Wavelet view on the Landau poles in quantum field theory

M.V.Altaisky[1]
Space Research Institute, Moscow

in collaboration with Dr. M. Hnatich (JINR)

*

Exact Renormalization Group 2024
Sep 22 – 28, 2024, SwissMAP

[1]e-mail: altaisky@rssi.ru

Abstract

Following a series of papers [Phys. Rev. D **93**, 105043 (2016), Err:**105**, 049901; **102**(2020)125021; **108** (2023)085023], we develop an approach to RG, where the effective action functional $\Gamma_A[\phi]$ is a sum of all fluctuations of scales from the size of the system L down to the scale of observation A. It is shown that RG flow equation of the type $\frac{\partial \Gamma_A}{\partial \ln A} = -Y(A)$ is a limiting case of such consideration, when the running coupling constant is assumed to be a differentiable function of scale. In this approximation, the running coupling constant, calculated at one-loop level, suffers from the Landau pole. In general case, when the scale-dependent coupling constant is a non-differentiable function of scale, the Feynman loop expansion results in a difference equation, which keeps the coupling constant finite for any finite value of scale A. Examples of the Euclidean ϕ^4 and QED are considered.

Reference

MA & M.Hnatich *Phys. Rev. D* **108** (2023) 085023

Outline

- UV divergences in quantum field theory
- Continuous wavelet transform in quantum field theory
- Wavelets and scale-dependent functions: $\phi(x) \to \phi_a(x), dx \to dx d\ln a$
- Quantum field theory without divergences
- Scale-dependent coupling constant for scale-dependent fields
- Renormalization group equation
- Differentiablity as a cause of Landau poles

UV divergences in QFT

ϕ^4 in $\mathbb{R}^d$ example:

$$Z[J] = \mathcal{N} \int e^{-\int d^d x \left[\frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi\right]} \mathcal{D}\phi,$$

n-point Green functions

$$\Delta^{(n)} \equiv \langle \phi(x_1) \ldots \phi(x_n) \rangle_c = \left. \frac{\delta^n \ln Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0}$$

The divergences of Feynman graphs in the perturbation expansion of the Green functions with respect to the small coupling constant λ emerge at coinciding arguments $x_i = x_k$. For instance, the bare two-point correlation function

$$\Delta_0^{(2)}(x-y) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{\imath p(x-y)}}{p^2 + m^2}$$

is divergent at $x = y$ for $d \geq 2$.

Why do we need scale-dependent functions?

$L^2(\mathbb{R})$ or not $L^2(\mathbb{R})$?

$\phi(x) \to \phi_a(x), \quad dx \to \frac{dxda}{a}$

- To localize a particle in an interval Δx the measuring device requires a momentum transfer of order $\Delta p \sim \hbar/\Delta x$. $\phi(x)$ at a point x has no experimental meaning. What is meaningful, is the vacuum expectation of product of fields in certain region around x

[M.V.A. *Phys. Rev. D* **81**(2010)125003]

Why do we need scale-dependent functions?

$L^2(\mathbb{R})$ or not $L^2(\mathbb{R})$?

$\phi(x) \to \phi_a(x), \quad dx \to \frac{dxda}{a}$

- To localize a particle in an interval Δx the measuring device requires a momentum transfer of order $\Delta p \sim \hbar/\Delta x$. $\phi(x)$ at a point x has no experimental meaning. What is meaningful, is the vacuum expectation of product of fields in certain region around x
- If the particle, described by $\phi(x)$, have been initially prepared on the interval $(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2})$, the probability of registering it on this interval is ≤ 1: for the registration depends on the strength of interaction and the ratio of typical scales related to the particle and to the equipment.

[M.V.A. *Phys. Rev. D* **81**(2010)125003]

Why do we need scale-dependent functions?

$L^2(\mathbb{R})$ or not $L^2(\mathbb{R})$?

$\phi(x) \to \phi_a(x), \quad dx \to \frac{dxda}{a}$

- To localize a particle in an interval Δx the measuring device requires a momentum transfer of order $\Delta p \sim \hbar/\Delta x$. $\phi(x)$ at a point x has no experimental meaning. What is meaningful, is the vacuum expectation of product of fields in certain region around x
- If the particle, described by $\phi(x)$, have been initially prepared on the interval $(x - \frac{\Delta x}{2}, x + \frac{\Delta x}{2})$, the probability of registering it on this interval is ≤ 1: for the registration depends on the strength of interaction and the ratio of typical scales related to the particle and to the equipment.
- Statement of existence: if a measuring equipment with a given resolution a fails to register an object, prepared on spatial interval of width Δx with certainty, then tuning the equipment to *all* possible resolutions a' would lead to the registration. $\int |\phi_a(x)|^2 d\mu(a,x) = 1$

[M.V.A. *Phys. Rev. D* **81**(2010)125003]

Continuous Wavelet Transform

Each square-integrable function $\phi(x) \in L^2(\mathbb{R}^d)$ can be represented in a form

$$\phi(x) = \frac{1}{C_\chi} \int_G \frac{1}{a^d} \chi\left(\frac{x-b}{a}\right) \phi_a(b) \frac{d^d b da}{a},$$

where

$$\phi_a(b) := \int_{\mathbb{R}^d} \frac{1}{a^d} \overline{\chi\left(\frac{x-b}{a}\right)} \phi(x) d^d x \equiv \langle a, b; \chi | \phi \rangle,$$

are known as wavelet coefficients of ϕ with respect to mother wavelet χ and the integration is performed over the affine group

$$G : x' = ax + b, x, b \in \mathbb{R}^d, a \in \mathbb{R}_+,$$

For isotropic wavelets

$$C_\chi = \int_0^\infty |\tilde{\chi}(ak)|^2 \frac{da}{a}.$$

Field theory for scale-dependent fields $\phi_a(b)$

Substitution of the CWT into field theory $Z[J]$ gives a theory for the fields $\phi_a(x)$

$$\begin{aligned} Z_W[J_a] &= \mathcal{N} \int \exp\Big[-\frac{1}{2}\int \phi_{a_1}(x_1) D(a_1,a_2,x_1-x_2)\phi_{a_2}(x_2)\frac{da_1 d^d x_1}{C_\psi a_1} \times \\ &\times \frac{da_2 d^d x_2}{C_\psi a_2} - \frac{\lambda}{4!}\int V_{x_1,\ldots,x_4}^{a_1,\ldots,a_4}\phi_{a_1}(x_1)\cdots\phi_{a_4}(x_4)\frac{da_1 d^d x_1}{C_\psi a_1} \times \\ &\times \frac{da_2 d^d x_2}{C_\psi a_2}\frac{da_3 d^d x_3}{C_\psi a_3}\frac{da_4 d^d x_4}{C_\psi a_4} + \int J_a(x)\phi_a(x)\frac{da d^d x}{C_\psi a}\Big]\mathcal{D}\phi_a, \end{aligned}$$

with $D(a_1,a_2,x_1-x_2)$ and $V_{x_1,\ldots,x_4}^{a_1,\ldots,a_4}$ denoting the wavelet images of the inverse propagator and that of the interaction potential.
The Green functions for scale component fields are given by functional derivatives

$$\langle \phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle_c = \left.\frac{\delta^n \ln Z_W[J_a]}{\delta J_{a_1}(x_1)\ldots\delta J_{a_n}(x_n)}\right|_{J=0}.$$

Feynman Diagram Technique

The Feynman rules

MA *IOP Conf. Ser.* **173**(2003)893; *Phys. Rev. D* **81**(2010)125003 :

- each field $\tilde\phi(k)$ will be substituted by the scale component $\tilde\phi(k) \to \tilde\phi_a(k) = \overline{\tilde\chi(ak)}\tilde\phi(k)$.
- each integration in momentum variable is accompanied by corresponding scale integration:
$$\frac{d^d k}{(2\pi)^d} \to \frac{d^d k}{(2\pi)^d} \frac{da}{a} \frac{1}{C_\chi}.$$
- each interaction vertex is substituted by its wavelet transform; for the N-th power interaction vertex this gives multiplication by factor $\prod_{i=1}^{N} \overline{\tilde\chi(a_i k_i)}$.

Since $\tilde\chi(p) \sim p^n e^{-\frac{p^2}{2}}, n \geq 1$, this gives an UV suppressing factors:

$$\frac{1}{p^2+m^2} \to \frac{\tilde\chi(a_1 p)\overline{\tilde\chi(a_2 p)}}{p^2+m^2}$$

What is observed?

If all states $\langle a \geq A, x; \chi|\phi\rangle$ are registered, but those with $a < A$ are not:

$$\begin{aligned} \phi^{(A)}(x) &= \frac{1}{C_g}\int_{a\geq A}\langle x|g;a,b\rangle d\mu(a,b)\langle g;a,b|\phi\rangle, \\ \mathcal{G}^{(A)}(x_1,\ldots,x_n) &\equiv \langle\phi^{(A)}(x_1),\ldots,\phi^{(A)}(x_n)\rangle \end{aligned}$$

The running coupling constant $\lambda(A)$, calculated with such cutoff, describes the interaction of all perturbations with scales up to A, but says nothing about the interaction of perturbations with scales about$(A, A-\delta A)$.

Causality assumption

There should be no scales a_i in internal lines of a Feynman diagram smaller than the minimal scale of all external lines A

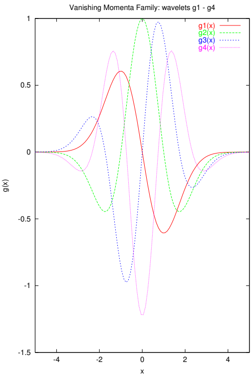
The integration $\int_A^\infty \frac{da}{a}$ in all internal lines harnesses a factor $f^2(Ak)$ on each line, where

$$f(x) = \frac{1}{C_\chi}\int_x^\infty |\tilde{\chi}(a)|^2\frac{da}{a}, \quad f(0) = 1$$

is effective cutoff function.

Examples of Cutoff Functions
Gaussian derivatives wavelets

$$\chi_n(x) = (-1)^{n+1}\frac{d^n}{dx^n}\frac{e^{-x^2/2}}{\sqrt{2\pi}},$$
$$\tilde\chi_n(k) = -(-\imath k)^n e^{-k^2/2}, \quad n > 0$$



$$C_{\chi_n} = \int_0^\infty a^{2n} e^{-a^2}\frac{da}{a} = \frac{\Gamma(n)}{2}$$

Effective cutoff function

$$f(n,x) = \frac{2}{\Gamma(n)}\int_x^\infty |\tilde\chi_n(a)|^2 \frac{da}{a}$$

Cutoff functions for $\chi_1 - \chi_4$:

$$f(1,x) = e^{-x^2}$$
$$f(2,x) = (x^2+1)e^{-x^2}$$
$$f(3,x) = (x^4+2x^2+2)e^{-x^2}/2$$
$$f(4,x) = (x^6+3x^4+6x^2+6)e^{-x^2}/6$$

Effective action functional

Effective action for scale-dependent fields:

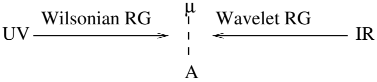
$$\Gamma[\phi_a] = -W_W[J_a] + \int J_a(x)\phi_a(x)\frac{da}{C_\psi a}d^dx$$

where $W_W[J_a] = \ln Z_W[J_a]$
Vertex functions

$$\Gamma_{(A)}[\phi_a] = \Gamma^{(0)}_{(A)} + \sum_{n=1}^{\infty}\frac{1}{n!}\Gamma^{(n)}_{(A)}(a_1,b_1,\ldots,a_n,b_n)\phi_{a_1}(b_1)\ldots\phi_{a_n}(b_n)\times$$
$$\times\frac{da_1 d^d b_1}{C_\psi a_1}\ldots\frac{da_n d^d b_n}{C_\psi a_n}$$

The observation scale $A = \min_i a_i$ plays the role of normalization scale

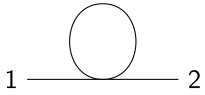
$A\frac{\partial}{\partial A}\Gamma_{(A)} = \ldots$ – Flow equation



Example of ϕ^4 in one loop

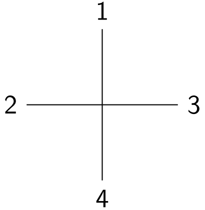
In one-loop approximation, the two-point and the four-point vertex functions, $\Gamma^{(2)}$ and $\Gamma^{(4)}$, respectively, are given by the following diagrams:

$\Gamma^{(2)} = \Delta_{12} - \frac{1}{2}$

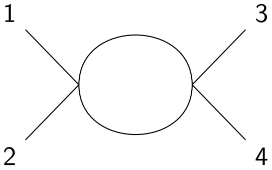


$,\quad \frac{\Gamma^{(2)}_{(A)}(a_1,a_2,p)}{\tilde\chi(a_1 p)\tilde\chi(-a_2 p)} = p^2 + m^2 + \frac{\lambda}{2} T^d_\chi(A)$

$\Gamma^{(4)} = -$



$- \frac{3}{2}$



$$\frac{\Gamma^{(4)}_{(A)}}{\tilde\chi(a_1 p_1)\tilde\chi(a_2 p_2)\tilde\chi(a_3 p_3)\tilde\chi(a_4 p_4)} = \lambda - \frac{3}{2}\lambda^2 X^d_\chi(A)$$

ϕ^4 calculations with $\chi_1(x) = -x\exp(-x^2/2)/\sqrt{2\pi}$ wavelet

"Tadpole" integral, χ_1 wavelet, $d = 4$

$$T^d_\chi(A) = \frac{S_d m^{d-2}}{(2\pi)^d}\int_0^\infty f^2_\chi(Amx)\frac{x^{d-1}dx}{x^2+1}$$

$$T^4_{\chi_1}(A) = \frac{m^2}{8\pi^2}\int_0^\infty e^{-2\alpha^2x^2}\frac{x^3dx}{x^2+1} = \frac{m^2}{32\pi^2}\left(\frac{1}{\alpha^2} - 2e^{2\alpha^2}\mathrm{Ei}_1(2\alpha^2)\right)$$

where $\alpha = Am$ and $\mathrm{Ei}_1(z) := \int_1^\infty \frac{e^{-xz}}{x}dx$ is the exponential integral of 1st kind.

"Fish" integral

$$X^d_\chi(A) = \int \frac{d^dq}{(2\pi)^d}\frac{f^2_\chi(qA)f^2_\chi((q-s)A)}{[q^2+m^2]\,[(q-s)^2+m^2]}$$

In relativistic limit $s^2 \gg 4m^2$, $s = p_1 + p_2$ the dimensionless scale is $\alpha = As$. In $d = 4$ in this limit the value of tadpole integral is

$$X^4_{\chi_1}(A) = \frac{1}{16\pi^2}\left[2\mathrm{Ei}_1(2\alpha^2) - \mathrm{Ei}_1(\alpha^2) + e^{-\alpha^2}\frac{1-e^{-\alpha^2}}{\alpha^2}\right]$$

Summing up over scale range $[A,\infty]$ we get

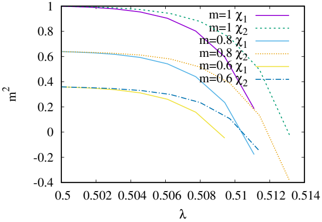
$$\lambda^{(1)}_{eff}(\alpha^2) = \lambda + \frac{3}{2}\frac{\lambda^2 e^{-\alpha^2}}{16\pi^2}\left[e^{\alpha^2}(2\mathrm{Ei}_1(2\alpha^2)-\mathrm{Ei}_1(\alpha^2)) + \frac{1-e^{-\alpha^2}}{\alpha^2}\right],$$

$$\lambda^{(2)}_{eff}(\alpha^2) = \lambda + \frac{3}{2}\frac{\lambda^2 e^{-\alpha^2}}{16\pi^2}\left[e^{\alpha^2}(2\mathrm{Ei}_1(2\alpha^2)-\mathrm{Ei}_1(\alpha^2)) + \frac{1-e^{-\alpha^2}}{\alpha^2} + \frac{\alpha^6+18\alpha^4+134\alpha^2+384-e^{-\alpha^2}(128\alpha^2+384)}{256\alpha^2}\right], \alpha\equiv As$$

In reality we need to sum up fluctuations up to the system size (L):

$$\lambda_A = \lambda_L + \frac{3}{2}\lambda_L^2[X^d_\chi(A) - X^d_\chi(L)],$$

$$m^2_A = m^2_L - \frac{\lambda_L}{2}[T^d_\chi(A) - T^d_\chi(L)]$$



Iteration of the finite shell renormalization goes from $L=4$ with the value $\lambda=\frac{1}{2}$ at the left of the picture by setting $L_{i+1}=L_i/\sqrt{2}\equiv A_i$. The right side of the picture corresponds to the UV limit of iteration. An arbitrary value of $s=2$ was chosen.

Continuous RG limit of $\lambda_A = \lambda_L + \frac{3}{2}\lambda_L^2[X_\chi^d(A) - X_\chi^d(L)]$

If the scales A and L are sufficiently close to each other, the difference equation

$$-\frac{\Delta\lambda}{\lambda^2} = -\frac{3}{2}\Delta X$$

can be transformed to the differential equation $d\frac{1}{\lambda} = -\frac{3}{2}dX$, which has the solution

$$\lambda(A) = \frac{\lambda_L}{1 - \frac{3}{2}\lambda_L(X(A) - X(L))},$$

which coincides with the solution of the original equation *only for small values of* λ_L – otherwise it suffers from the pole.
The differentiation of $\lambda(\alpha^2)$ with respect to scale gives the equation [χ_1 wavelet]

$$\alpha^2\frac{\partial\lambda}{\partial\alpha^2} = \frac{3}{2}\lambda^2\alpha^2\frac{\partial X^4_{\chi_1}}{\partial\alpha^2} = \frac{3\lambda^2}{32\pi^2}\frac{e^{-\alpha^2}-1}{\alpha^2}e^{-\alpha^2}$$

Its asymptote for small values $\alpha \ll 1$ coincides with standard result

$$\frac{\partial\lambda_{eff}}{\partial\mu} \approx \frac{3\lambda^2}{16\pi^2}, \quad \mu = -\ln\alpha.$$

Scale dependence for different wavelets

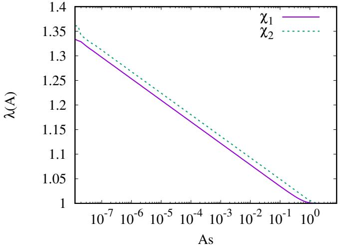
For different wavelets of the same Gaussian derivatives family the logarithmic slopes are the same for $\alpha^2 \ll 1$:

$$d_i := \alpha^2 \frac{\partial X^4_{\chi_i}}{\partial \alpha^2} = -\frac{1}{16\pi^2} + O(\alpha^2)$$

For the first two wavelets the small scale Taylor series gives

$$d_1 = -\frac{1}{16\pi^2} + \frac{3\alpha^2}{32\pi^2} - \frac{7\alpha^4}{96\pi^2} + O(\alpha^6),$$
$$d_2 = -\frac{1}{16\pi^2} - \frac{13\alpha^2}{1024\pi^2} + \frac{139\alpha^4}{3072\pi^2} + O(\alpha^6).$$

The shape of the mother wavelet works as an aperture of the microscope used to study the details of different scales



Dependence of the coupling constant on the logarithm of the dimensionless scale $\alpha = As$ calculated in one-loop approximation, with χ_1 and χ_2 wavelets. The value of the coupling constant is normalized to $\lambda_L = 1$ at infinity. The parameter $s = 2$ was taken.

Landau poles

In the limit of small α, the RG equation for the coupling constant,

$$\frac{\partial \lambda}{\partial \ln \alpha} = -\frac{3\lambda^2}{16\pi^2},$$

has a well-known solution

$$\lambda(\alpha) = \frac{\lambda_1}{1 + \frac{3\lambda_1}{16\pi^2} \ln \frac{\alpha}{\alpha_1}},$$

where $\lambda_1 \equiv \lambda(\alpha_1)$ is a reference value of the coupling constant at a certain reference value α_1.

This solution suffers from a Landau pole.

In wavelet theory the differential equation

$$\alpha^2 \frac{\partial \lambda}{\partial \alpha^2} = \frac{3\lambda^2}{32\pi^2} \frac{e^{-\alpha^2} - 1}{\alpha^2} e^{-\alpha^2}$$

can be solved as an RG-type equation,

$$d\left(\frac{1}{\lambda}\right) = -\frac{3}{32\pi^2} \frac{e^{-\alpha^2}(e^{-\alpha^2} - 1)}{\alpha^4} d\alpha^2$$

If the value of λ is known at certain scale $\lambda_1 = \lambda(x_1 = (A_1 s)^2)$, then at $x = (As)^2$

$$\lambda(x) = \frac{1}{\frac{1}{\lambda_1} + \frac{3}{32\pi^2}[F(x) - F(x_1)]},$$

where $F(x) := 2\Gamma(-1, 2x) - \Gamma(-1, x)$, with $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ being the incomplete gamma-function.

Thank You for your attention!

Conclusion

Differential theory makes $1 + x$ into $\frac{1}{1-x}$. This results in Landau poles. It seems better to avoid such procedures ...