From the Kronecker quiver to Painlevé III

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Introduction

Update on a long-running project about geometric structures defined by DT invariants.

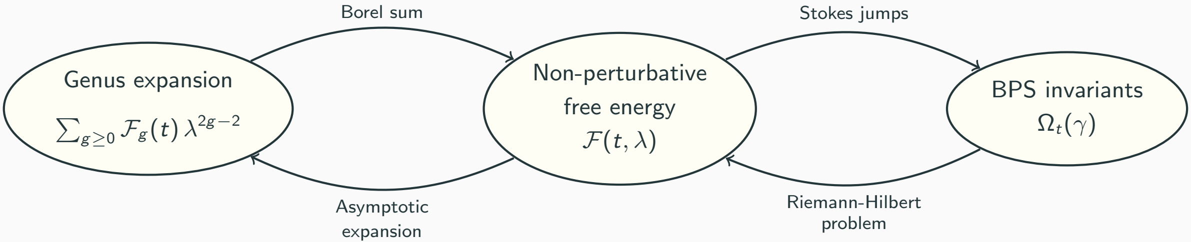
Interacts with ideas around non-perturbative topological string partition functions.

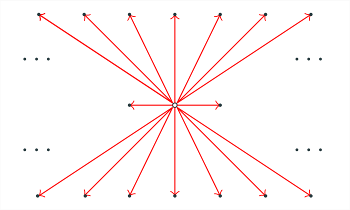
Owes much to Gaiotto, Moore, Neitzke, and many others!

We will give an overview and discuss a new example worked out with **Fabrizio Del Monte**.

Related to the quiver $\bullet \rightrightarrows \bullet$, Painlevé III_3, and pure SU(2) gauge theory ($N = 2$, $d = 4$).

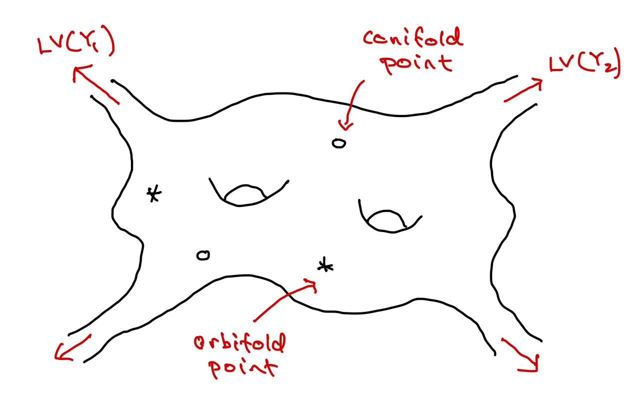
Different perspective to the usual resurgence story





Analytic continuation of $\mathcal{F}(t, \lambda)$?

The *A*-model partition function should exist globally on the stringy Kähler moduli space.



On the other hand, the GW invariants are associated to some particular large volume limit.

An attempt at an abstract global approach

Our starting data is a CY_3 triangulated category $\mathcal{D}$ such as $D^b \operatorname{Lag}(Y)$ or $D^b \operatorname{Coh}(Y)$.

(These will relate to the B-model and A-model partition functions respectively.)

A key role is played by the complex manifold of stability conditions $\operatorname{Stab}(\mathcal{D})$.

When $\mathcal{D} = \mathcal{D}^b \operatorname{Coh}(Y)$ this is expected to contain the stringy Kähler moduli space.

The aim is to build a global geometric structure on $\operatorname{Stab}(\mathcal{D})$.

*** So far we only understand a few simple examples (e.g. class $S[A_1]$). ***

Ray diagrams

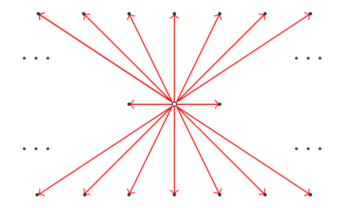
Associated to $\mathcal{D}$ is a lattice $\Gamma \cong \mathbb{Z}^{\oplus n}$ and a skew-symmetric form $\langle -, - \rangle \colon \Gamma \times \Gamma \to \mathbb{Z}$.

At each point $\sigma \in \mathrm{Stab}(\mathcal{D})$ there is a map $Z \colon \Gamma \to \mathbb{C}$ called the central charge.

Fixing a basis $\Gamma = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_n$ gives local co-ordinates $z_i = Z(\gamma_i)$.

Under further assumptions there are also DT invariants $\Omega(\gamma) \in \mathbb{Q}$ for each class $\gamma \in \Gamma$.

We consider the rays spanned by the central charges $Z(\gamma) \in \mathbb{C}^*$ for classes with $\Omega(\gamma) \neq 0$.



Riemann-Hilbert problem

Introduce the torus $\mathbb{T} = \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$.

For each ray $\ell \subset \mathbb{C}^*$ there is a partially-defined $\mathbb{S}_\ell \in \mathrm{Aut}(\mathbb{T})$ satisfying

$$\mathbb{S}_\ell^*(X_\beta) = X_\beta \cdot \prod_{Z(\gamma)\in\ell} (1 - X_\gamma)^{\Omega(\gamma)\cdot\langle\beta,\gamma\rangle}$$

We consider a RH problem which depends on $\sigma \in \mathrm{Stab}(\mathcal{D})$ and an element $\xi = \exp(\theta) \in \mathbb{T}$.

Find a piecewise holomorphic map $X \colon \mathbb{C}^* \to \mathbb{T}$ with jumps and asymptotics:

$$X(\epsilon) \mapsto \mathbb{S}_\ell(X(\epsilon)) \quad \text{as } \epsilon \in \mathbb{C}^* \text{ crosses a ray } \ell,$$

$$X(\epsilon) \cdot \exp(Z/\epsilon) \to \xi \in \mathbb{T} \text{ as } \epsilon \to 0, \qquad |\epsilon|^{-k} < \|X(\epsilon)\| < |\epsilon|^k \text{ as } \epsilon \to \infty.$$

Categories of class $S[A_1]$

There is an interesting special class of CY_3 categories $\mathcal{D} = \mathcal{D}(g, m)$.

They are indexed by $g \geq 0$ and $m = (m_1, \cdots, m_d)$ with $m_i \geq 3$.

There is an identification

$$\mathrm{Stab}(\mathcal{D}(g, m))/\mathrm{Aut}(\mathcal{D}(g, m)) \cong \mathrm{Quad}(g, m) := \{(C, Q_0)\}$$

- C a compact, connected Riemann surface of genus g,
- Q_0 a quadratic differential on C with simple zeroes and poles of orders m_i.

Moroever $\Gamma \cong H_1(\Sigma, \mathbb{Z})^-$, where $\Sigma \to C$ is the double cover $\{y^2 = Q_0(x)\}$, and

$$Z(\gamma) = \oint_\gamma \sqrt{Q_0}.$$

Today's example: Kronecker quiver

We consider the special case $g = 0$ and $m = (3, 3)$ and set $\mathcal{D} = \mathcal{D}(g, m)$.

Then in fact $\mathcal{D} = \mathcal{D}_{CY_3}(\bullet \rightrightarrows \bullet)$ is the CY_3 category associated to the Kronecker quiver.

We have $\Gamma = \mathbb{Z}\gamma_1 \oplus \mathbb{Z}\gamma_2$ with $\langle \gamma_1, \gamma_2 \rangle = 2$.

We consider quadratic differentials on $\mathbb{P}^1$ of the form

$$Q_0(x)\, dx^{\otimes 2} = \left(\frac{1}{x} + H + tx \right) \frac{dx^2}{x^2}.$$

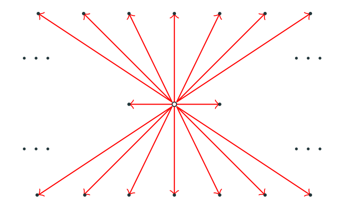
These are parameterised by the space

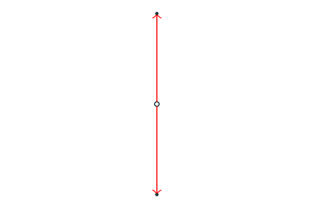
$$M := \mathrm{Stab}(\mathcal{D}) / \mathrm{Aut}(\mathcal{D}) \cong \left\{ (t, H) \in \mathbb{C}^2 : t(H^2 - 4t) \neq 0 \right\}.$$

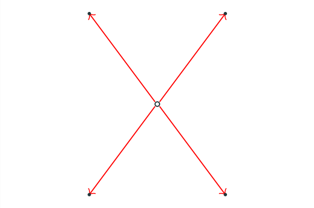
Kronecker ray diagrams

For class $S[A_1]$ categories the DT invariants are counts of finite-length trajectories.

In our Kronecker example there are three possible ray diagrams:







$$\Omega(\pm(\gamma_1 + n(\gamma_1+\gamma_2))) = 1$$
$$\Omega(\pm(\gamma_1+\gamma_2)) = -2$$

$$\Omega(\pm\gamma_1) = \Omega(\pm\gamma_2) = 1$$

Solving the RH problem: pencil of flat connections

As above, a stability condition on $\mathcal{D}$ defines a quadratic differential

$$Q_0(x)\,dx^{\otimes 2} = \left(\frac{1}{x} + \frac{H}{x^2} + \frac{t}{x^3}\right)dx^{\otimes 2}.$$

Take $(q,p) \in \mathbb{C}^2$ with $p^2 = Q_0(q)$ and also $r \in \mathbb{C}$ and consider the ODE

$$f''(x) = \left(\epsilon^{-2}Q_0(x) + \epsilon^{-1}Q_1(x) + Q_2(x)\right)f(x),$$

$$Q_1(x) = -\frac{pq^2}{x^2(x-q)} + \frac{2pqr}{x^2}, \qquad Q_2(x) = \frac{3}{4(x-q)^2} + \frac{rq-x}{x^2(x-q)} + \frac{r^2}{x^2}.$$

The associated linear system is gauge equivalent to the connection

$$\nabla_\epsilon = d - \begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\frac{dx}{x} - \frac{1}{\epsilon}\begin{pmatrix} pq & 1-qx^{-1} \\ tx-q^{-1} & -pq \end{pmatrix}\frac{dx}{x}.$$

Expected solution to the RH problem

Now take $X_i(\epsilon)$ to be the Fock-Goncharov co-ordinates of the above connection.

We take the WKB triangulation for the differential $\epsilon^{-2} Q_0(x) dx^{\otimes 2}$.

This gives the correct discontinuities across rays as in Gaiotto-Moore-Neitke.

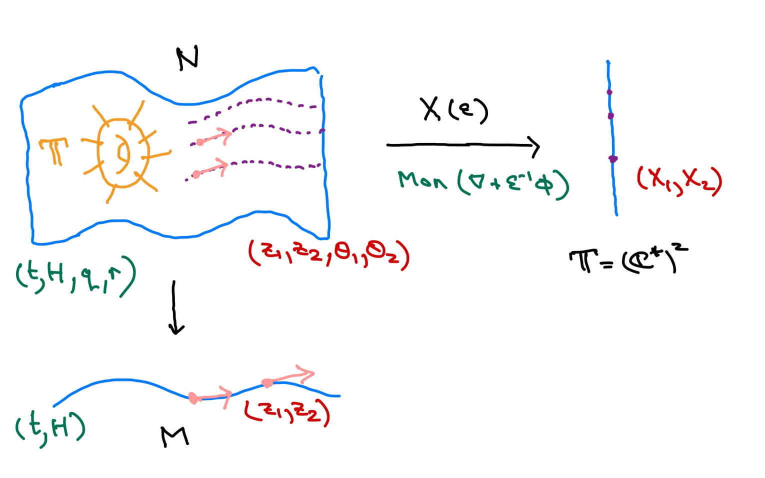
Exact WKB analysis should give the required asymptotics as $\epsilon \to 0$ if we set

$$\xi(\gamma) = \exp(\theta(\gamma)), \qquad \theta(\gamma) = -\int_\gamma \frac{Q_1(x)\, dx}{2\sqrt{Q_0(x)}}.$$

The asymptotcs as $\epsilon \to \infty$ are easy if $r = 0$ but are not proved in general.

Level sets of the solution to the RH problem

For each $\epsilon \in \mathbb{C}^*$ the kernel of $dX(\epsilon)$ defines a half-dimensional sub-bundle $H_\epsilon \subset T_X$.



Joyce structure

Take a basis $\Gamma = \mathbb{Z}\gamma_1 \oplus \cdots \oplus \mathbb{Z}\gamma_n$ and write $z_i = Z(\gamma_i)$, $\theta_i = \theta(\gamma_i)$, etc.

The sub-bundle $H_\epsilon \subset T_N$ is spanned by vector fields $h_i + \epsilon^{-1} v_i$ of the form

$$h_i = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial}{\partial \theta_q}, \qquad v_i = \frac{\partial}{\partial \theta_i}, \qquad \eta_{pq} = \langle \gamma_p, \gamma_q \rangle,$$

where $W \colon N \to \mathbb{C}$ satisfies Plebański's second heavenly equations

$$\frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} = \sum_{p,q} \eta_{pq} \cdot \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q}.$$

There is a complex HK structure on N given in the basis (v_i, h_i) by block-diagonal matrices

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad \omega = \eta^{-1}.$$

Complex 2-forms

There are closed 2-forms $\Omega_I(w_1, w_2) = g(I(w_1), w_2)$ etc. given explicitly by

$$\Omega_0 := \Omega_J + i\Omega_K = \frac{1}{2} \cdot \sum_{p,q} \omega_{pq} \cdot dz_p \wedge dz_q, \qquad 2i\Omega_I = -\sum_{p,q} \omega_{pq} \cdot dz_p \wedge d\theta_q,$$

$$\Omega_\infty := \Omega_J - i\Omega_K = \frac{1}{2} \cdot \sum_{p,q} \omega_{pq} \cdot d\theta_p \wedge d\theta_q + d\left(\sum_{p,q} \frac{\partial W}{\partial \theta_q} \cdot dz_q\right).$$

We also pull back the symplectic form on $\mathbb{T}$ via the solution to the RH problem

$$\Omega_\epsilon = \frac{1}{2} \cdot \sum_{p,q} \omega_{pq} \cdot dx_p(\epsilon) \wedge dx_q(\epsilon), \qquad X_p(\epsilon) = \exp(x_p(\epsilon)).$$

For $\epsilon \in \mathbb{C}^*$ there is then a relation

$$\Omega_\epsilon = \epsilon^{-2}\Omega_0 + 2i\epsilon^{-1}\Omega_I + \Omega_\infty.$$

Tau function

Define a function locally on $N \times \mathbb{C}^*$ by the relation

$$d \log(\tau) = \Theta_0 + 2i\Theta_I + \Theta_\infty - \Theta_\epsilon,$$

where we chose symplectic potentials $d\Theta_0 = \Omega_0$, etc. We can take

$$\Theta_0 = \frac{1}{2} \cdot \sum_{p,q} \omega_{pq} \cdot z_p \, dz_q, \qquad 2i\Theta_I = -\sum_{p,q} \omega_{pq} \cdot z_p \, d\theta_q,$$

$$\Theta_\epsilon = \frac{1}{2} \cdot \sum_{p,q} \omega_{pq} \cdot x_p(\epsilon) \, dx_q(\epsilon).$$

Defining Θ_∞ is more tricky, but we can restrict to a locus where $\Omega_\infty = 0$ and take $\Theta_\infty = 0$.

Back to the Kronecker example

The isomonodromy flows are

$$t\frac{\partial}{\partial t} - qt\frac{\partial}{\partial H} + \frac{2pq^2}{\epsilon}\frac{\partial}{\partial q} + 2qr\frac{\partial}{\partial q} - \frac{r}{\epsilon pq^2}(q^2t-1)\frac{\partial}{\partial r}, \qquad \frac{\partial}{\partial H} - \frac{1}{2\epsilon pq}\frac{\partial}{\partial r}.$$

The Plebański function is

$$W = \frac{pq}{6(H^2-4t)}\left(tq + (H+6tq)r + (6H+12tq)r^2 + 8p^2q^2r^3\right).$$

The form Ω_∞ vanishes on the locus $r=0$. With appropriate choices we have

$$d\log\left(\tau|_{r=0}\right) = -\frac{H}{\epsilon^2}\frac{dt}{t} + \frac{p\,dq}{\epsilon} + d\left(\frac{4H}{\epsilon^2} + \frac{2qp}{\epsilon}\right) + \frac{1}{4\pi i}x_1 dx_2.$$

It is a Painlevé III_3 τ-function.

Future directions

Menelaos Zikidis constructs the Joyce structures on all spaces $\mathrm{Quad}(g, m)$.

Are they regular along the locus $\theta = 0$? Has to do with zeroes of the Painlevé τ-function.

Can we construct the Joyce structure for $\mathcal{D} = \mathcal{D}^b \operatorname{Coh}(\omega_{\mathbb{P}^2})$?

Construct the twistor space of the Joyce structure directly using the DT invariants.

The twistor lines then give the correct solutions to the RH problem.