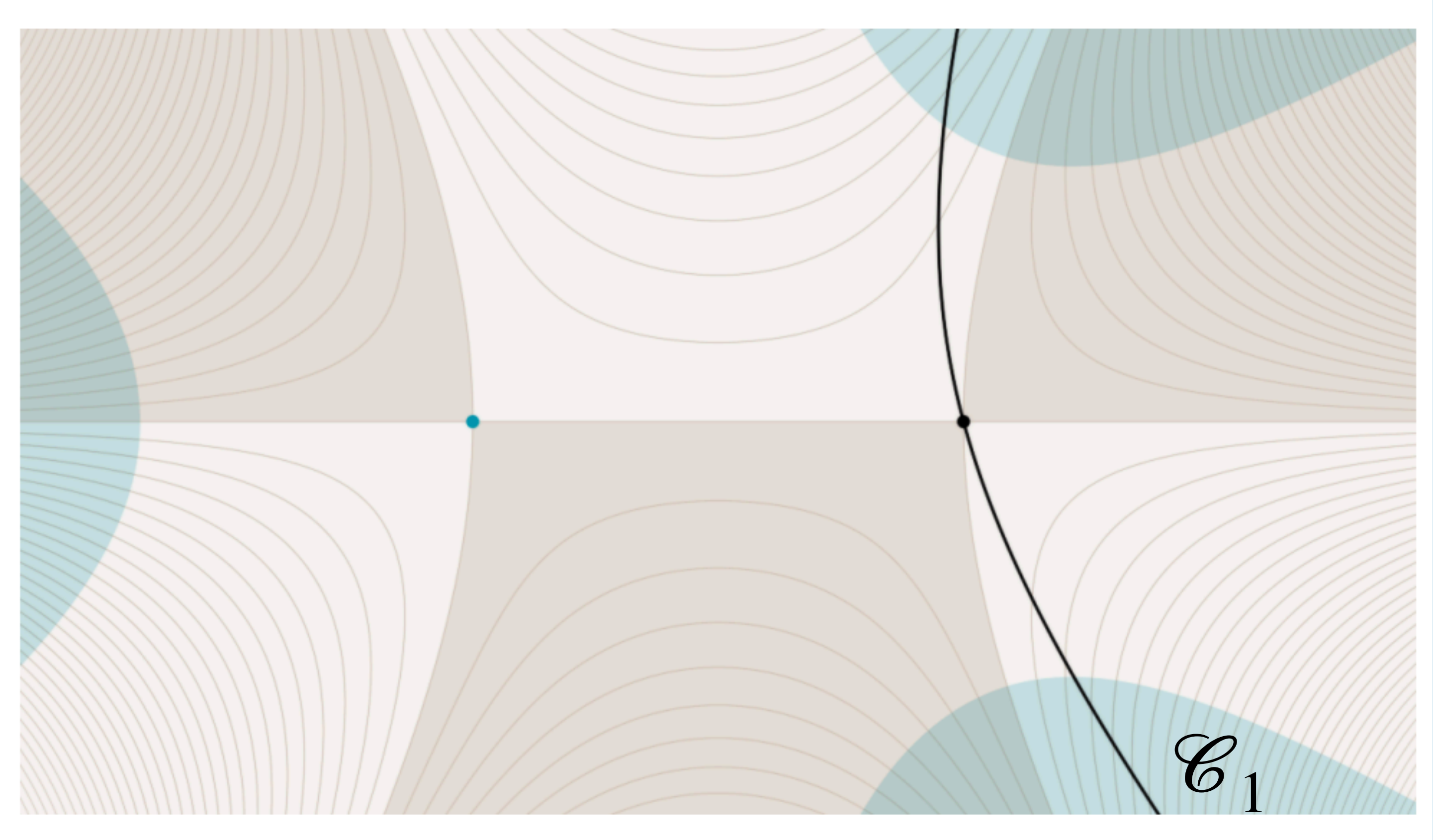
Warm-up: thimble integrals

Thimble integrals

The examples of the Airy function

- Thimble integrals are integrals over the steepest descent contours of a Morse function f. Thus, they define analytic functions

$$\mathrm{Ai}(\tau) = \int_{\mathscr{C}_1} e^{-\tau f}\, \nu\,, \quad \text{where } f(z) = \frac{z^3}{3} - z\,,\ \nu = dz$$



- The asymptotic expansion of $\mathrm{Ai}(\tau)$ as $\tau \to \infty$ is a divergent power series, whose Borel transform has singularities at the critical values of f

The assumption of f being Morse can be relaxed: isolated critical values, but they might be degenerate [Mistergard]

Thimble integrals

Thimbles integrals are Borel-Laplace summable

- Thimble integrals are integrals over the steepest descent contours. Thus, they define analytic functions

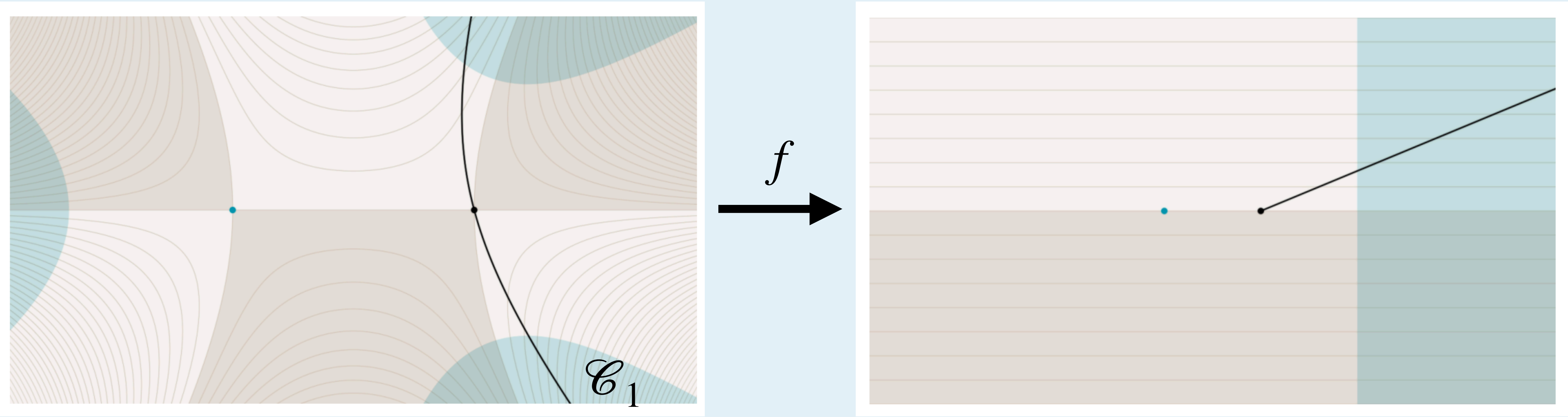
$$\mathrm{Ai}(\tau) = \int_{\mathscr{C}_1} e^{-\tau f}\, \nu\,, \quad \text{where } f(z) = \frac{z^3}{3} - z\,, \nu = dz$$

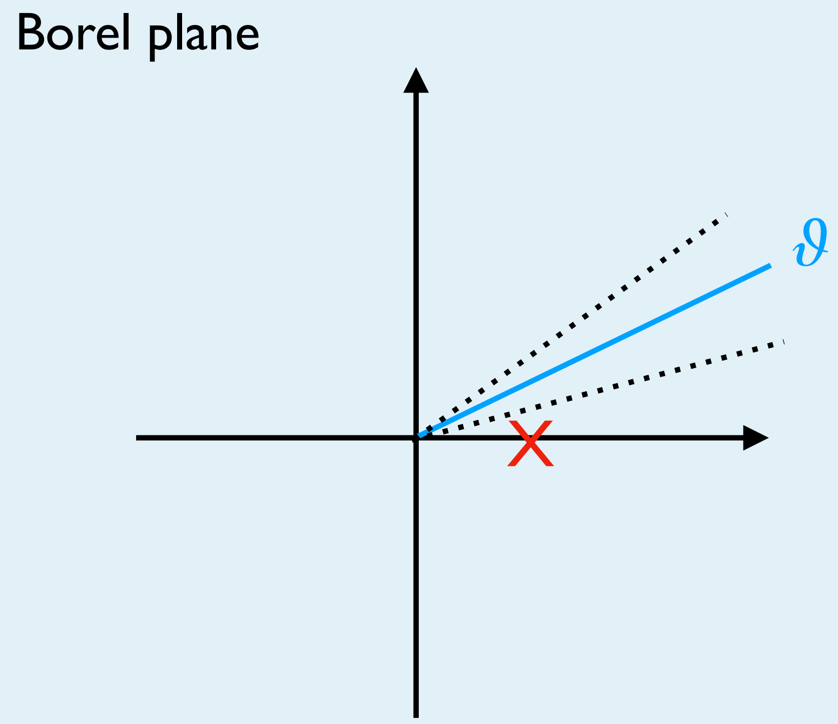
- The asymptotic expansion of $\mathrm{Ai}(\tau)$ as $\tau \to \infty$ is a divergent power series, whose Borel transform has singularities at the critical points of f

- Thimble integrals are Borel-Laplace transforms, i.e. they are the sum of their asymptotics

    - Change of coordinates $\zeta = f$. Indeed the dependence on the variable τ is only at the exponent [**F.** —Fenyes]

    - Use Nevanlinna (or Watson) theorem. Indeed the asymptotics of $\mathrm{Ai}(\tau)$ holds for $\theta = \arg(\tau)$ in a sector of opening angle $\geq \pi$





Thimble integrals

The Stokes constants counts saddle points connections

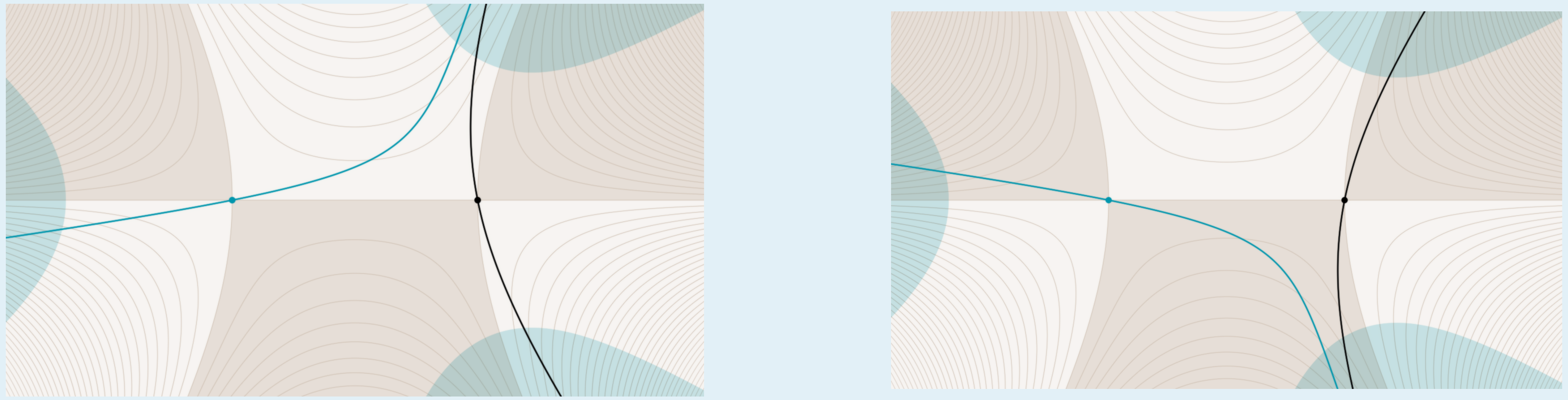
- Thimble integrals are integrals over the steepest descent contours. Thus, they define analytic functions

$$\mathrm{Ai}(\tau) = \int_{\mathscr{C}_1} e^{-\tau f}\, \nu\,, \quad \text{where } f(z) = \frac{z^3}{3} - z\,,\, \nu = dz$$

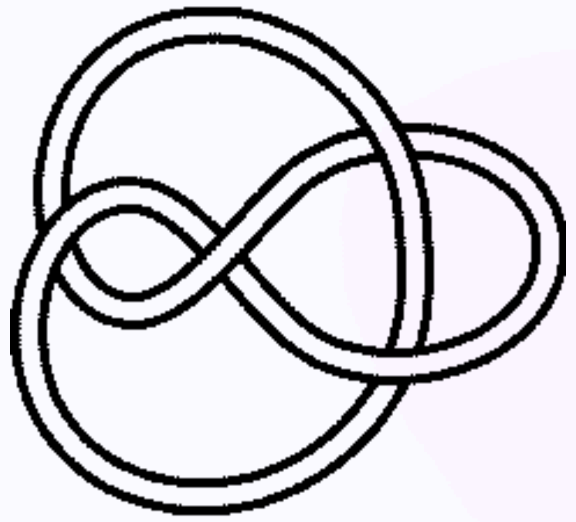
- The asymptotic expansion of $\mathrm{Ai}(\tau)$ as $\tau \to \infty$ is a divergent power series, whose Borel transform has singularities at the critical points of f

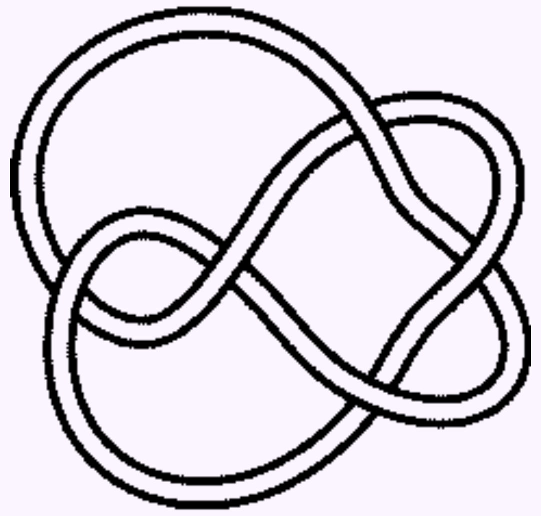
- Thimble integrals are Borel-Laplace transforms, i.e. they are the sum of their asymptotics

- The **Stokes constants** count saddle connections between different critical points and they are computed by Picard—Lefschetz formula



State integrals for the 4_1 and 5_2 knots





The formal invariant $\tilde{\Phi}_K$

Recap

- The formal series we want to resum is

$$\tilde{\Phi}_{\Xi}(\tau) = \int \tilde{\Psi}(z,\tau)^{B}\mathbf{e}\Big(-\frac{A}{2}z\tau\Big(z+1-\frac{1}{\tau}\Big)+m_0 z\tau\Big)dz\,,$$

where $\Xi = (A, B, p_0)$ and $p_0 = (z_0, m_0)$ is a critical point of the CS functional

- Recall $\tilde{\Psi}(z,\tau)$ is the asymptotic of the Faddeev's quantum dilogarithm $\tilde{\Psi}(z,\tau) = \mu_8\,\mathbf{e}\Big(-\frac{\tau}{24}-\frac{1}{24\tau}-\sum_{k=0}^{\infty}(2\pi \mathrm{i})^k\frac{B_k}{k!}\mathrm{Li}_{2-k}(\mathbf{e}(z))\tau^{1-k}\Big)$

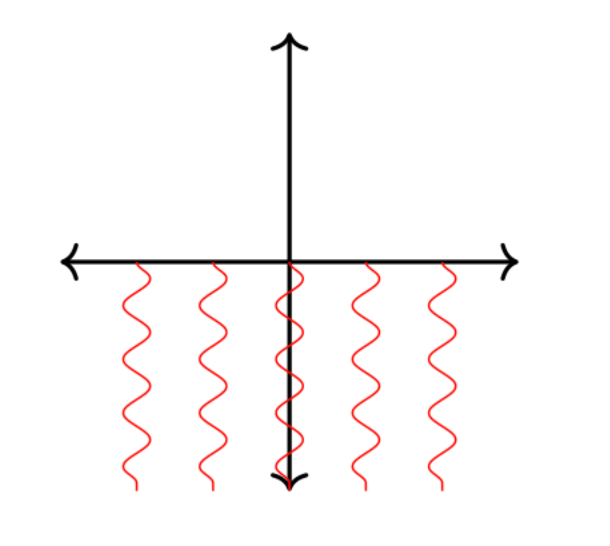
- The potential function V is defined as

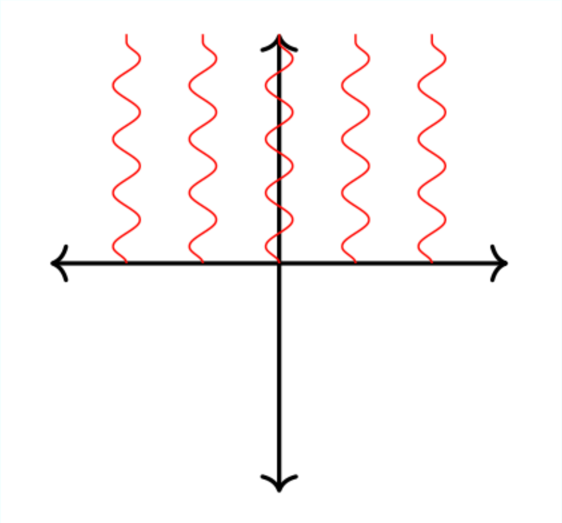
$$V(z,m) = B\,\frac{\mathrm{Li}_2(\mathbf{e}(z))}{(2\pi \mathrm{i})^2} + \underbrace{\frac{B}{24}+\frac{A}{2}z(z+1)+mz}_{\text{Gaussian}}\,,$$

which is multivalued!!!

The volume

The Riemann surface of V





- Choose a branch of $\mathrm{Li}_2(\mathbf{e}(z))$ and restrict the potential V to the Riemann surface Σ

- Thus, we restrict the image of V to the cylinder $\mathbb{C}/\mathbb{Z}$ to get a holomorphic Morse function $V : \Sigma \to \mathbb{C}/\mathbb{Z}$ such that

$$V(z, m) = B\,\frac{\mathrm{Li}_2(\mathbf{e}(z))}{(2\pi \mathrm{i})^2} + \frac{B}{24} + \frac{A}{2} z(z+1) + mz\,,$$

where $m = 1, \ldots, A$

- The function V computes the **volumes** of the 4_1 (reps. 5_2) knots with the parameters $(A, B) = (1,2)$ and (resp. $(2,3)$)

- For 4_1, we find that $(z_1, m_1) = (-1/6, 0)$ and $(z_2, m_2) = (-5/6, 0)$ are the two critical points and these have volumes

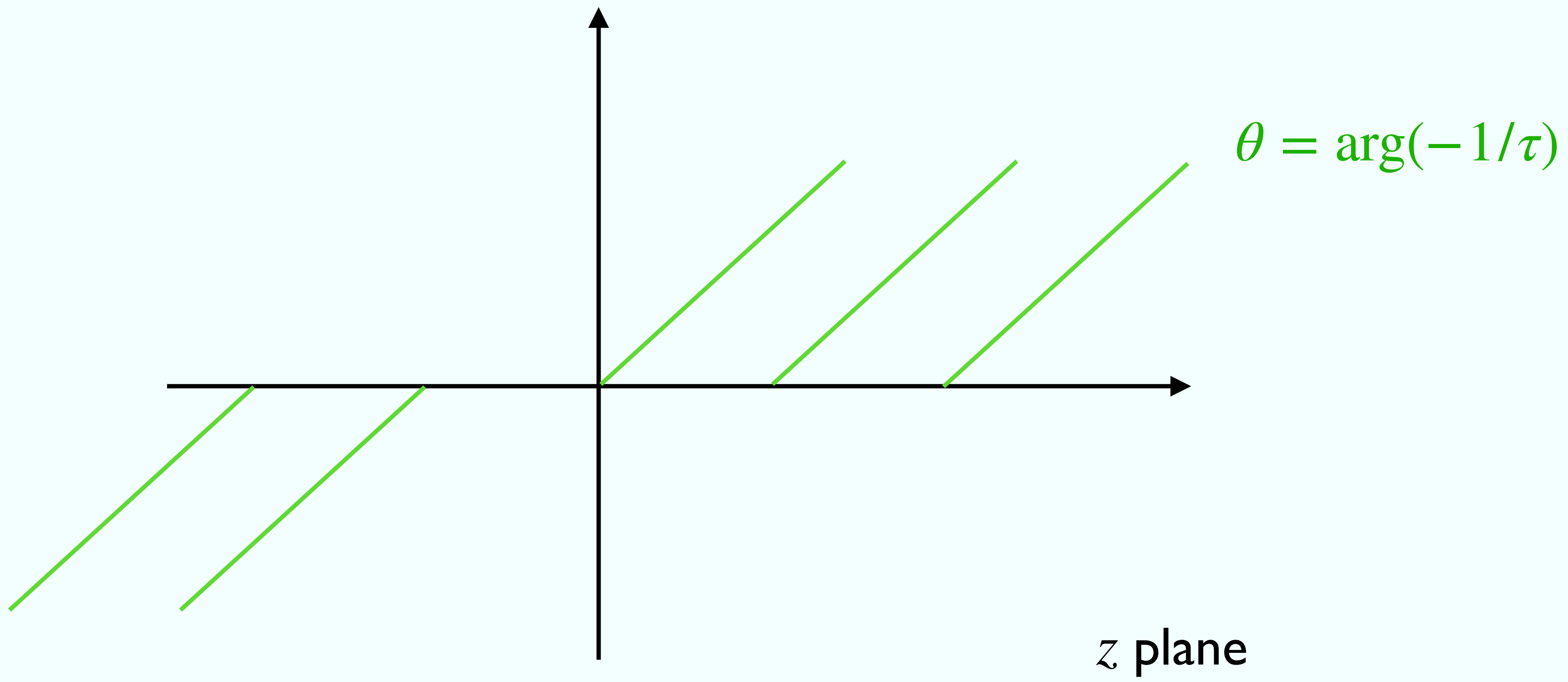
$$V(-1/6, 0) = 0.051418\cdots \mathrm{i}\,, \qquad V(-5/6, 0) = -\,0.051418\cdots \mathrm{i}\,.$$

The asymptotics of Faddeev's quantum dilogarithm

Different asymptotics in different sectors

- Choose a determination of $\mathrm{Li}_2(\mathbf{e}(z))$

- The Faddeev's quantum dilogarithm $\Phi(z;\tau)$ is uniform Gevrey asymptotic to $\tilde{\Psi}(z,\tau)$ as $|\tau| \to \infty$ for fixed argument of z bounded away from the green lines



- There is a non-trivial dependence on z, which is the integration variable

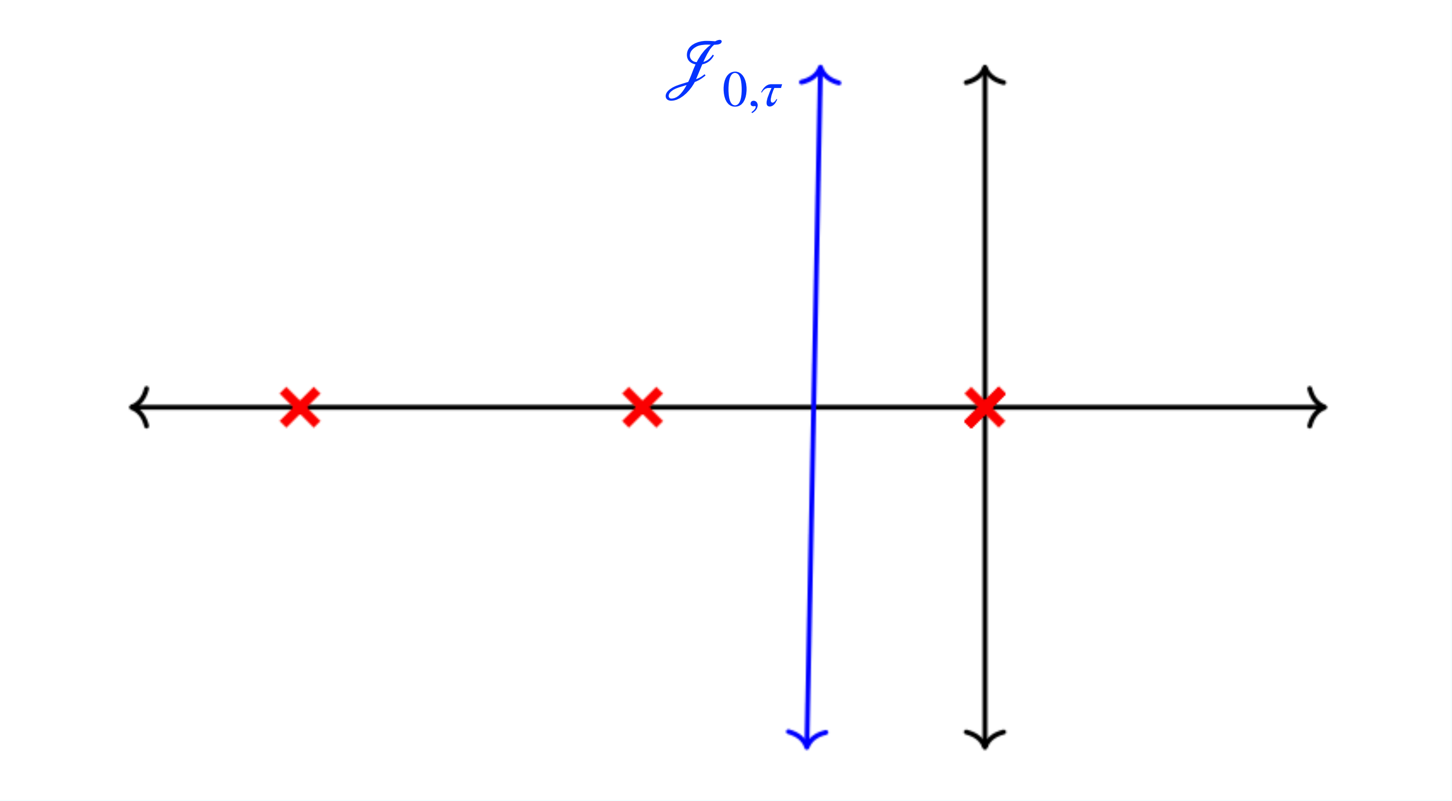
State integrals

An analytic candidate

- For every $m, \ell \in \mathbb{Z}_{>0}$, the descendants of the **state integrals** are defined as

$$I_{m,\ell}(\tau) := \mu_8^B \tilde{q}^{-B/24} q^{B/24} \int_{\mathcal{J}_{\ell,\tau}} \Phi((z-\ell)\tau;\tau)^B \mathbf{e}\Big(\frac{A}{2} z(z\tau+\tau+1) + mz\tau\Big) dz\,,$$

where the contour $\mathcal{J}_{\ell,\tau} := \left(\frac{\mathrm{i}}{\sqrt{\tau}} e^{-\mathrm{i}A\epsilon}\mathbb{R}_{\geq 0} - \frac{1}{2} + \ell\right) \cup \left(\frac{\mathrm{i}}{\sqrt{\tau}} e^{-\mathrm{i}(A-B)\epsilon}\mathbb{R}_{\leq 0} - \frac{1}{2} + \ell\right)$ and $m, \ell \in \mathbb{Z}$

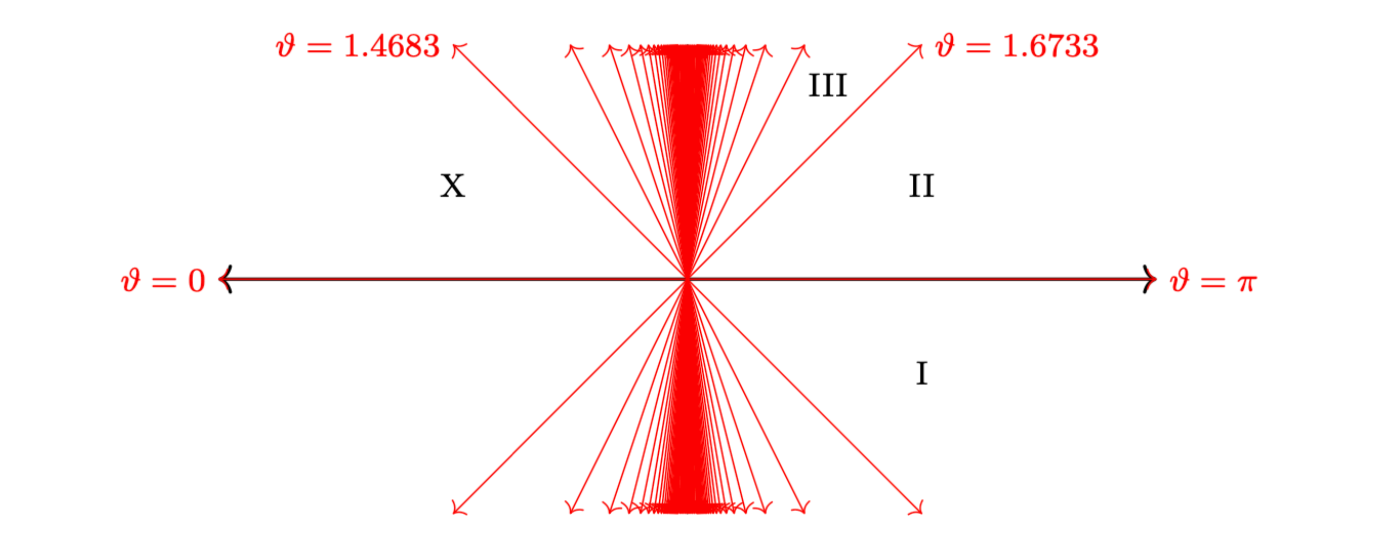


- $I_{0,0}(\tau)$ corresponds to AK state integral, where $4_1 : (A=1, B=2)$ and $5_2 : (A=2, B=3)$
- Since $\Phi(z;\tau)$ is meromorphic, the state integrals $I_{m,\ell}(\tau)$ are **analytic** functions
- There are linear relations between $I_{m,\ell}$ — the $\max\{A,B\}$ are independent

- **Theorem** [Andersen—Kashaev]: AK state integrals are topological invariants of the knot K, and they are the partition function of the Teichmüller TQFT

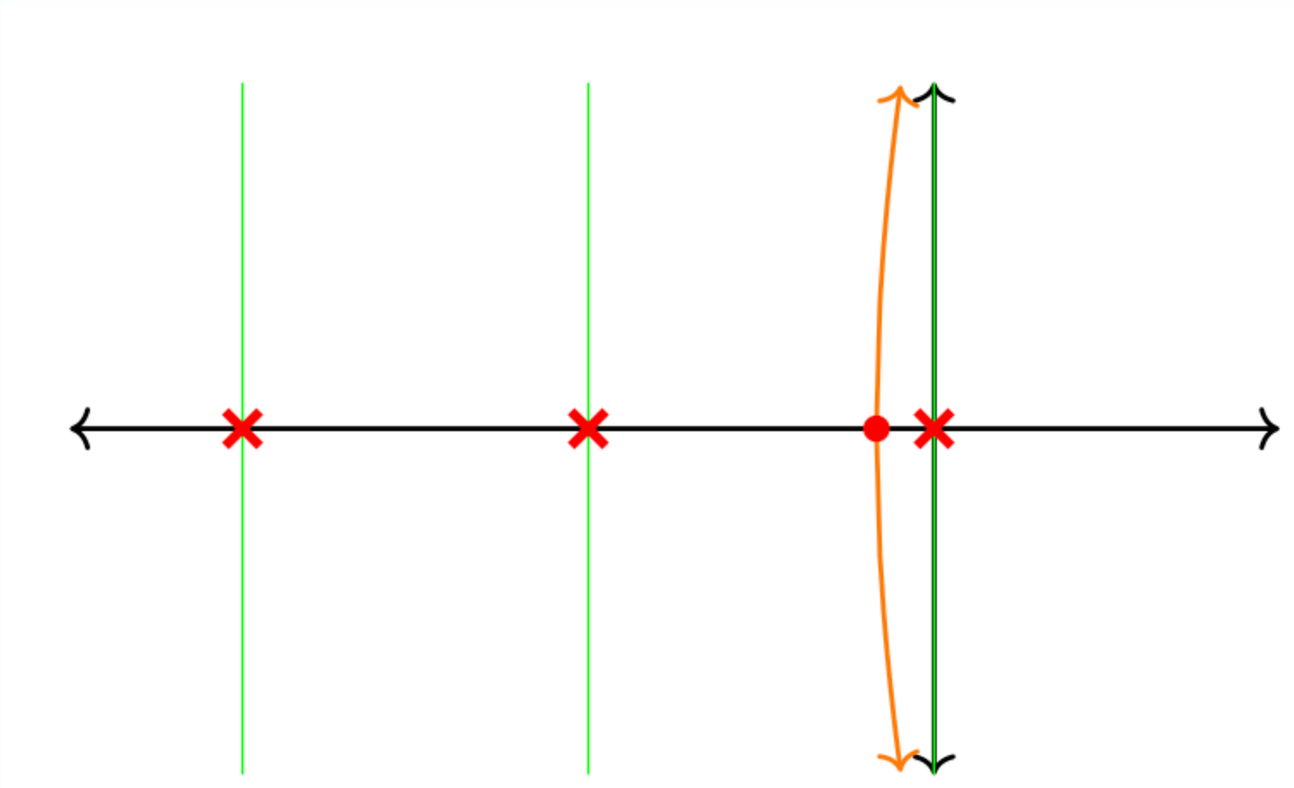
The steepest descent contours

Steepest descent contours vs state integrals



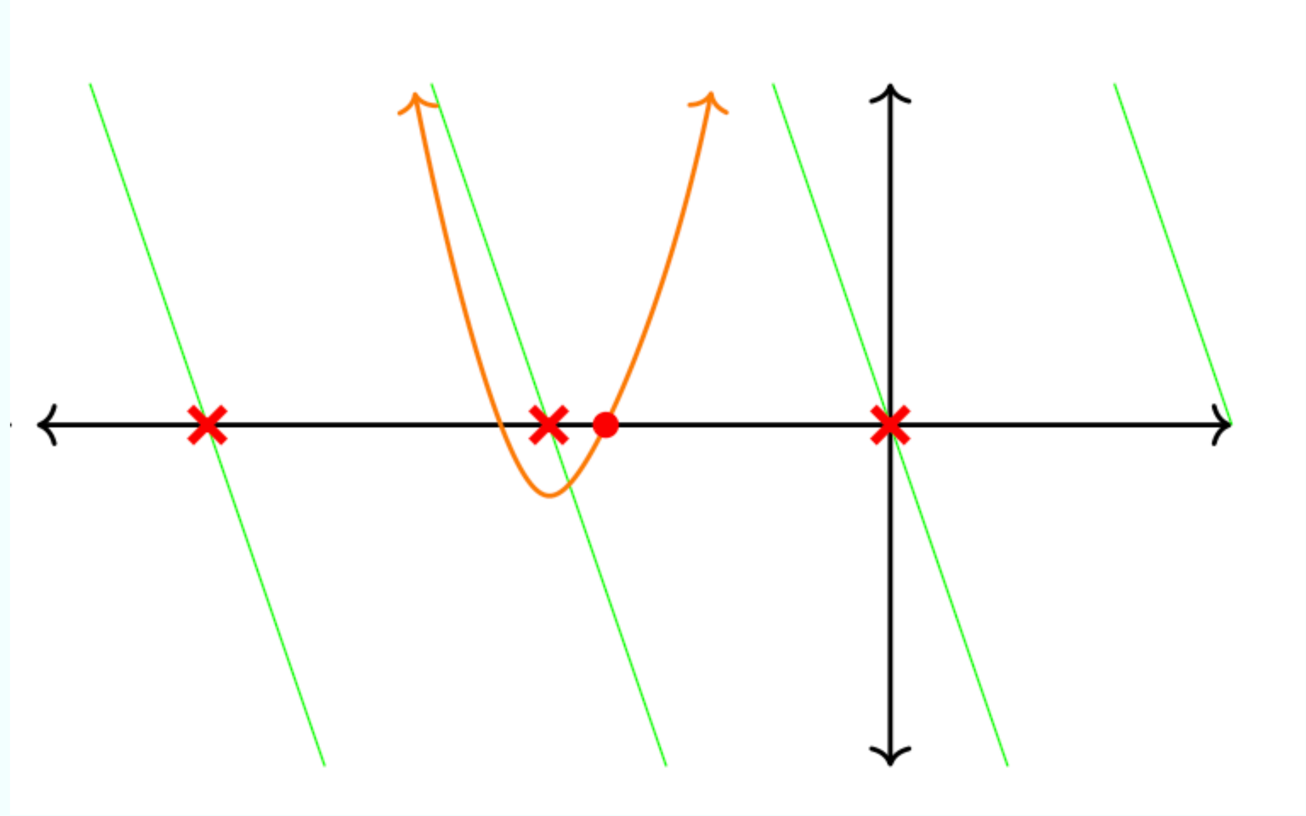
- The steepest descent contours are the level set of the imaginary part of V, drawn in orange

For $\vartheta = \pi$



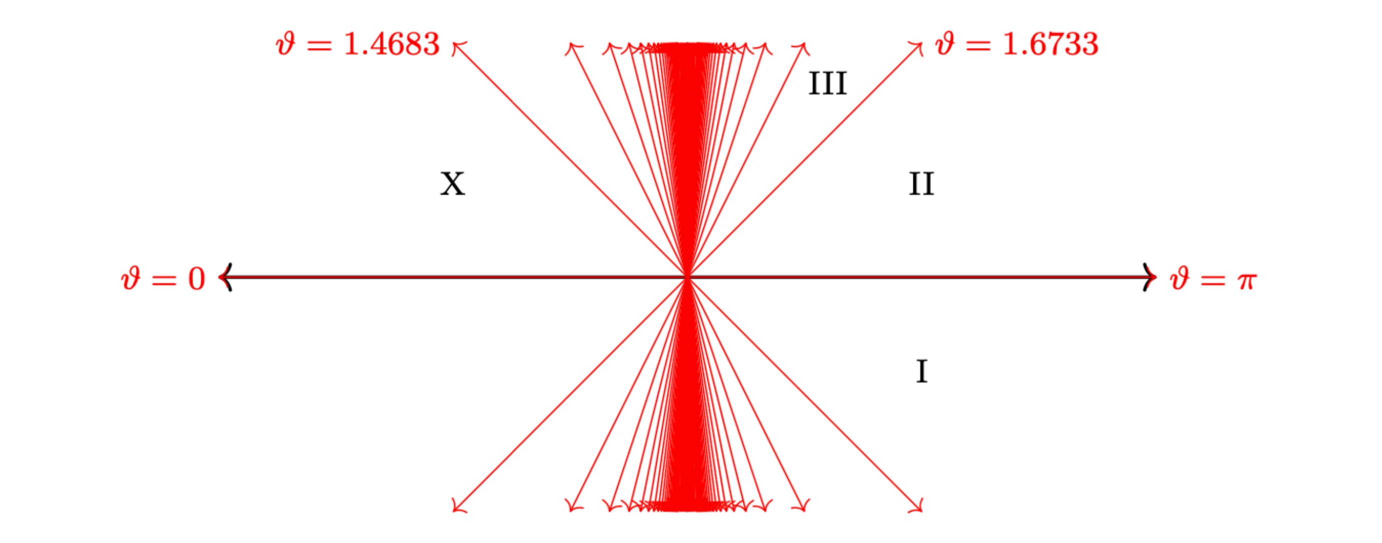
The thimble through $-1/6$

For $\vartheta \in I$



The thimble through $-5/6$

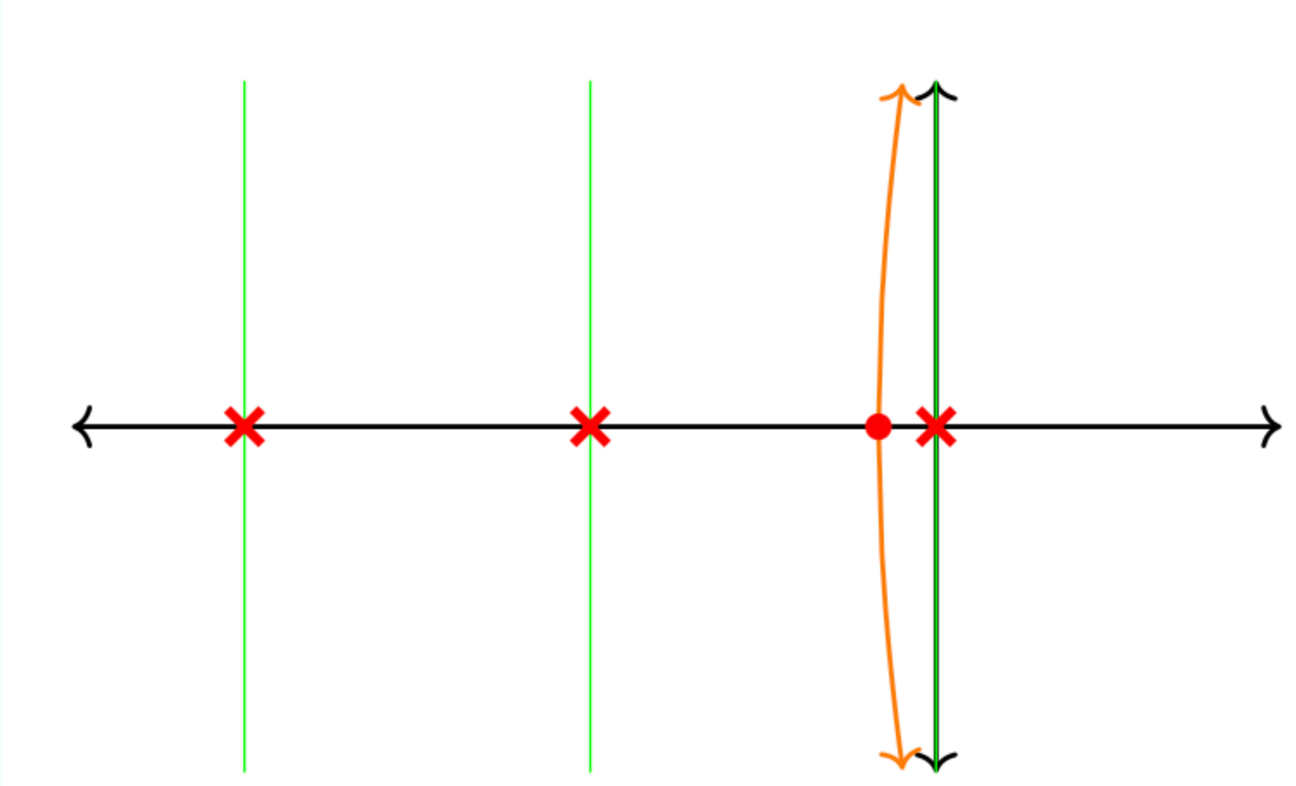
The steepest descent contours



Steepest descent contours vs state integrals

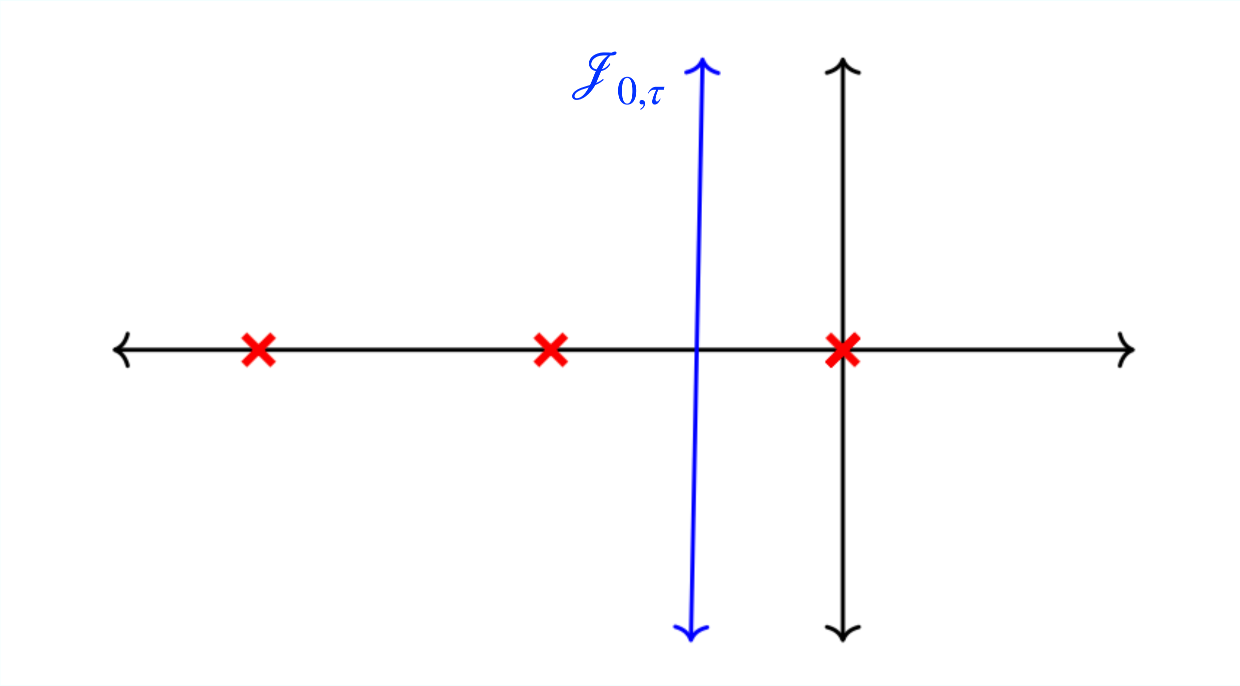
- The steepest descent contours are the level set of the imaginary part of V, drawn in orange
- Can we deform these contours to the state integral ones? NOT always — the state integral contours do not live on the surface Σ
- Crossing green lines $\Leftrightarrow$ state integrals will have different leading asymptotics

For $\vartheta = \pi$

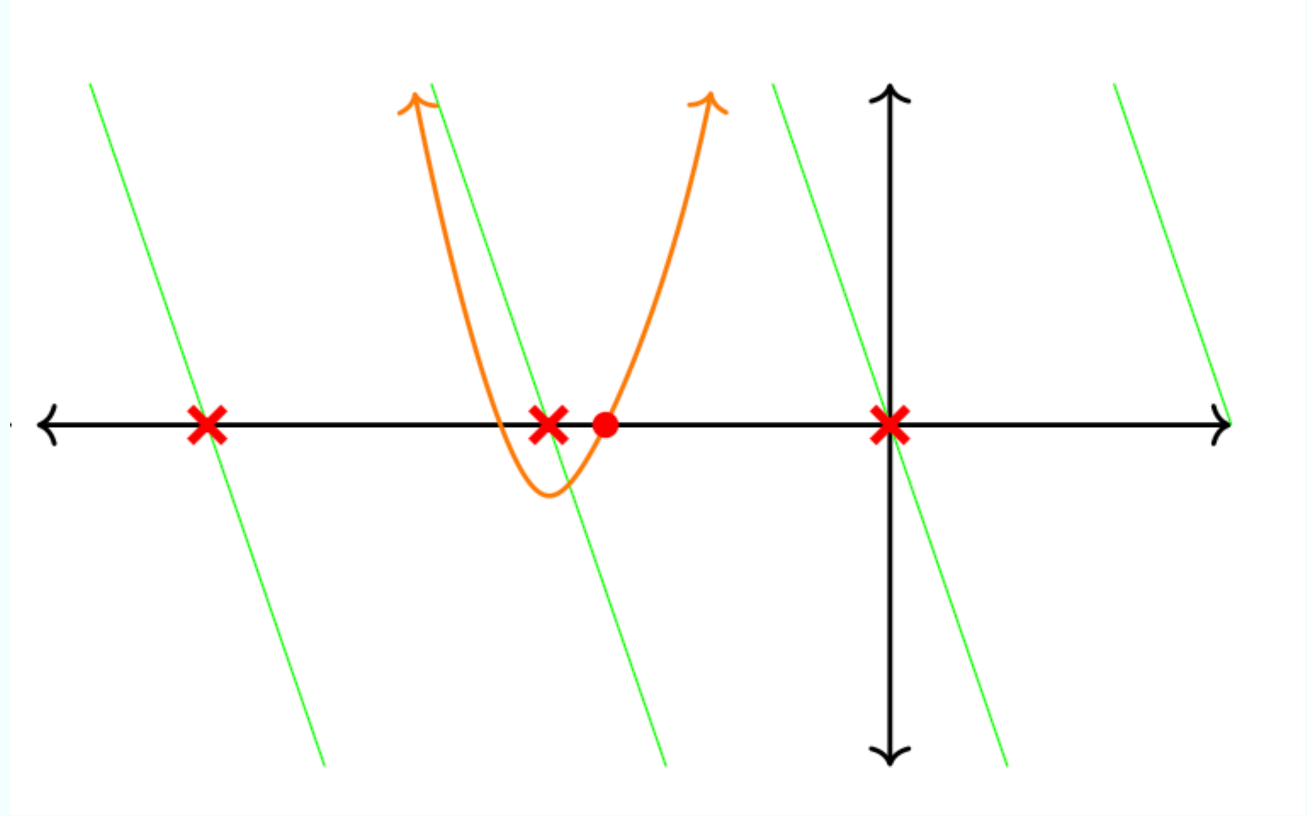


The thimble through $-1/6$ can be deformed

✓



For $\vartheta \in I$



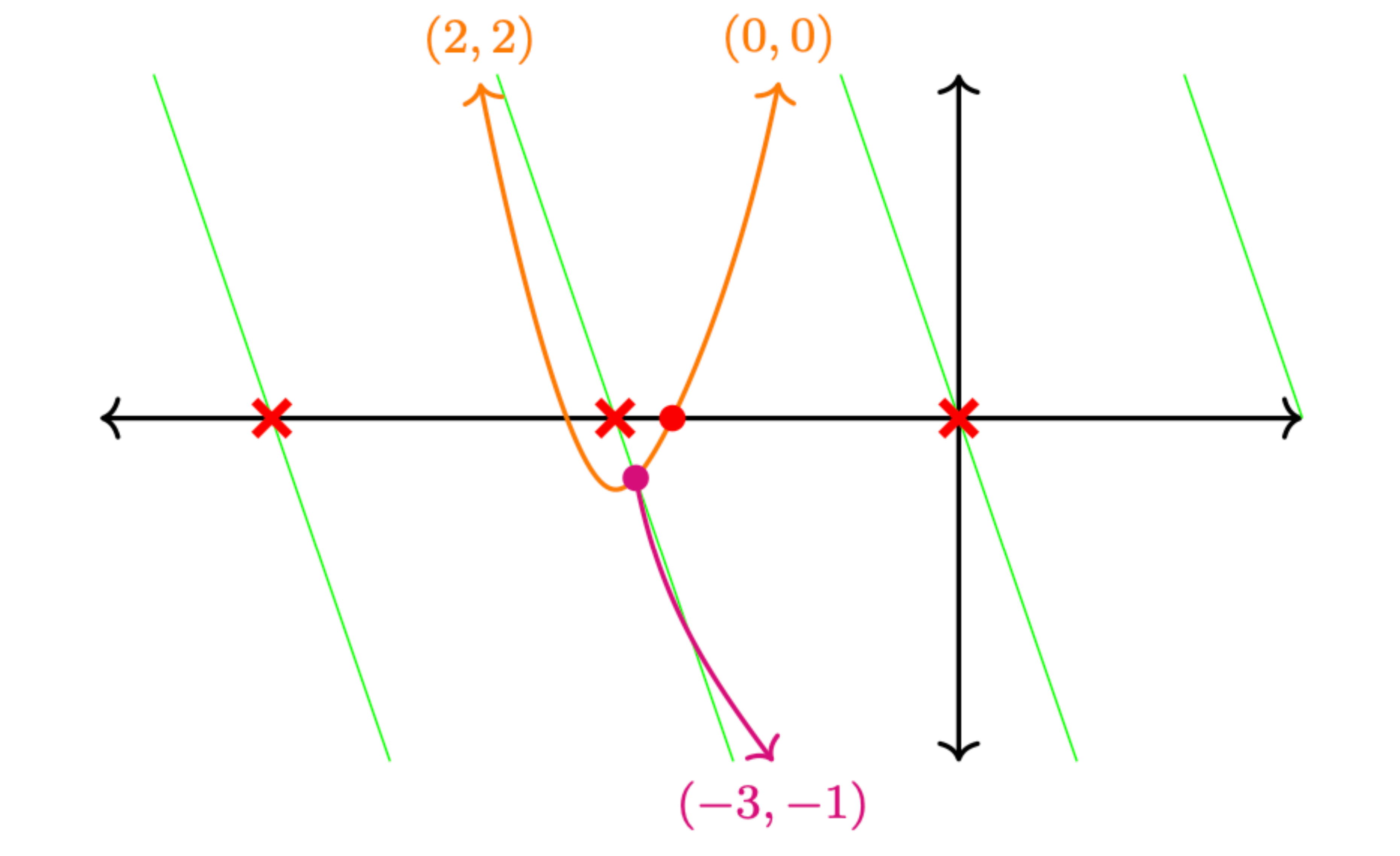
The thimble through $-5/6$ cannot be deformed

✗

The steepest descent contours

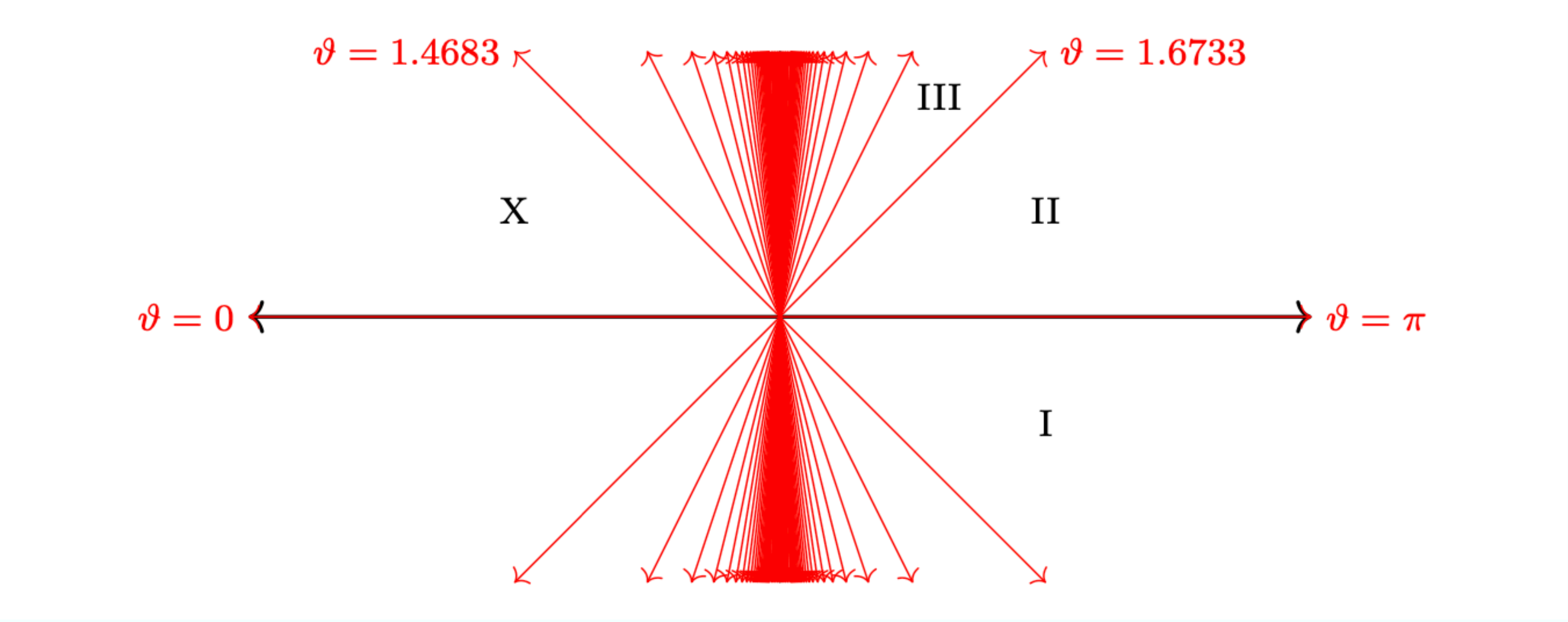
Algorithm [F.—Wheeler]

- Every time the thimble intersects a green line, we should flow again with a different volume
- Collect the contributions that might come from these new contours if they intersect the reals



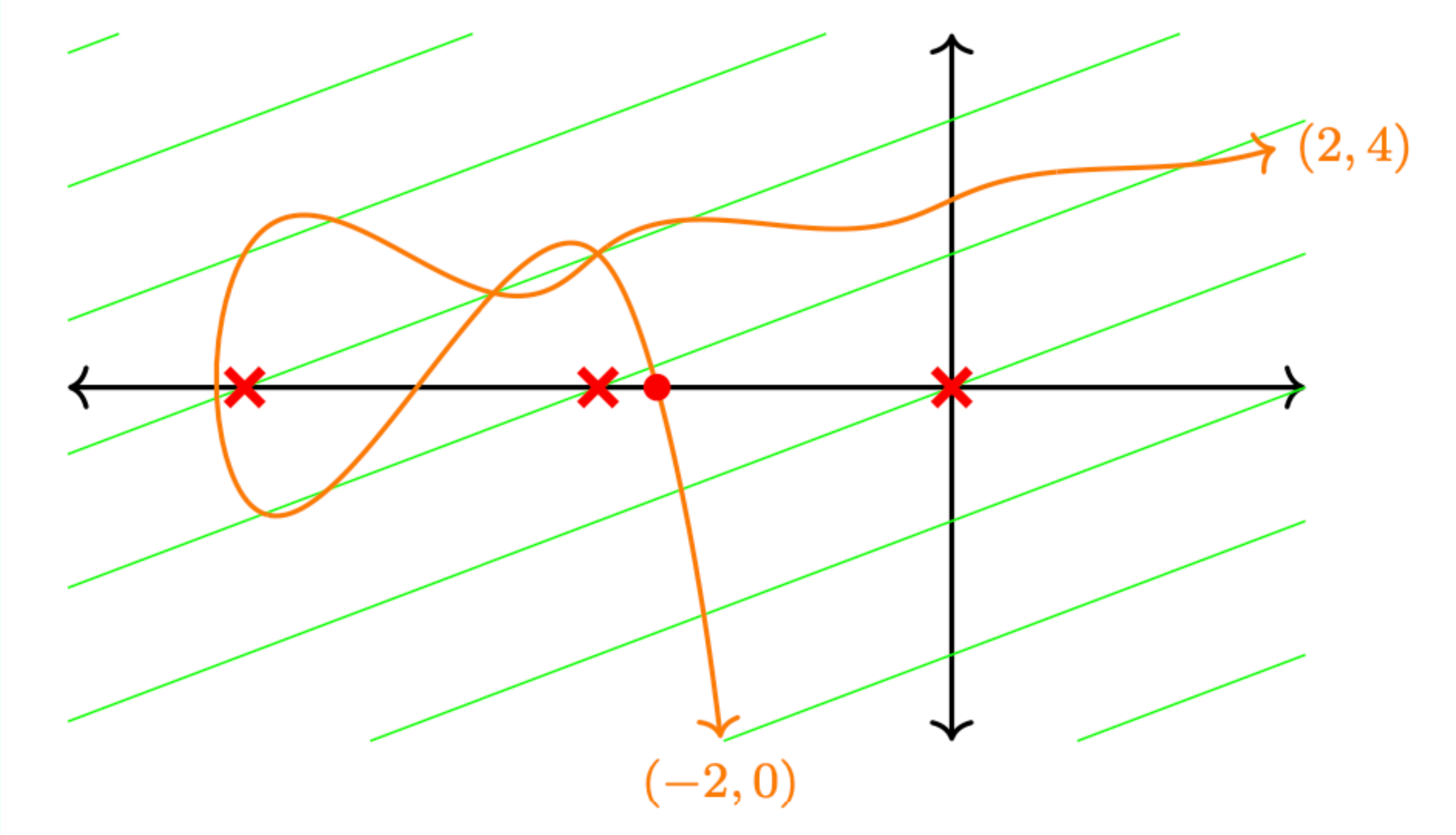
No more contributions from the magenta contour

The steepest descent contours

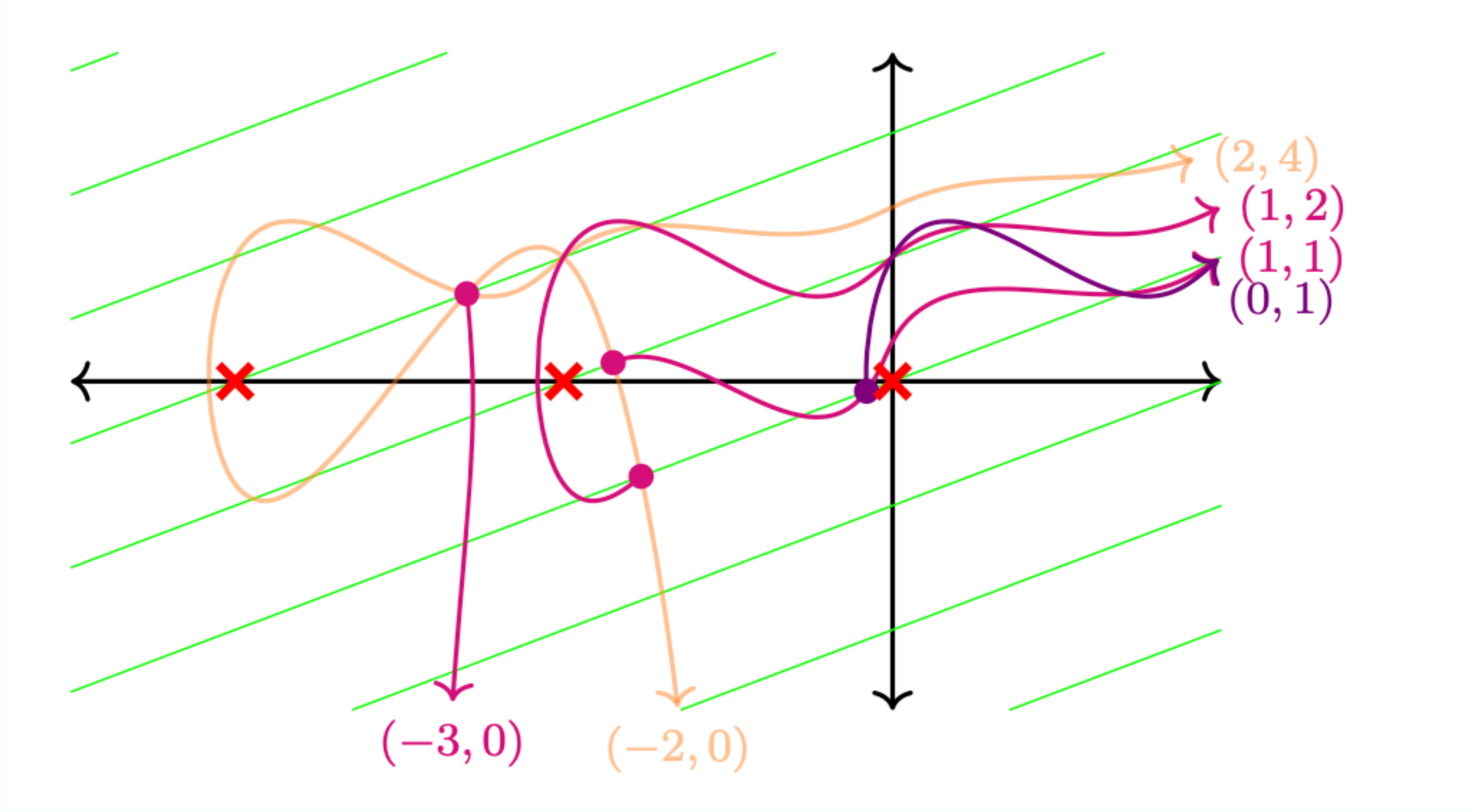


Algorithm [F.—Wheeler]

- For $\vartheta \in \mathrm{III}$ the thimble is



- More contributions are coming from the magenta and violet contours

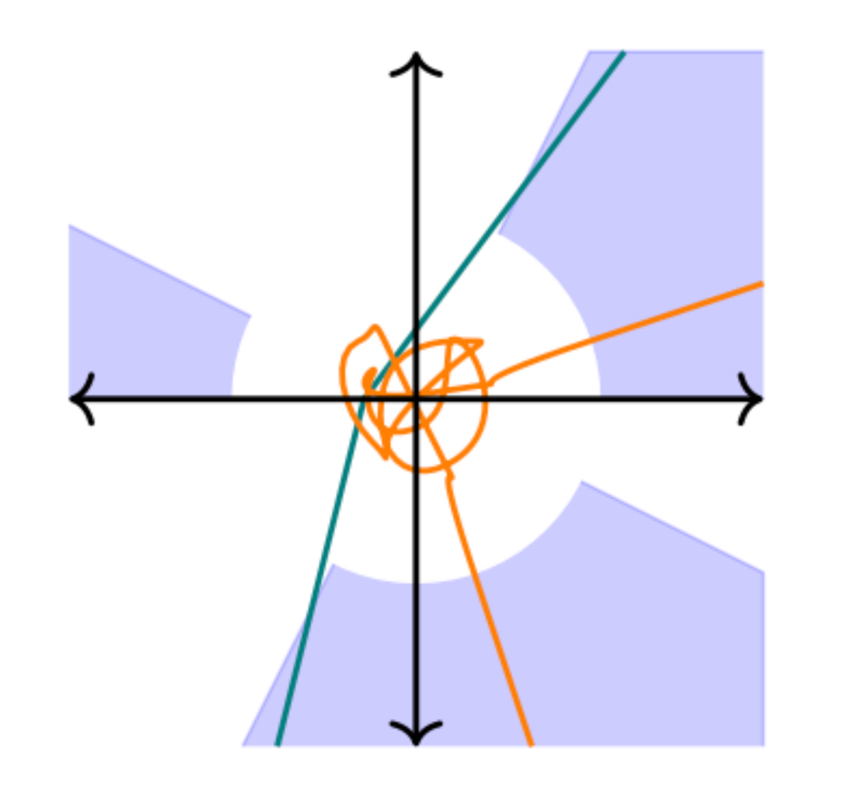


Main result

Summability of state integrals [F.—Wheeler]

- **Theorem** [F.—Wheeler]: The algorithm terminates.

Indeed, the Gaussian part of the volume V dominates at infinity

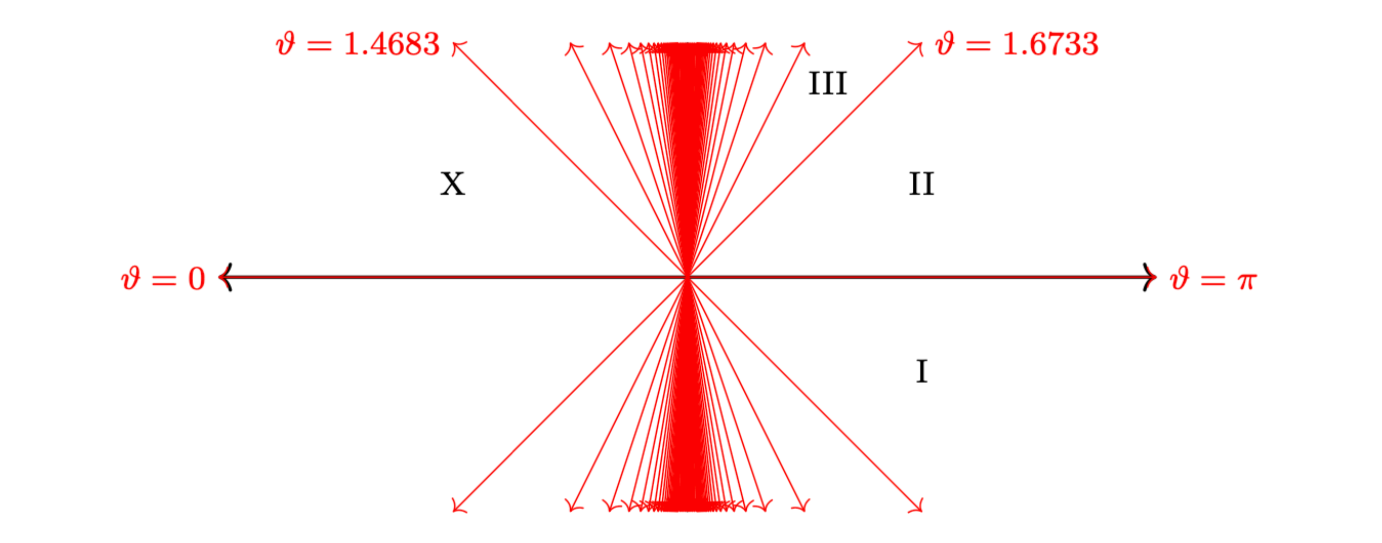


The thimble contour

The state integral contour

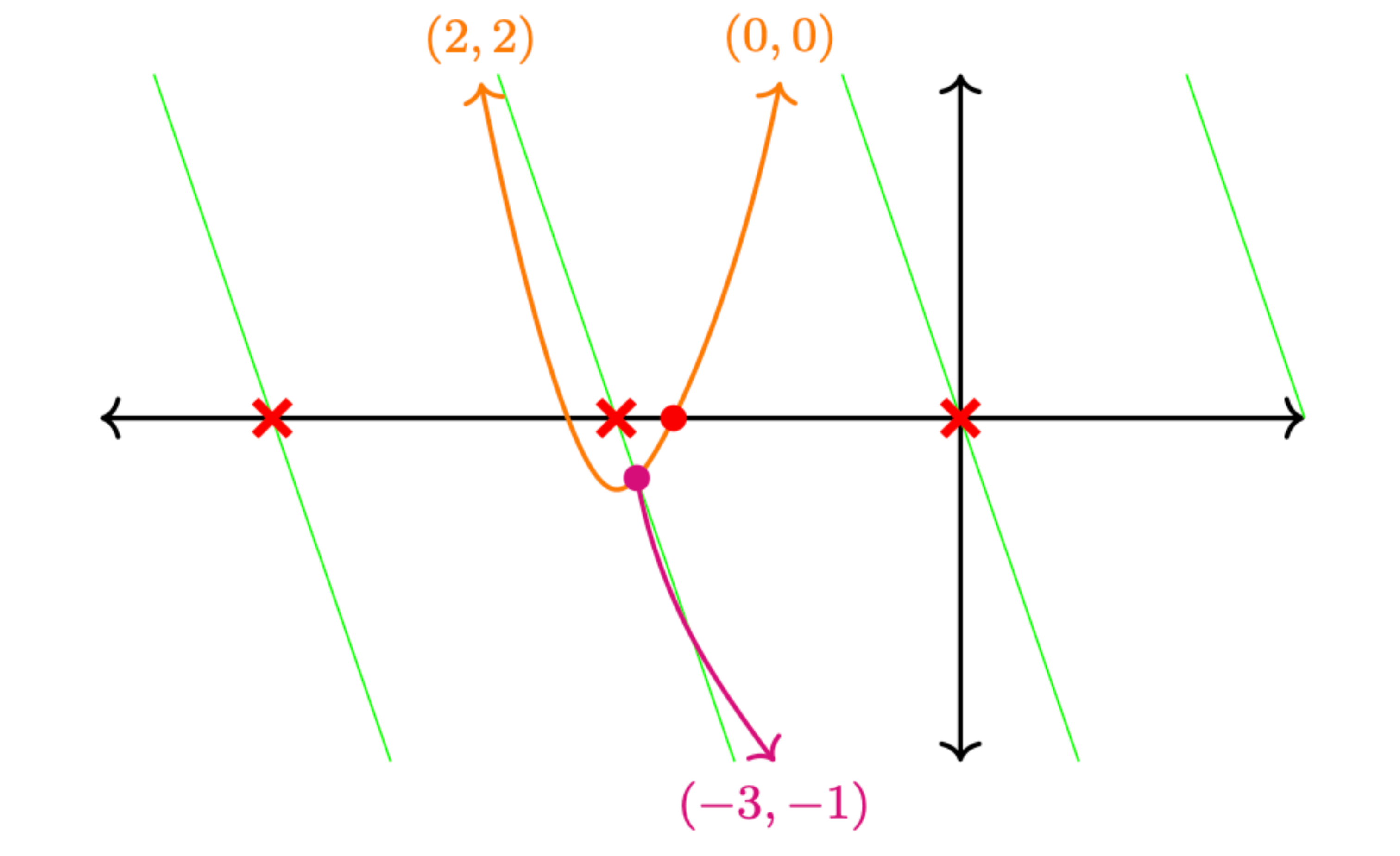
- **Theorem** [F.—Wheeler]: The thimble integral gives the Borel—Laplace resummation of $\tilde{\Phi}_{\Xi}$. In addition, the thimble integral decomposes into a finite sum of state integrals

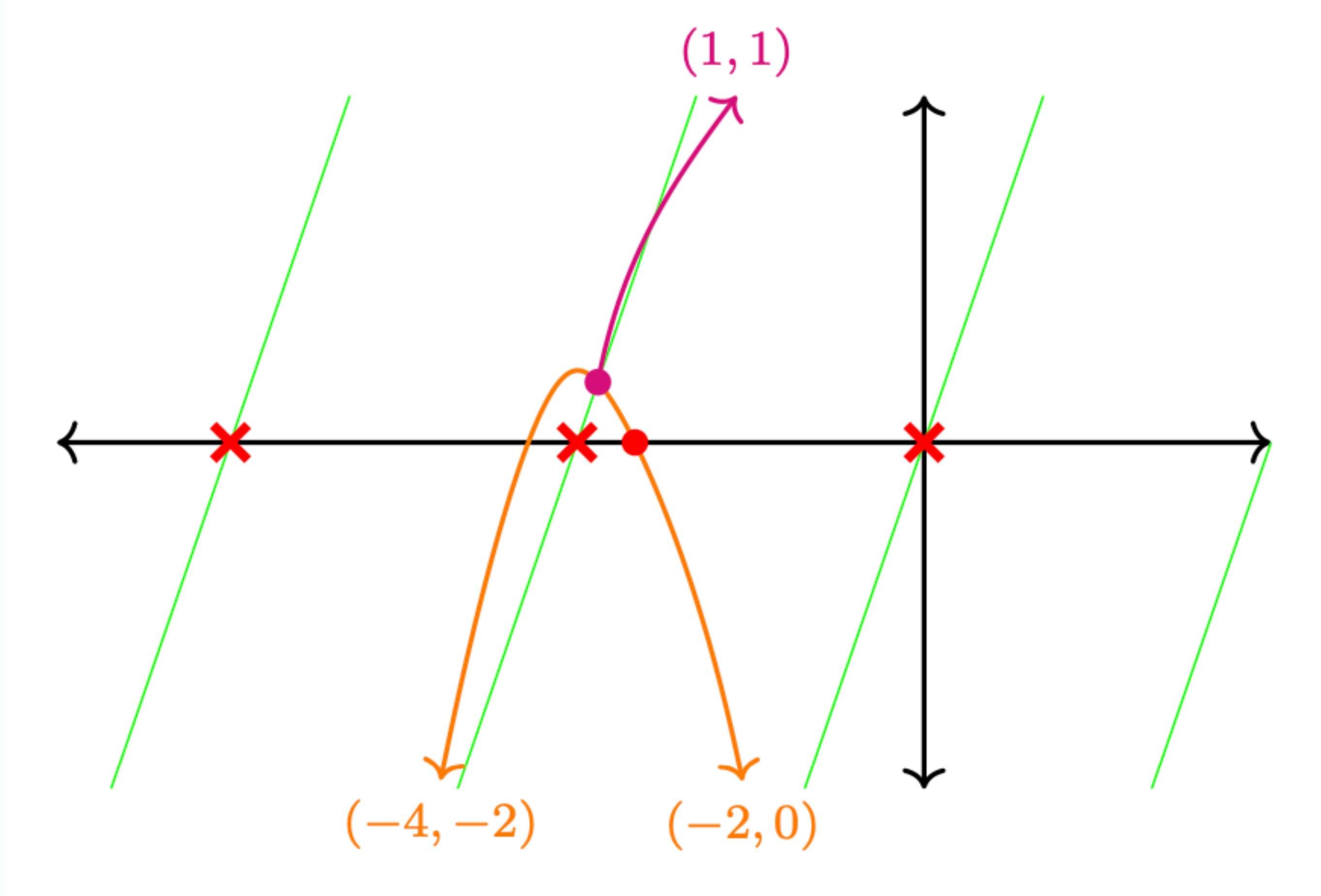
The Stokes constants

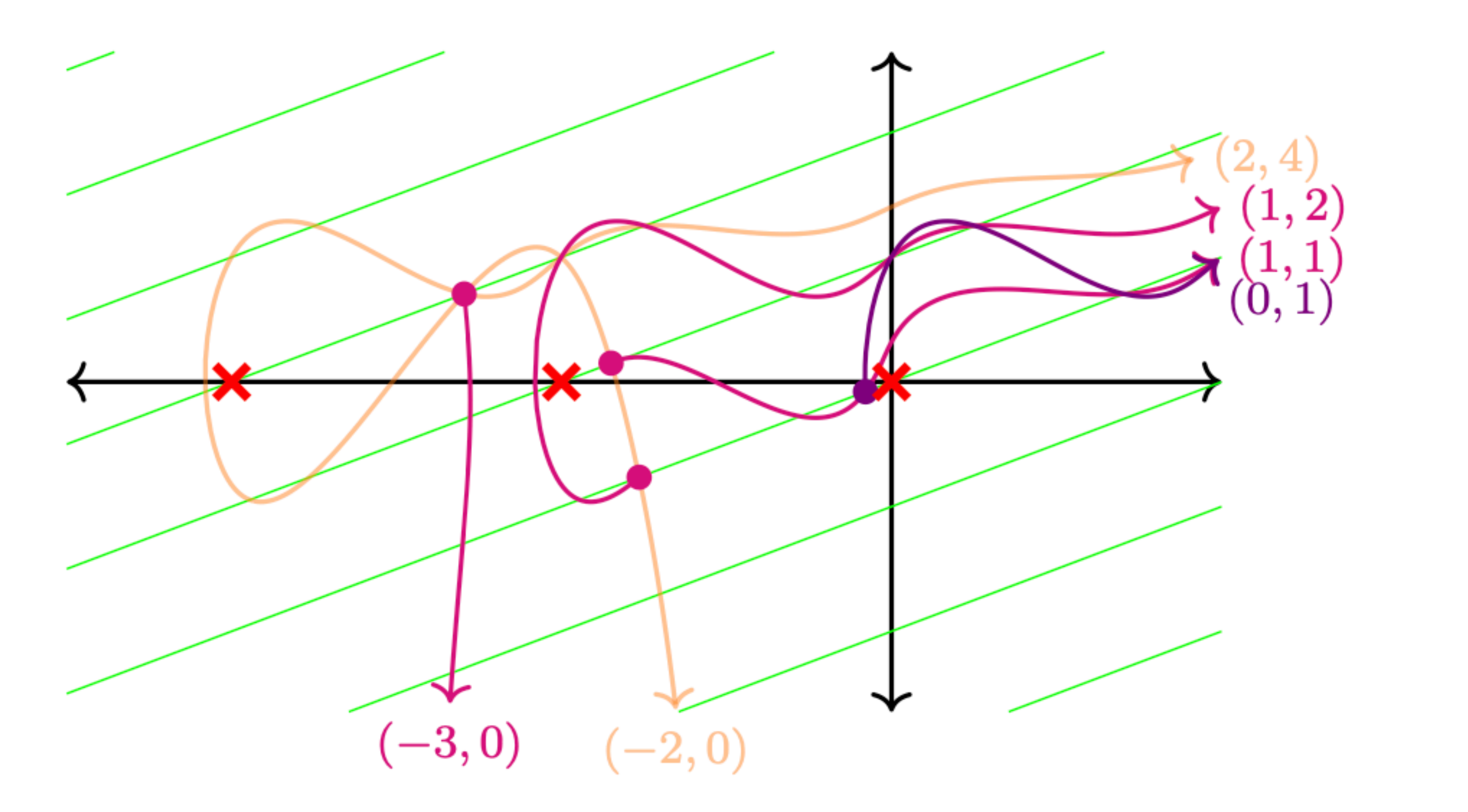


Geometric computation of the Stokes constants [F.—Wheeler]

- The output of the algorithm for the 4_1 knot







$\hat{\Phi}_{\mathrm{I}} = I_{0,0} + q^2 I_{2,-1} = I_{0,0} + I_{1,0}$

$\hat{\Phi}_{\mathrm{II}} = -I_{0,0} + I_{0,-1} = -I_{0,0} - I_{-1,0}$

$\hat{\Phi}_{\mathrm{III}} = -I_{0,0} + I_{0,-1} - q^4 I_{2,-2} + 2q^2 I_{1,-1} + 2q^2 I_{1,-1} - 4q I_{0,0}$
$= -I_{0,0} - I_{-1,0} - 9q I_{0,0}$

- To compute the **Stokes constants** is enough to compare the analytic functions $\hat{\Phi}_\bullet$ in adjacent sectors

- **Remark**: Stokes constants do not come only from saddle connections. Actually, the majority of them come from the way we patched different state integrals when the thimble crosses a green line

Conclusion

Conclusion and open questions

- The formal series $\tilde{\Phi}_K$ is Borel—Laplace summable for $K = 4_1$ and 5_2, and its sum is given by a combination of state integrals as prescribed by our algorithm
- The decomposition is computed following the steepest descent contours + patching different asymptotics of Faddeev's quantum dilogarithm
- The Stokes constants can be computed geometrically

- **Fermionic traces** in topological strings on toric CY 3-folds are integrals similar to the state integrals, so their asymptotics should be Borel—Laplace summable
- **Higher dimensional** state integrals, i.e. with $z \in \mathbb{C}^n$?—— *in progress with J. Andersen, M. Kontsevich and C. Wheeler*

Conclusion and open questions

- The formal series $\tilde{\Phi}_K$ is Borel—Laplace summable for $K = 4_1$ and 5_2, and its sum is given by a combination of state integrals as prescribed by our algorithm
- The decomposition is computed following the steepest descent contours + patching different asymptotics of Faddeev's quantum dilogarithm
- The Stokes constants can be computed geometrically

- **Fermionic traces** in topological strings on toric CY 3-folds are integrals similar to the state integrals, so their asymptotics should be Borel—Laplace summable
- **Higher dimensional** state integrals, i.e. with $z \in \mathbb{C}^n$?—— *in progress with J. Andersen, M. Kontsevich and C. Wheeler*

Thank you for your attention