Some result of critical dynamics of the O(4) critical point in QCD

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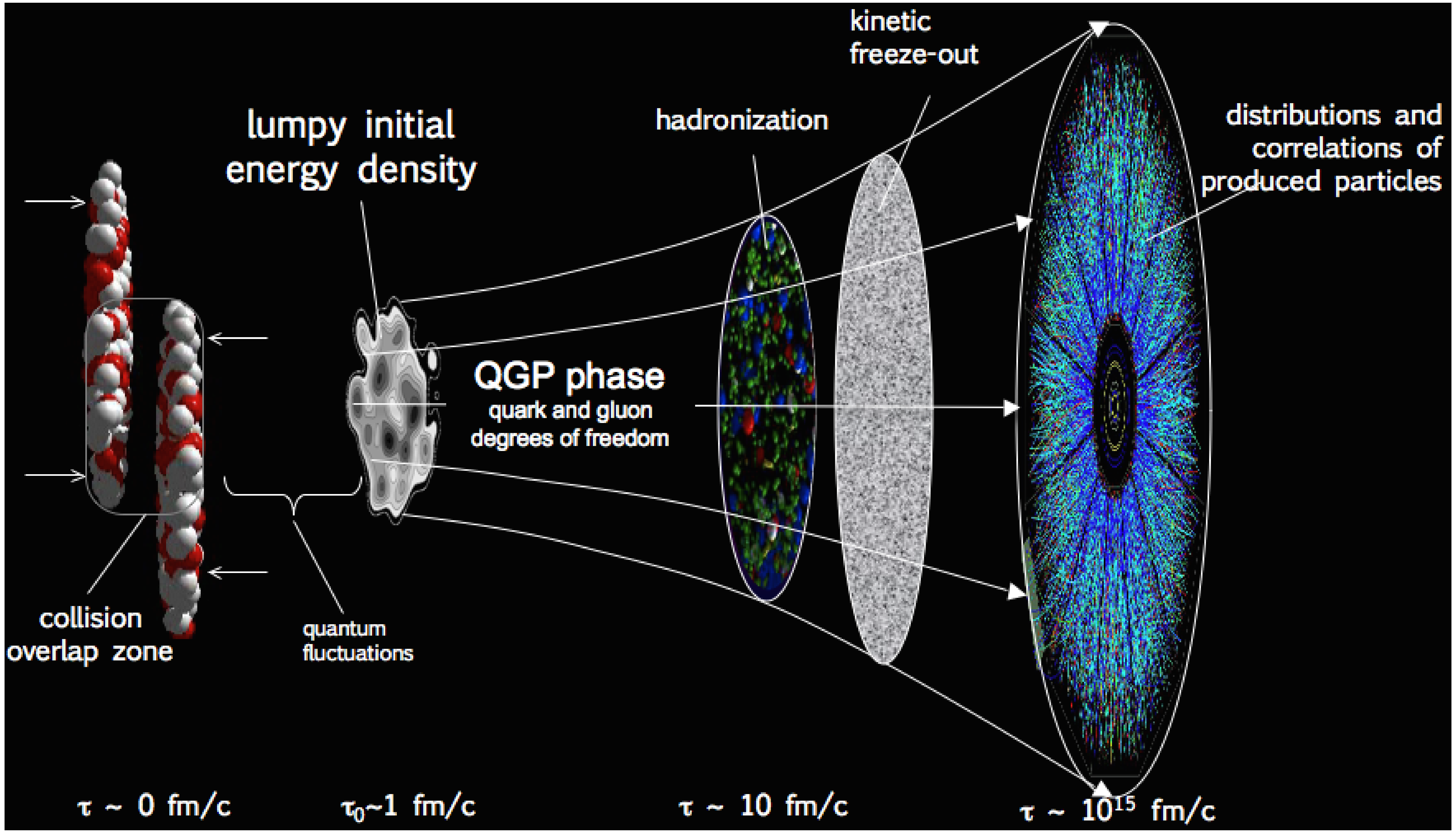
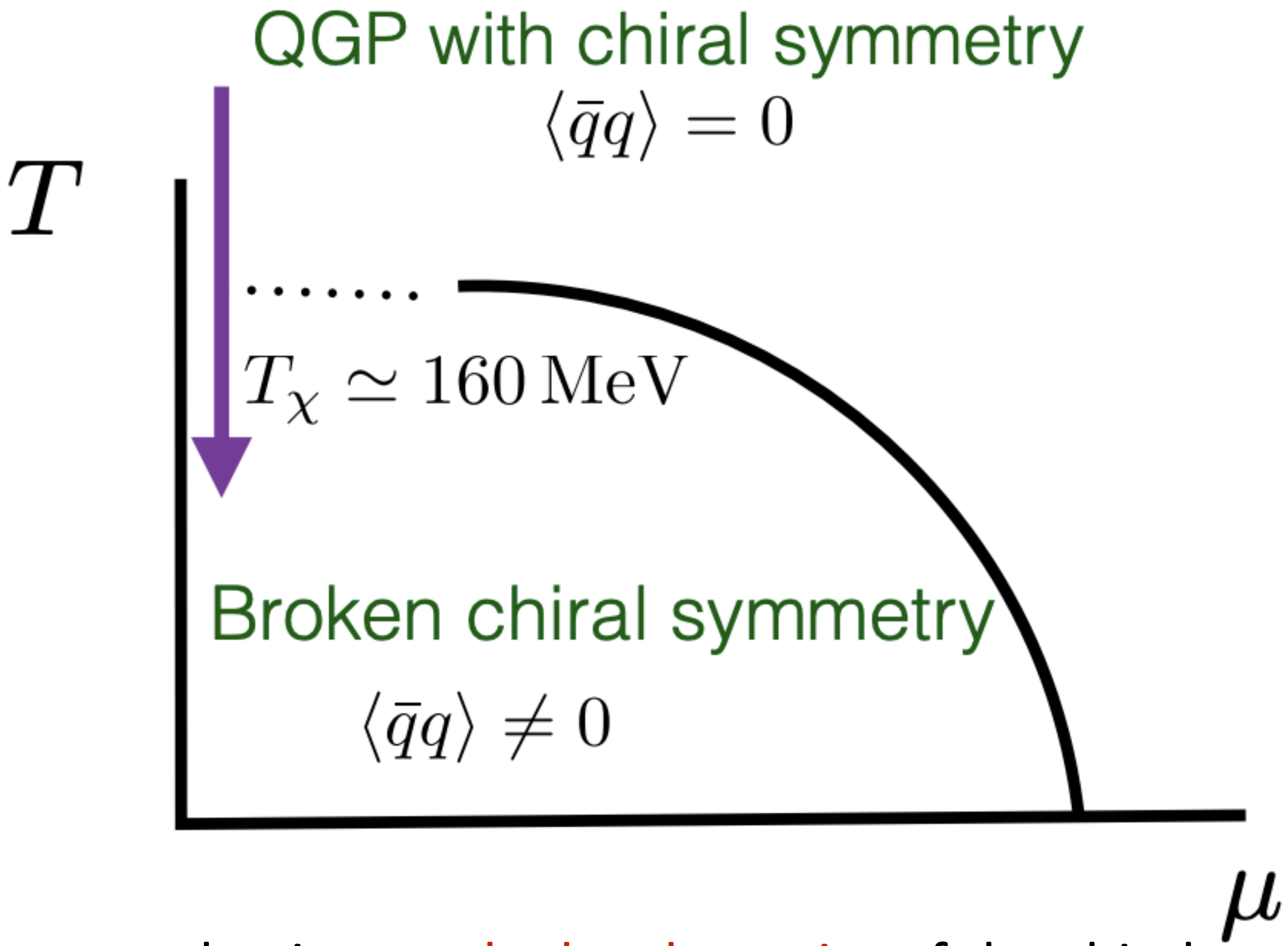
A.Florio, E.G. A.Mazeliauskas, A. Soloviev, D. Teaney in preparation
L. Batini, E. G. and N. Wink, PRD 108 (2023)

ERG, 25/9/2024

Heavy ion Collision



Motivation



We are neglecting any hydro-dynamics of the chiral condensate !

Setup: the O(4) phase transition

The (approximated) conserved quantities of 2 flavour QCD are

$T^{\mu\nu}$	J^μ_V	J^μ_A
Stress	Iso-vector (isospin)	Iso-axial
(T, u^μ)	μ_V	μ_A
	$\bar{q}\gamma^0 t_I q$	$\bar{q}\gamma^0\gamma_5 t_I q$

The approximate flavour symmetry $SU(2)_L \times SU(2)_R \sim O(4)$

The order parameter is the chiral condensate

$$\langle \bar{q}q \rangle \sim \phi_\alpha = (\sigma, \varphi_\alpha) = (\text{sigma}, \text{pions})$$

We need the hydrodynamic theory of the charge and the order parameter

Rajagopal Wilczek (93)
Son and Stephanov (02)

Son; Jensen et al 1203.3556, Bhattacharya et al

Ideal equation

The pressure is the normal plus the potential and a kinetic term of the condensate

$$\mathcal{W}[g_{\mu\nu}, A_\mu, H] = \int_x \sqrt{g}\left\{p(T) + \frac{\chi_0}{4}\mu_{\alpha\beta}\mu_{\alpha\beta} - \frac{1}{2}\Delta^{\mu\nu}(D_\mu\phi)_\alpha(D_\mu\phi)_\alpha - V(\Phi) + H_\alpha\phi_\alpha\right\}$$

In the ideal case we have a dependence of the field and its spatial derivative (Ginzburg-Landau)

Stress

$$(T^{\mu\nu}) = \frac{2}{\sqrt{g}}\frac{\delta\mathcal{W}}{\delta g_{\mu\nu}}$$

Currents

$$(J^\rho)_{\alpha\beta} = \frac{2}{\sqrt{g}}\frac{\delta\mathcal{W}}{\delta(A_\rho)_{\alpha\beta}}$$

Order parameter

$$\phi_\alpha = \frac{1}{\sqrt{g}}\frac{\delta\mathcal{W}}{\delta H_\alpha}$$

Form symmetry we have

quark mass

$$\partial_\mu T^\mu_\nu = 0$$

$$\partial_\mu J^\mu_{\alpha\beta} = (\phi_\alpha H_\beta - \phi_\beta H_\alpha)$$

We need an equation for the condensate

Entropy production

Entropy

$$s = \frac{1}{T}(e + p - \frac{1}{2}\mu_{\alpha\beta}n_{\alpha\beta})$$

$$\vec{\xi} = \Delta^{\mu\nu}\partial_\mu \phi$$

$$d = u \cdot \partial$$

Modified Gibbs relation

$$dp = sdT + \frac{1}{2}n_{\alpha\beta}d\mu_{\alpha\beta} - \frac{1}{2}d\vec{\xi}^2 + \left(-\frac{\partial V}{\partial \phi_\alpha} + H_\alpha\right) d\phi_a$$

Using the equation of motion one can find the entropy production

new terms!

$$\partial_\mu(su^\mu - \frac{\mu_{\alpha\beta}}{T}(q^\mu)_{\alpha\beta}) = \left[\frac{1}{T}(d\phi)_\alpha + \frac{\mu_{\alpha\beta}}{T}\phi_\beta\right]\left[\partial_\mu \vec{\xi}^\mu_\alpha - \frac{1}{\Phi}\frac{\partial V}{\partial \Phi}\phi_\alpha + H_\alpha\right]$$
$$-\nabla_\mu\left(\frac{u_\nu}{T}\right)\Pi^{\mu\nu} - \partial_\mu\left(\frac{\mu_{\alpha\beta}}{2T}\right)(q^\mu)_{\alpha\beta}$$

Stress

Diffusion

Josephson constrain

In the ideal case the time derivative of the field is "locked" to the chemical potential

$$(d\phi)_\alpha + \mu_{\alpha\beta}\phi_\beta = 0$$

The same equation can be obtain imposing the stability of the condensate

$$[\langle \bar{q}q \rangle, H - \mu N] = 0$$ ← Poisson bracket

In the dissipative case

$$(d\phi)_\alpha + \mu_{\alpha\beta}\phi_\beta = \text{ gradient corrections}$$

such that the entropy is increasing

Dissipative equations

Relaxation equation for the order parameter

$$u^\mu \partial_\mu \phi_\alpha + \mu_{\alpha\beta}\phi_\beta = \Gamma\left[\partial_\mu(\Delta^{\mu\nu}\partial_\nu\phi_\alpha) - \frac{1}{\Phi}\frac{\partial V}{\partial \Phi}\phi_\alpha + H_\alpha\right] + \zeta^{(1)}\phi_\alpha \nabla\cdot u$$

Dissipative energy momentum tensor

$$T^{\mu\nu} = u^\mu u^\nu (e+p) + p g^{\mu\nu} + (\partial_\mu\phi)_\alpha(\partial_\nu\phi)_\alpha - u^\mu u^\nu u^\sigma u^\rho (\partial_\sigma\phi)_\alpha(\partial_\rho\phi)_\alpha$$
$$-\eta\sigma^{\mu\nu} - \Delta^{\mu\nu}\left[\zeta^{(0)}\nabla\cdot u - \zeta^{(1)}\phi_\alpha\left(\partial_\mu(\Delta^{\mu\nu}\partial_\nu\phi_\alpha) - \frac{1}{\Phi}\frac{\partial V}{\partial\Phi}\phi_\alpha + H_\alpha\right)\right]$$

Current with diffusion

$$(J^\mu)_{\alpha\beta} = n_{\alpha\beta}u^\mu + (J^\mu_\perp)_{\alpha\beta} - T\sigma\Delta^{\mu\nu}\partial_\nu\left(\frac{\mu_{\alpha\beta}}{T}\right)$$

Superfluid component

$$(J^\mu_\perp)_{\alpha\beta} = \Delta^{\mu\nu}[(D_\nu\phi)_\alpha\phi_\beta - (D_\nu\phi)_\beta\phi_\alpha]$$

Wilczek-Rajagopal hep-ph/9210253 + little bit us

Linearized equation

We linearize the equation around equilibrium (mean field)

$$\phi_\alpha = (\bar{\sigma} + \delta\sigma, \bar{\sigma}\varphi_a)$$

Axial charge and pion equation (assuming zero vector chemical potential)

$$\partial_t \varphi = -\mu_A + \Gamma\left(\nabla^2 - m^2\right)\varphi,$$
$$\partial_t \mu_A = v^2(-\nabla^2 + m^2)\varphi + D_0 \nabla^2 \mu_A,$$

Sigma equation

$$\partial_t \delta\sigma = \Gamma\left[\nabla^2 - m_\sigma^2\right]\delta\sigma,$$

The parameter depends on the temperature and on the value of the fields at the minimum

$$v^2 = \frac{\bar{\sigma}^2}{\chi} \qquad m^2 = \frac{H}{\bar{\sigma}} \qquad \Gamma = \text{const} \qquad D_0 = \text{const}$$

Solving for the pion we get a dumped Klein Gordon equation

Equation of motion (Model G)

Rajagopal Wilczek (93) S. Schlichting, D. Smith, and L. von Smekal, Nucl. Phys. B (2020)

J. V. Roth, Y. Ye, S. Schlichting, and L. von Smekal, (2024).

Chiral condensate ϕ_a + Axial and Vector charge $n_{ab} = \chi_0 \mu_{ab}$

$$\partial_t \phi_a + g_0\, \mu_{ab}\phi_b = \Gamma_0 \nabla^2 \phi_a - \Gamma_0 (m_0^2 + \lambda\phi^2)\phi_a + \Gamma_0 H_a + \theta_a \,,$$

$$\partial_t n_{ab} + g_0\, \nabla \cdot (\nabla \phi_{[a}\phi_{b]}) + H_{[a}\phi_{b]} = D_0 \nabla^2 n_{ab} + \partial_i \Xi^i_{ab} \,.$$

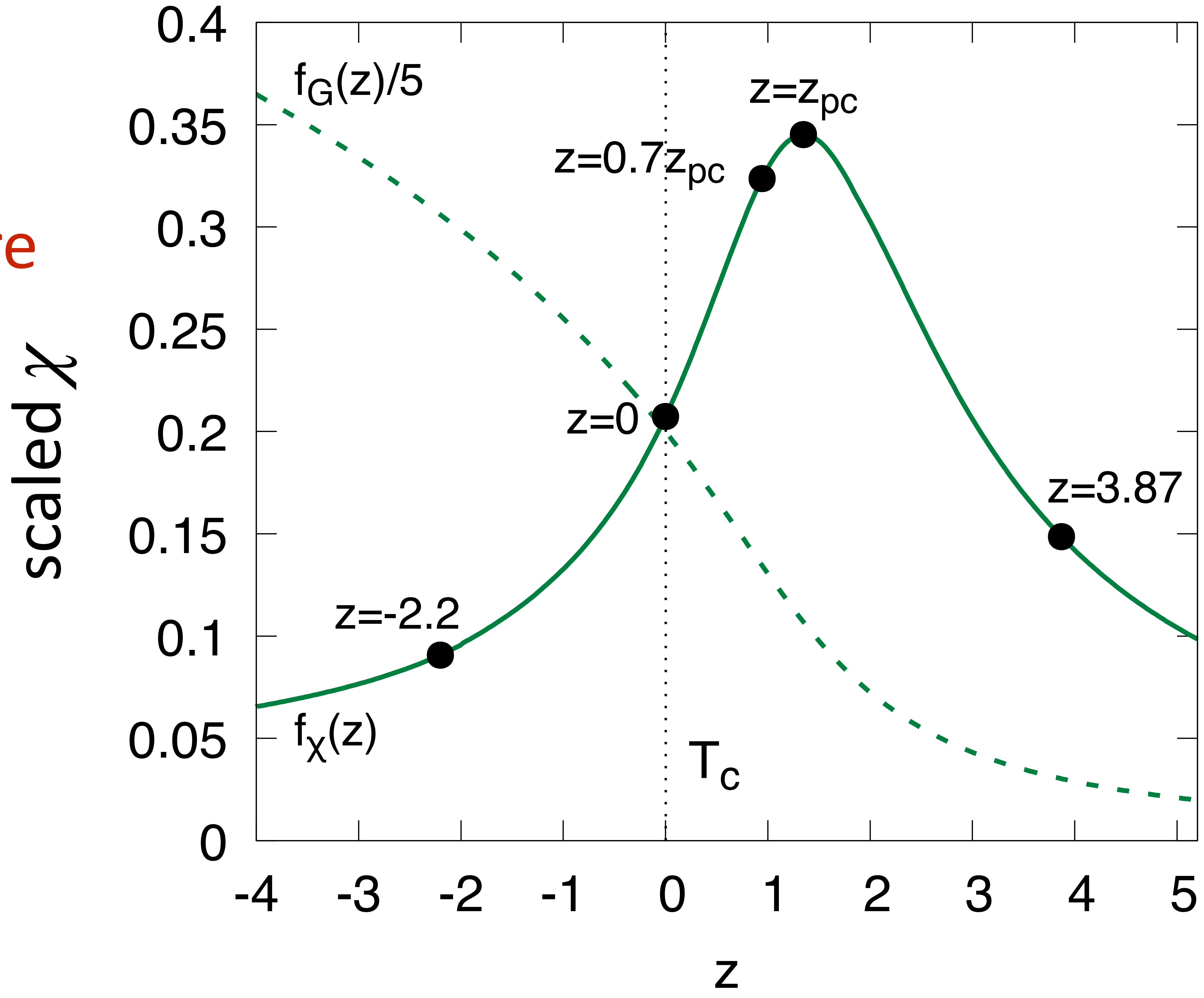
Ideal part — Dissipative part — Gaussian Noise

- The ideal part is charge conservation and Josephson constraint
- Two dissipative coefficient Γ_0 and D_0 and noise
- The simulation of the stochastic process is done with an ideal step and metropolis update. C.Chattopadhyay, J.Ott, T.Schaefer and V.V.Skokov, PRL.(2024)

- At high temperature: charge diffusion
- At low temperature: pion propagation as the vev develops

Simulating the O(4) crossover

Low temperature



High temperature

Susceptibility

$$\chi = h^{1/\delta - 1} f_\chi(z)$$

Vev

$$\bar{\sigma} = h^{1/\delta} f_G(z)$$

Scaling variabile

$$z = h^{-1/\beta\delta} \left(\frac{T - T_c}{T_c} \right)$$

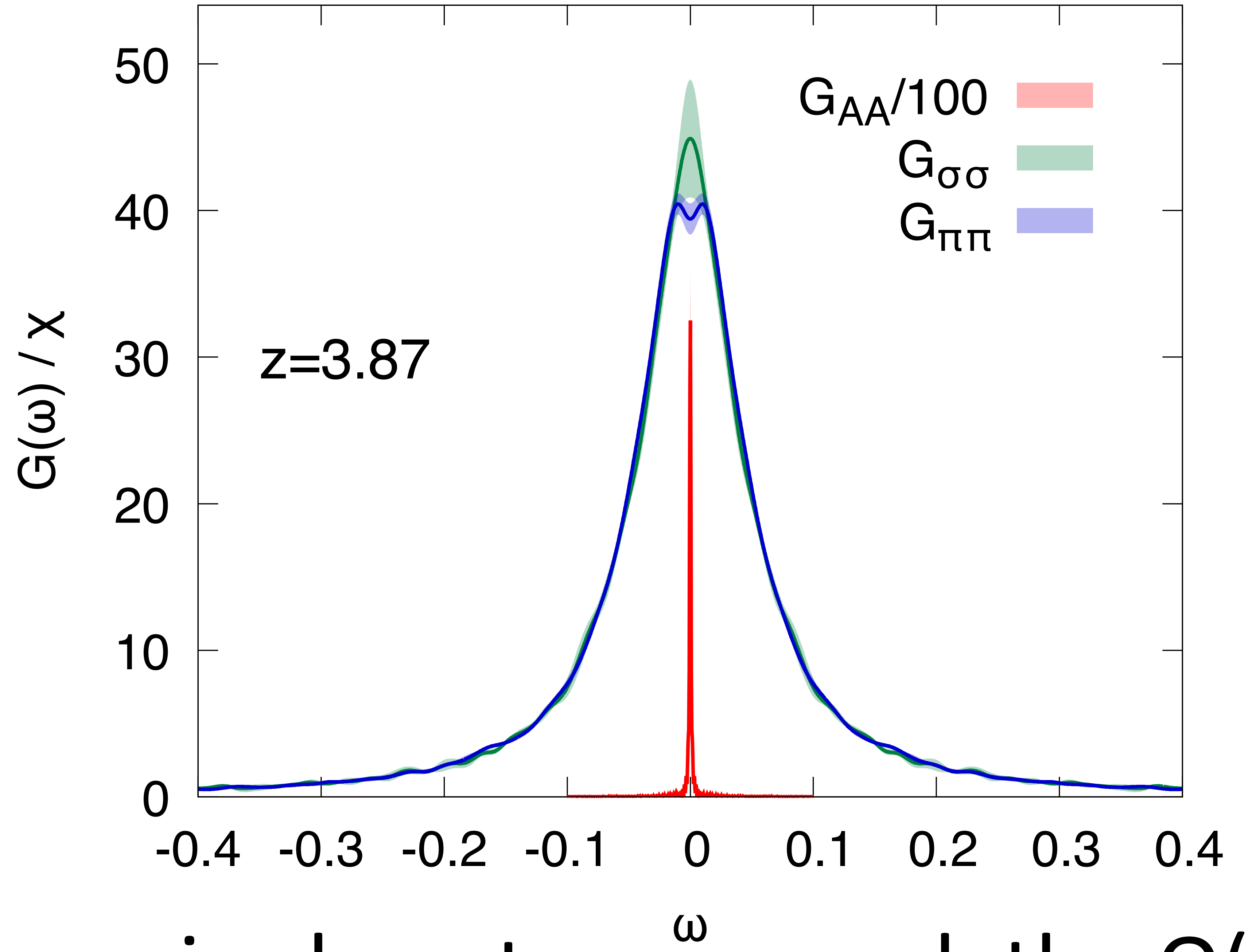
High temperature

$$G_{\sigma\sigma}(t,k) \equiv \frac{1}{V}\langle \sigma(t,\mathbf{k})\sigma(0,-\mathbf{k})\rangle_c \,,$$

$$G_{\pi\pi}(t,k) \equiv \frac{1}{3V}\sum_s \langle \pi_s(t,\mathbf{k})\pi_s(0,-\mathbf{k})\rangle_c \,,$$

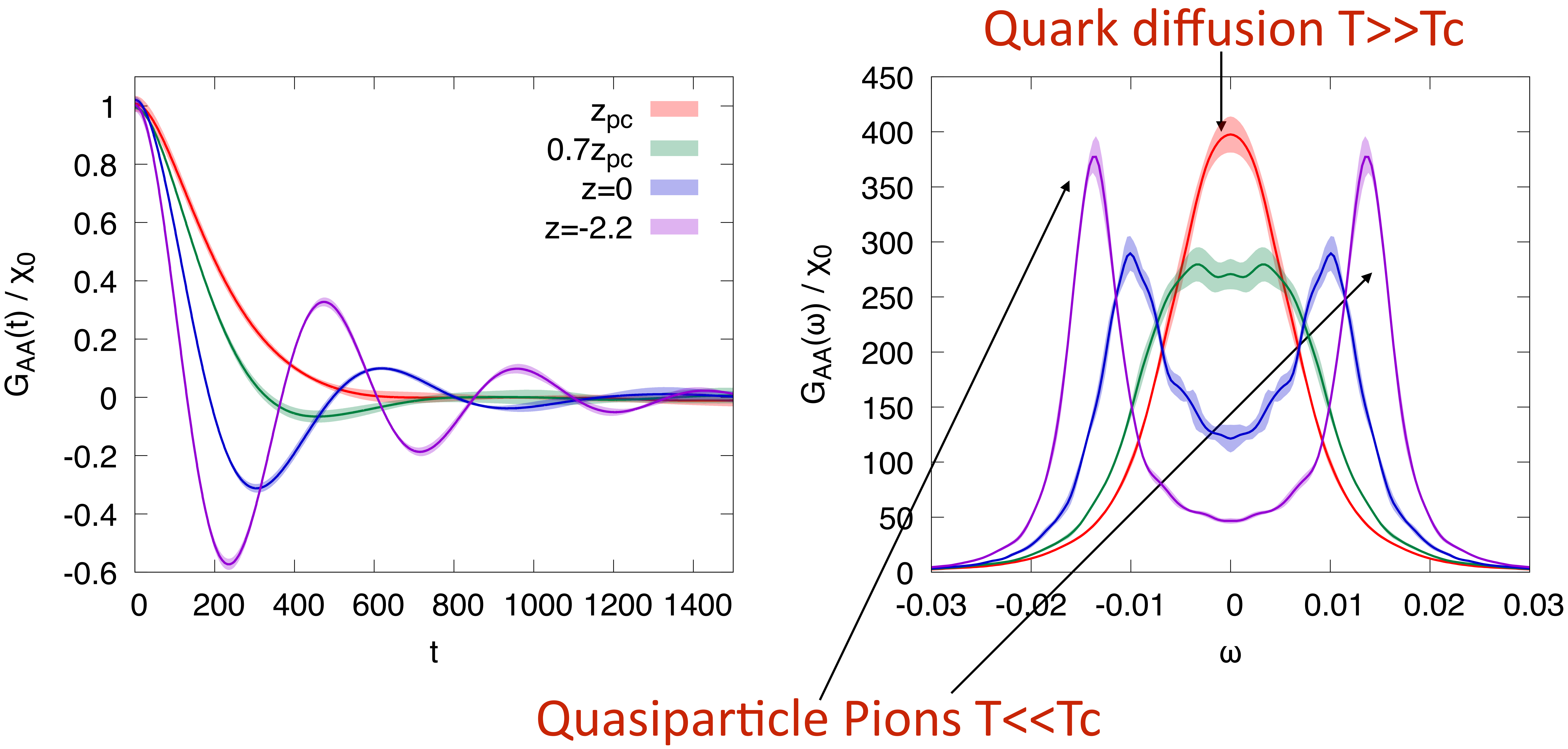
$$G_{AA}(t,k) \equiv \frac{1}{3V}\sum_s \langle n_A^s(t,\mathbf{k})n_A^s(0,-\mathbf{k})\rangle_c \,,$$

We focus on the statistical correlator at $k = 0$



The axial charge is almost conserved, the O(4) field are simply dissipate with a broad width

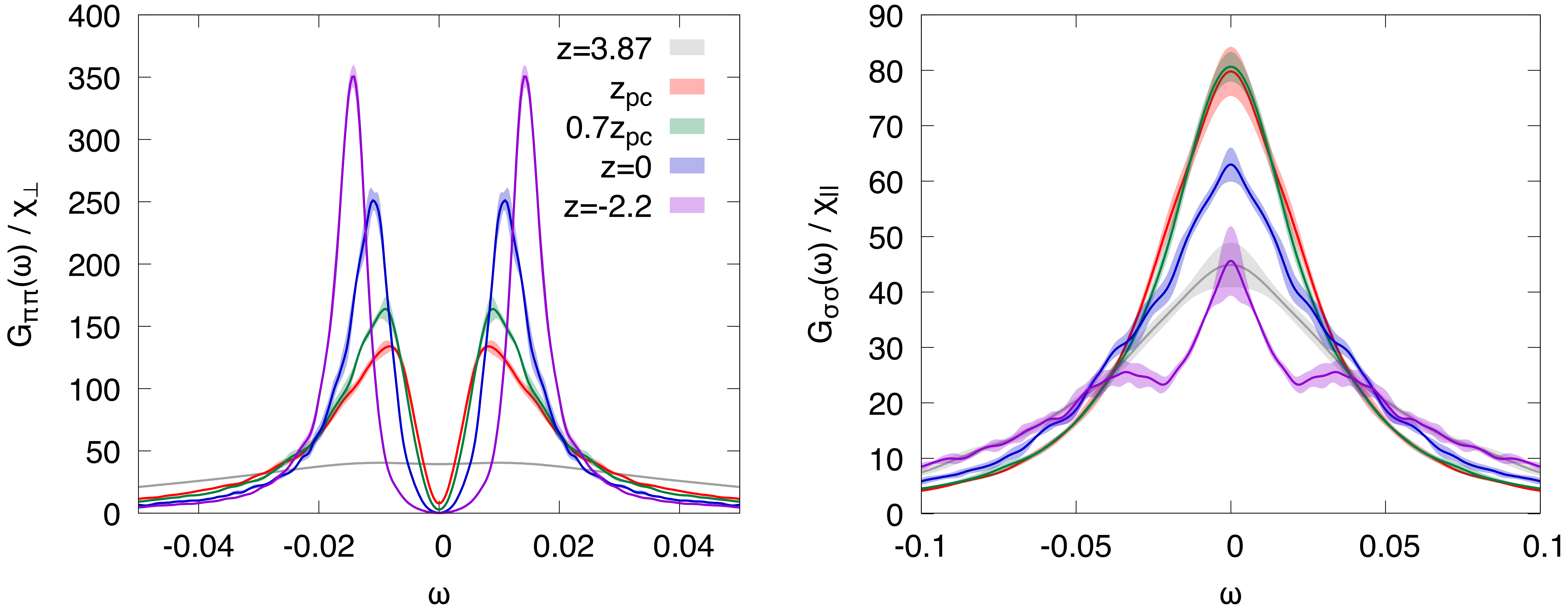
Propagation of axial charge across the transition



Around T_{pc} the axial charge starts changing form a diffusive form

to a quasiparticle one

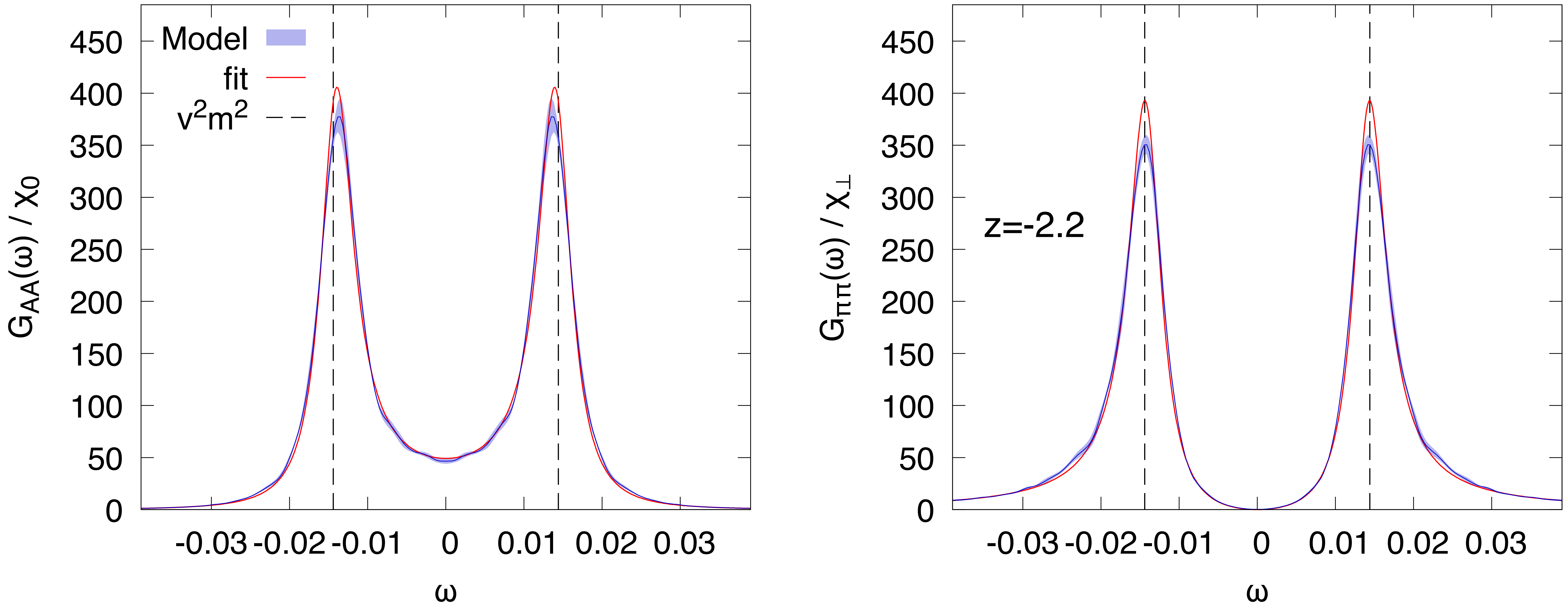
Scalar and Pseudo scalar channel



- The pion correlation function follows the axial current one, from a diffusive behaviour at high temperature the quasiparticle peak are formed at low temperature

- In the sigma channel the damping constant Γ change as function of the temperature

Pion propagator fit



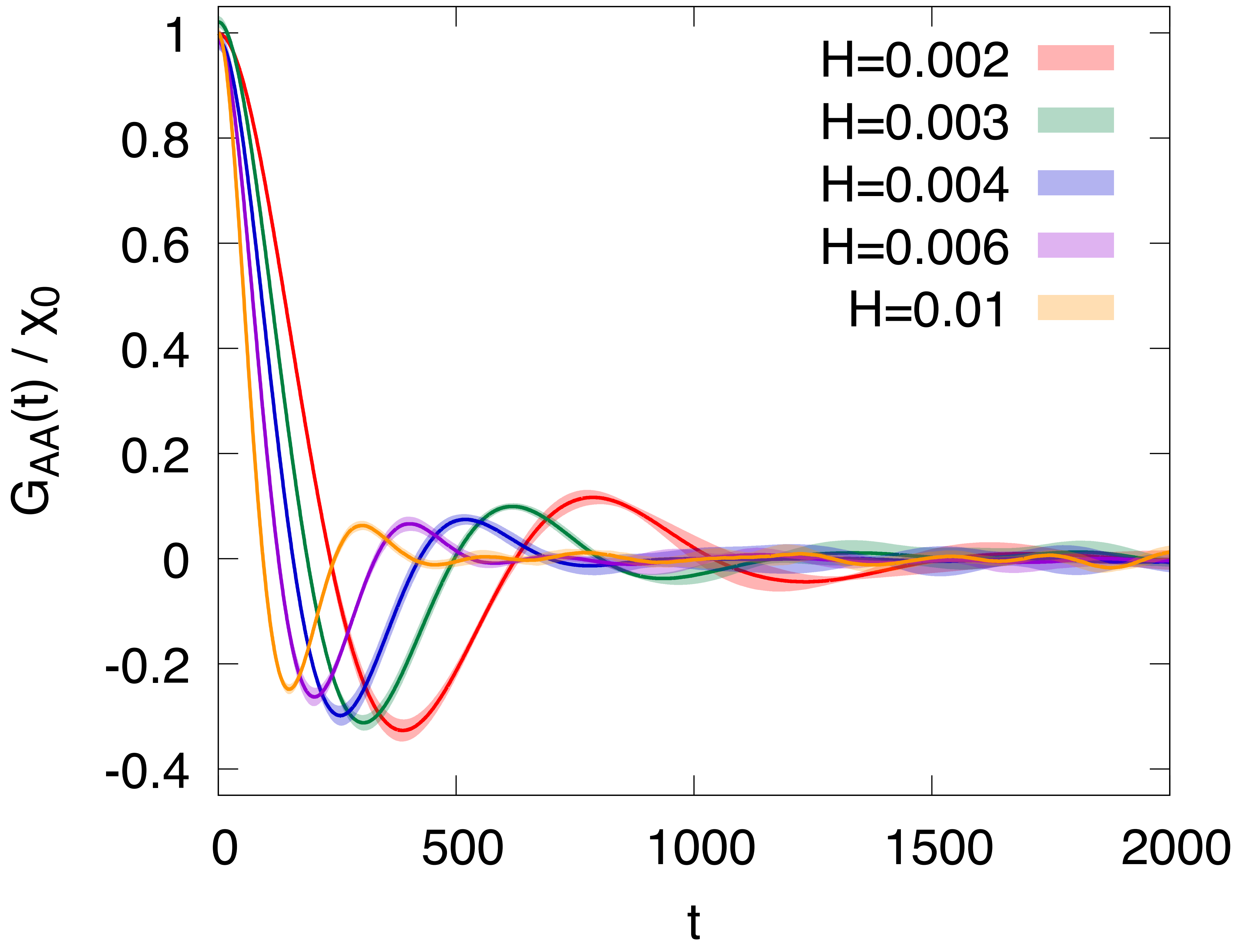
$$G_{AA}(\omega) = \frac{2\chi_0 \Gamma v^2 m^4}{(-\omega^2 + v^2 m^2)^2 + (\omega \Gamma m^2)^2} \quad G_{\pi\pi}(\omega) = \frac{2\chi_\perp \Gamma m^2 \omega^2}{(-\omega^2 + v^2 m^2)^2 + (\omega \Gamma m^2)^2}$$

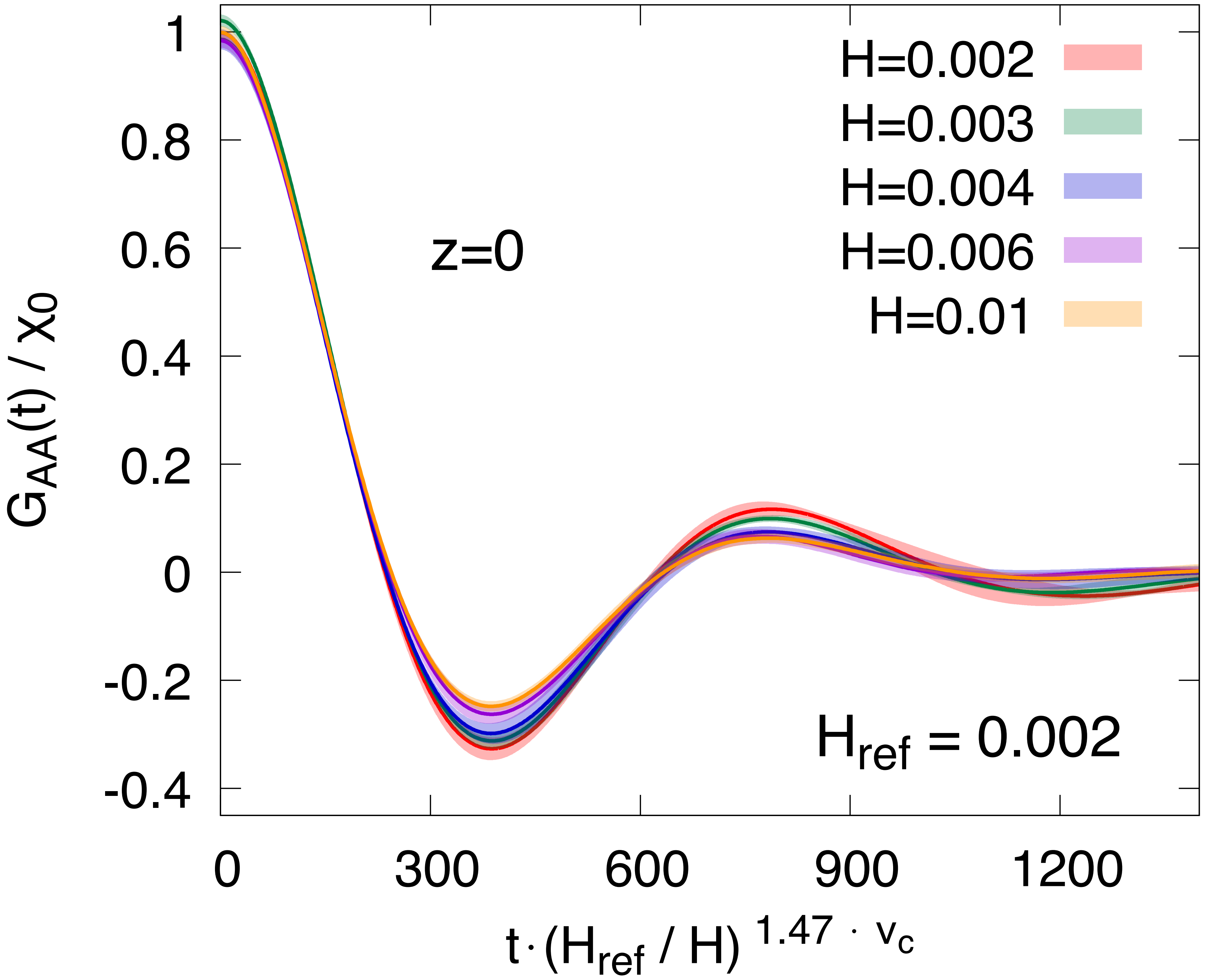
The pole mass satisfy the GOR (Gell-Mann Oakes Renner) relation

$$v^2 m^2 = \frac{H\sigma_0}{\chi_0}$$

Dynamical scaling on the critical line

On the Critical line z=0 we should have scaling with a dynamical critical exponent ζ





$$\frac{G_{AA}(t,H)}{\chi_0} = Y_A^c\left(H^{\zeta\nu_c} t\right) \ ,$$

Mesure: $\zeta = 1.47 \pm 0.01$

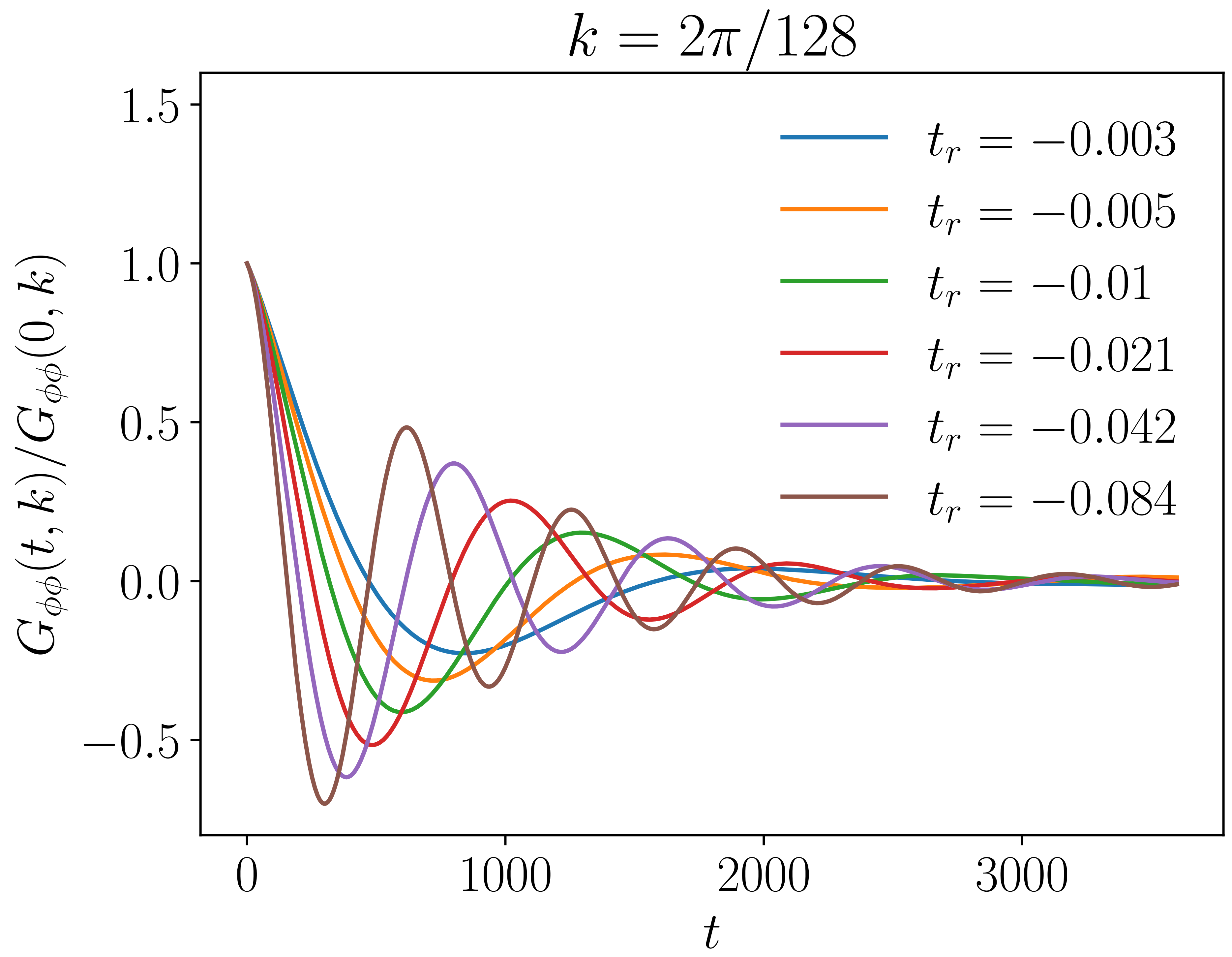
Expected: $\zeta = d/2 = 1.5$

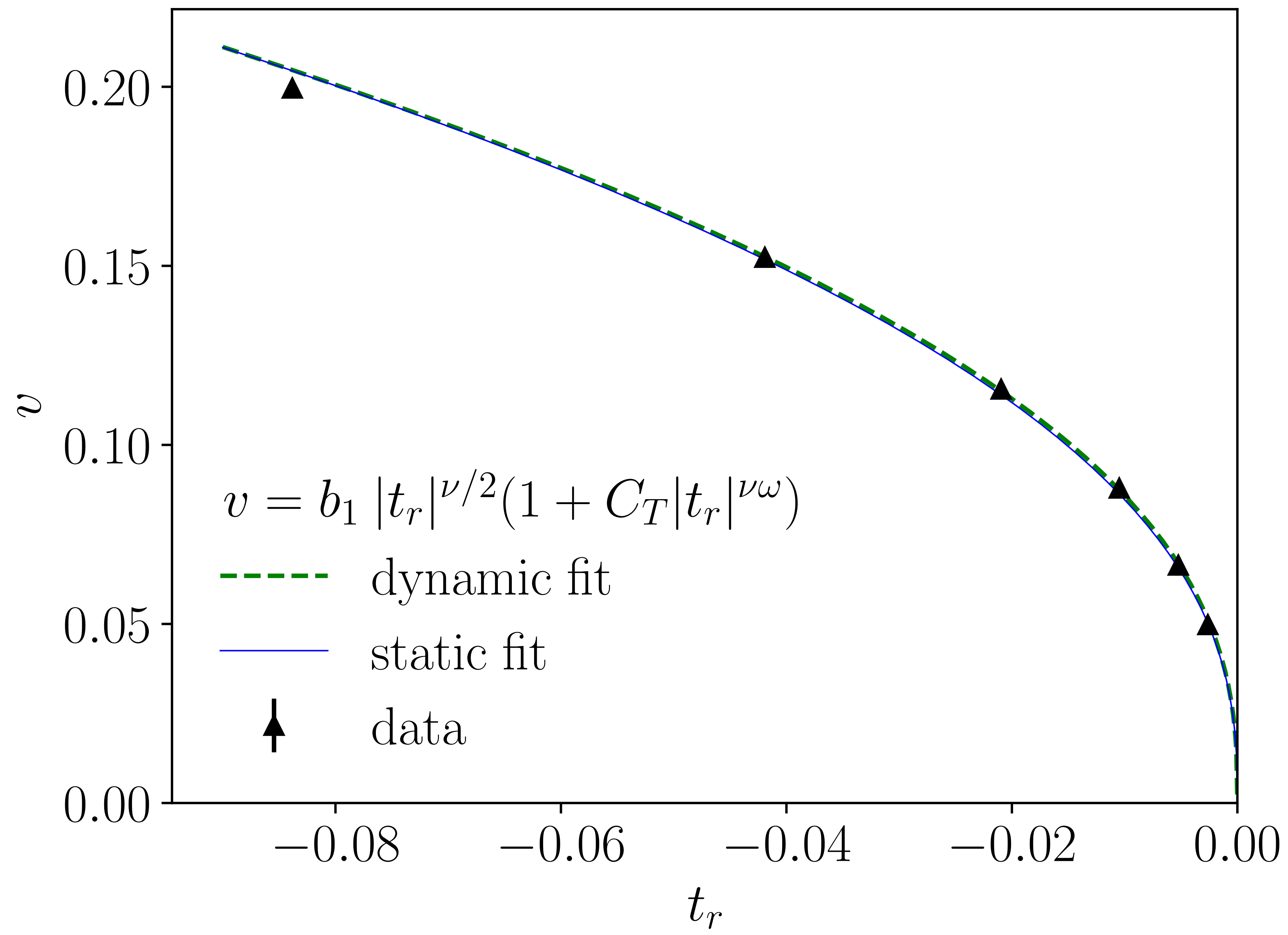
Rajagopal Wilczek (93)

Finite k broken phase m_q=0

A. Florio, E.G., D. Teaney PRD (2023)

In the broken phase one has pion waves



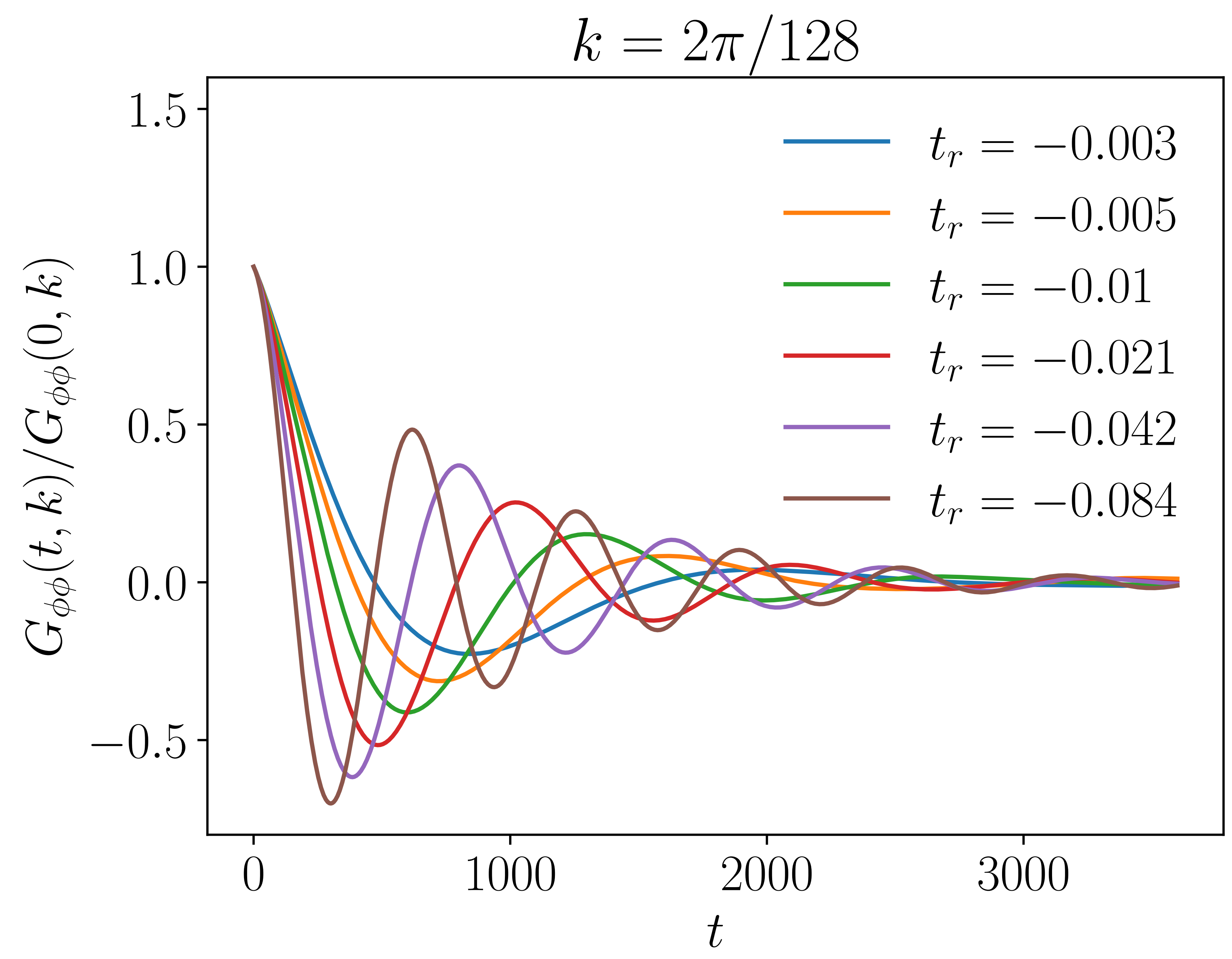


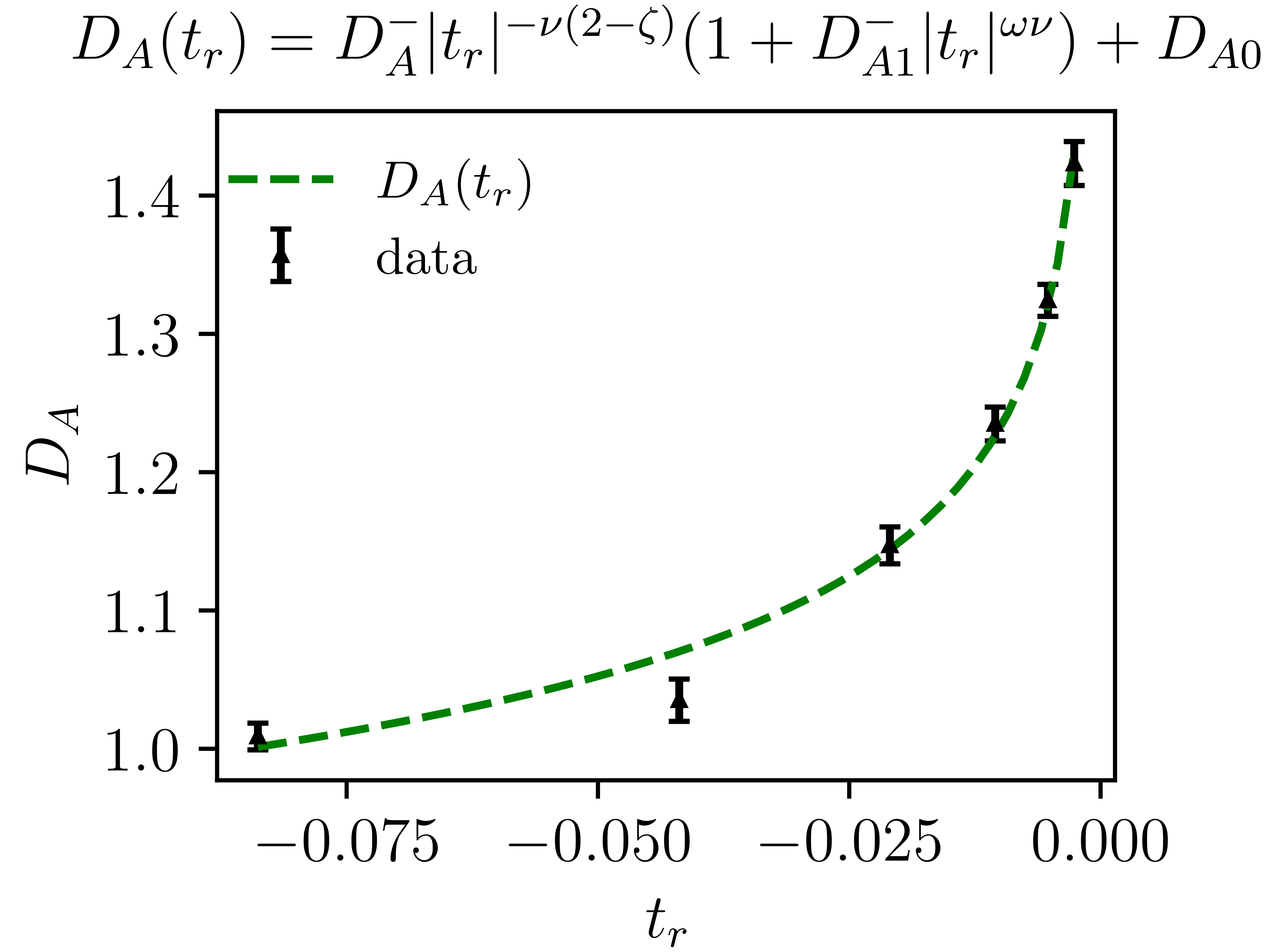
$$\omega(k) = vk + v_1 k^2.$$

The dispersion relation of the of the waves actually is determine by the GOR relation

$$v^2 \propto \langle\phi^2\rangle$$

Axial Diffusion coefficient m_q=0





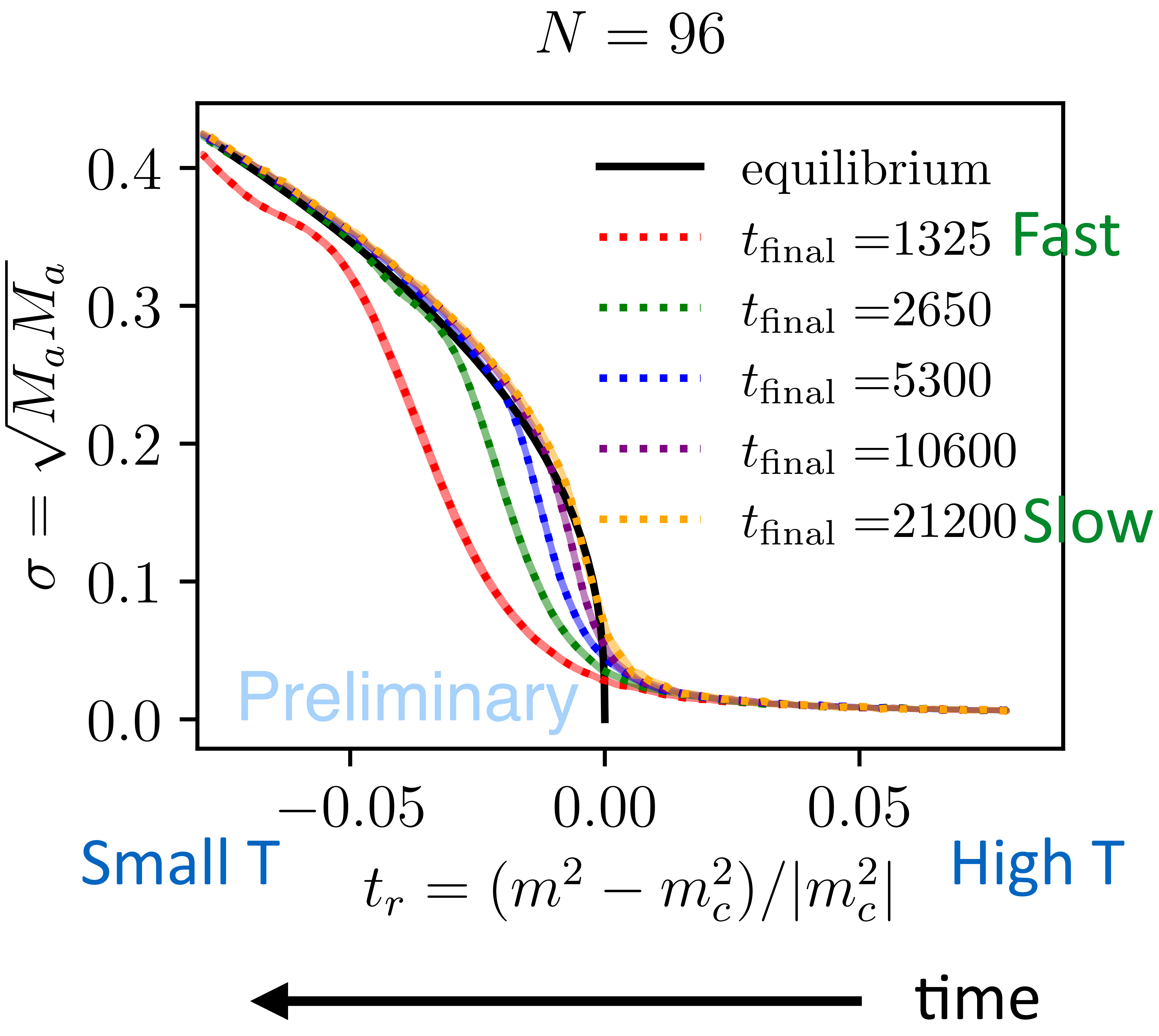
- The axial diffusion in the broken phase satisfies the scaling relation
- The constat part is big respect to the critical part
- There are severe dependencies form the finite volume effects, such as

Damping rate of the pions $\quad \Gamma(k) = D_A k^2(1 + d_1/L)$

Kibble-Zureck Protocol

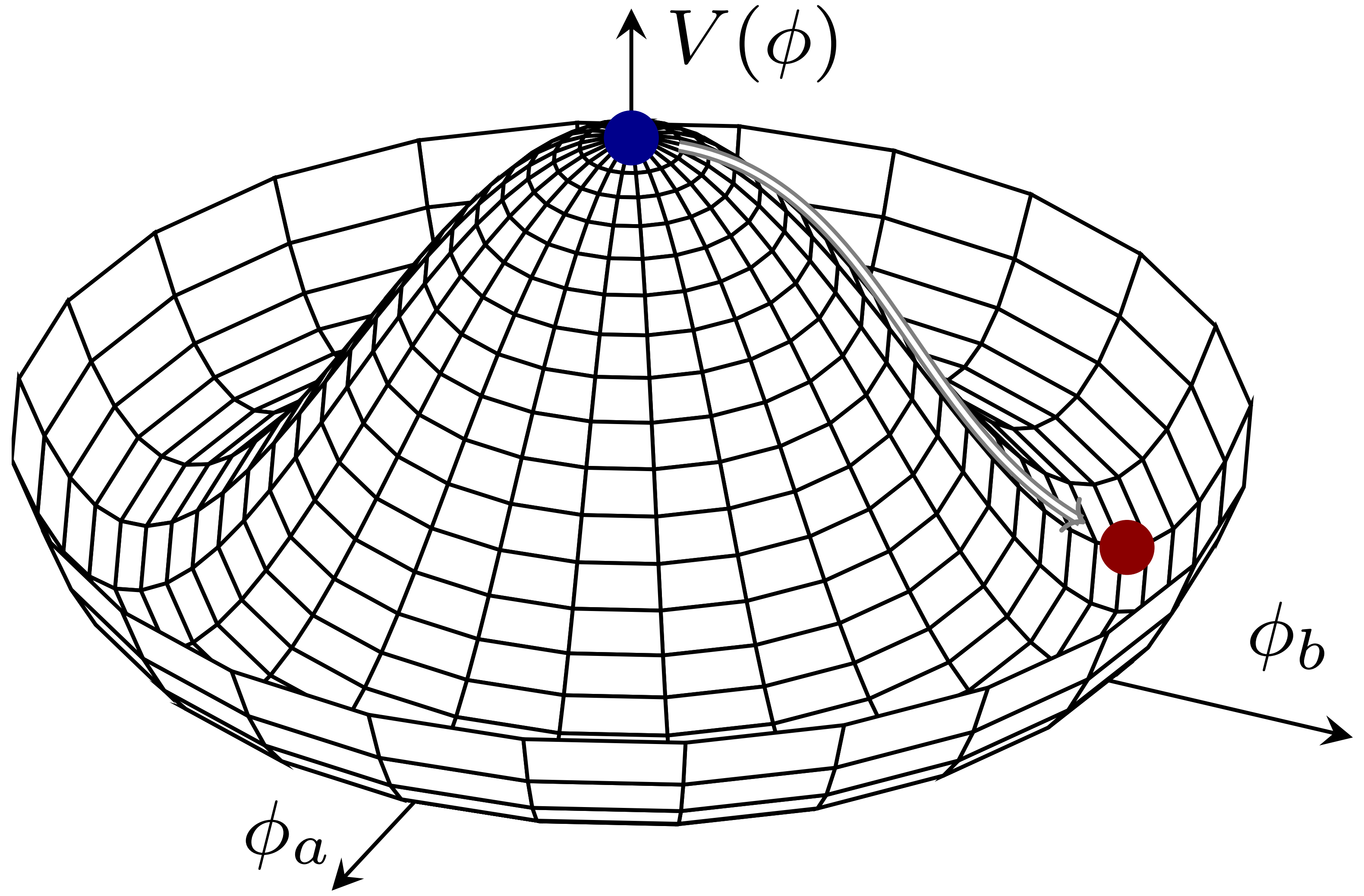
A.Florio, E.G., A.Mazeliauskas, A. Soloviev, D. Teaney in preparation

Starting form high temperature we performed a series of simulations where the temperature is lowerd at constat rate each time step



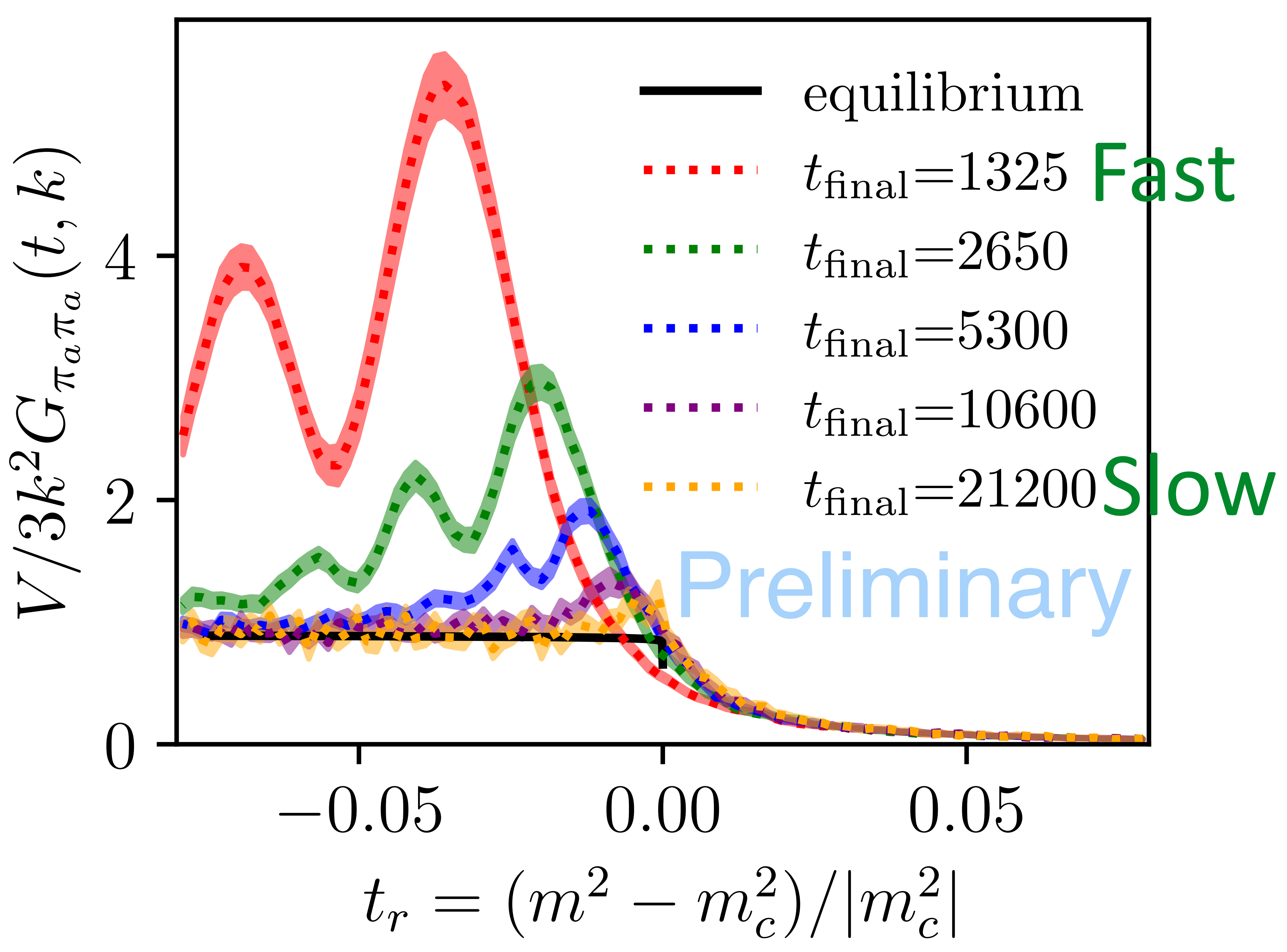
For slow rate the system has always time to requilibrate

For fast rate the system falls out of equilibirum

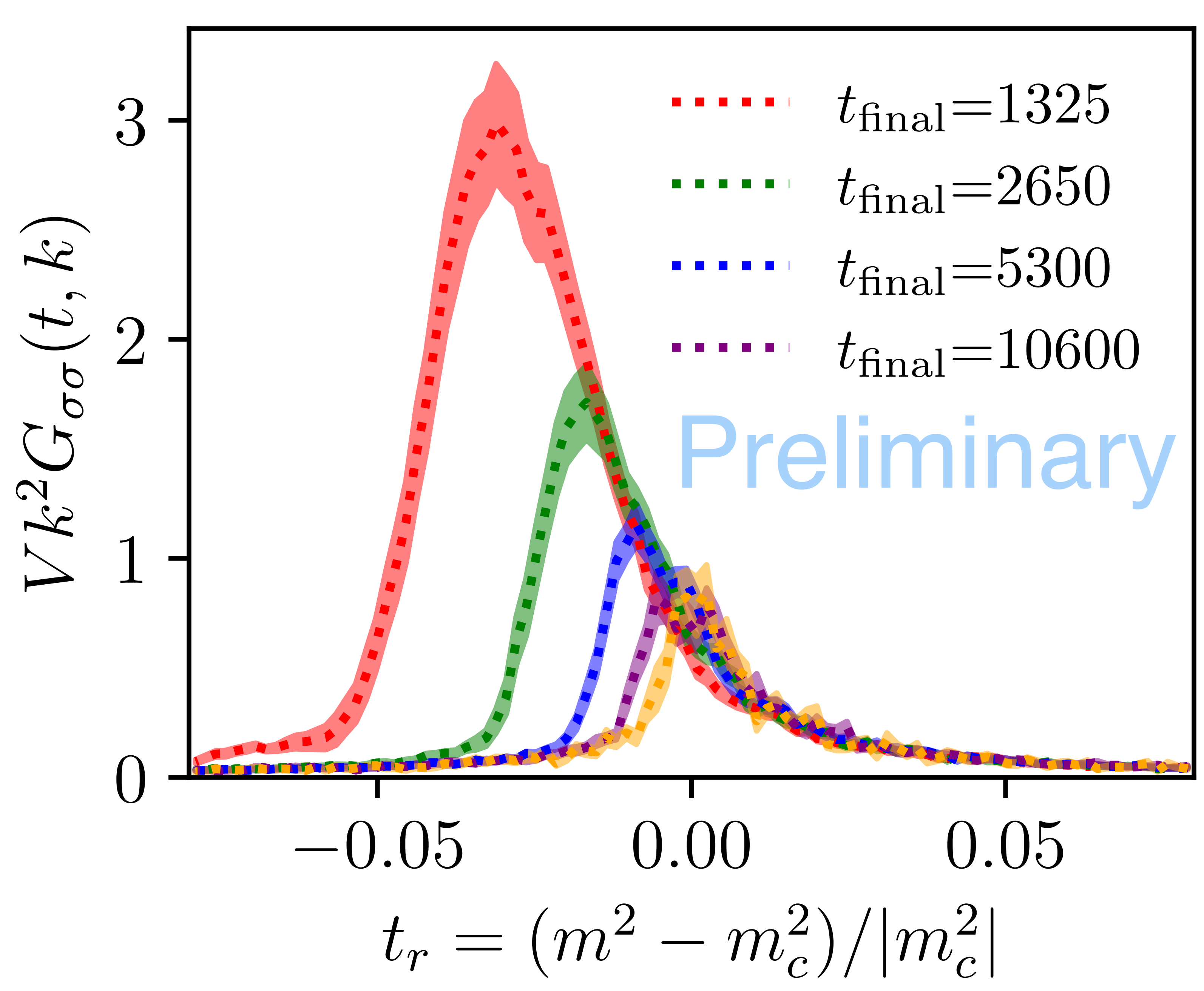


2-point function for small momenta

The equal time correlator



$G_{\pi\pi}(t,k) \propto f_\pi(t,k)$



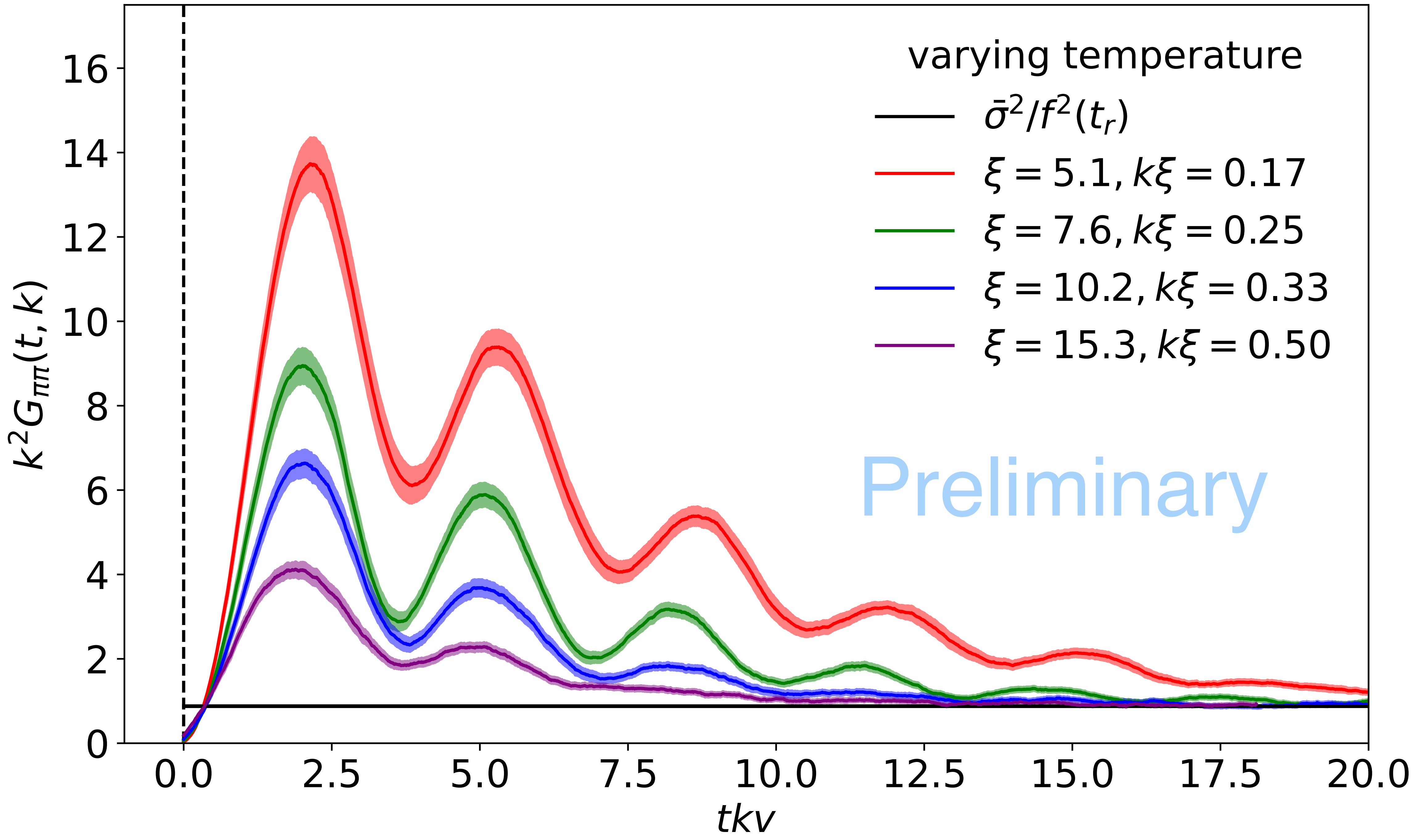
- Both corrlator have an exponetial rise due to the rolling down of the potential
- The longitudianl model relaxes quickly to equilibrium
- The pion correlation starts oscillating and relaxes much slower
- The non-equilibirum contributions are big respect to equilibirium

Quench

A way simpler protocal is sudden quench form restore to broken phase

From a thermal state in the restore phase we flip the temperature

$$m_0^2 \to -m_0^2$$



- Qualitative similar to the Kibble-Zureck
- Exponential rising for small time
- Oscillation with pion energy dispersion
- Tiny delay of reaching the maximum

Mean field equations

The mean field equation in gaussian approxiamation are not hard to find

$$\partial_t \sigma = -\Gamma_0 (m_0^2 + \lambda \sigma^2)\sigma + \Gamma_0 H$$

$$\partial_t G_{\sigma\sigma}(t,k) = -2\Gamma_0 (k^2 + m_\sigma^2) G_{\sigma\sigma}(t,k) + 2\Gamma_0$$

$$\partial_t G_{\pi\pi}(t,k) = \frac{2\sigma}{\chi_0} G_{A\pi}(t,k) - 2\Gamma_0 (k^2 + m_\pi^2) G_{\pi\pi}(t,k) + 2\Gamma_0$$

$$\partial_t G_{AA}(t,k) = -2(\sigma k^2 + H) G_{A\pi}(t,k) - 2D_0 k^2 G_{AA}(t,k) + 2D_0 \chi_0 k^2$$

$$\partial_t G_{A\pi}(t,k) = -(\sigma k^2 + H) G_{\pi\pi}(t,k) + \frac{\sigma}{\chi_0} G_{AA}(t,k) - \Gamma_0 (k^2 + m_\pi^2) G_{A\pi}(t,k) - D_0 k^2 G_{A\pi}(t,k)$$

- The pions and sigma are decopled
- The axial charge and the pions are entangled
- The vector charge is unimportant
- Can be used to figure the relevant scale in the quench
- The solution is qualitativelty in agreement with the simulations

Analytic estiamte for a sudden quench

To have an analytical control over the scale of the problem one can study
A quench from the unbroken pahse to the broken phase for really small quark mass

The condesate start almost at zero and it expoenentiate to a finite value

$$\frac{\sigma(t)}{\sigma_{\mathrm{eq}}} = \frac{1}{\sqrt{C^2 e^{-2\tau} + 1}} \qquad C^2 = \frac{\sigma_{\mathrm{eq}}}{2\sigma_{\mathrm{initial}}} = e^{\tau_0}$$

Everthing is controlled by a big parmeter C if the quark mass is small

$$G_{\pi\pi}(\tau) = \frac{2T}{m_{\mathrm{ini}}^2} \left(\frac{z_Q^3}{4} \right) \frac{e^{2(\tau-\tau_0)}}{e^{2(\tau-\tau_0)} + 1}$$

The correlation function gets enhanced

$$\frac{G_{\pi\pi}}{G_{\pi\pi}^{\mathrm{eq}}} \simeq \frac{1}{2} C$$

A much more simple model: Model A

- Schwinger Keldysh contour
- Semiclassical approximation
- Equilibrium ($\Longrightarrow$ Fluctuation-dissipation relation)
- Unitarity
- Derivative expansion up to ∇^2 and ∂_t

$$\Gamma_k[\Phi] = \int_{t,\mathbf{x}} \phi_a \Big(X_k(\phi_r)(\partial_t \phi_r - i\phi_a) - \nabla^2 \phi_r + U_k^{(1)}(\phi_r) \Big),$$

X_k ↓ Relaxation rate

$U_k^{(1)}$ ↓ Effective potential derivative

$$\Gamma_{k=\Lambda} = \mathcal{S} \longrightarrow \Gamma_{k=0} = \Gamma$$

$$U_\Lambda(\phi) \equiv -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

$$X_\Lambda(\phi) \equiv 1$$

Hohenberg and B. I. Halperin, Rev. Mod. Phys. 49, 435 (1977),
P. Glorioso and H. Liu, arXiv:1805.09331
L. Canet, H. Chaté, B. Delamotte,J. Phys. A: Math. Theor. 44 (201
J. V. Roth and L. von Smekal, JHEP (2023).
T.Schaefer and V.Skokov, PRD (2022)

A much more simple model: Model A

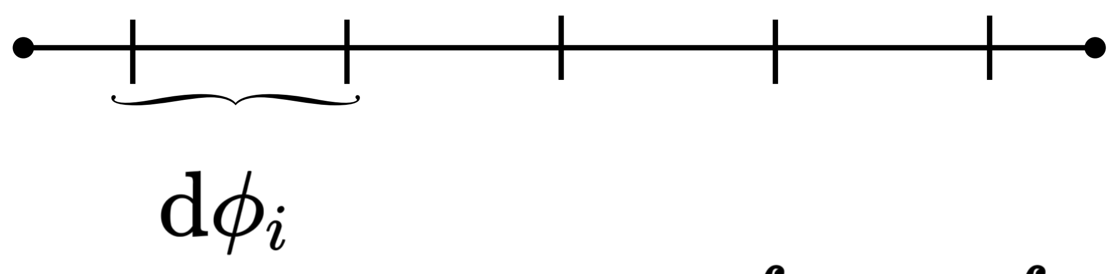
- Litim regulator

$$\partial_k U^{(1)}(\phi) = \frac{\partial}{\partial\phi}\left(\frac{\Omega_d}{(2\pi)^d}\frac{k^{d+1}}{d}\frac{1}{U^{(2)}+k^2}\right) \longrightarrow \partial_t u = \partial_\phi f$$

Transport equation

$$\partial_k X(\phi) = \frac{\Omega_d k^{d+1}}{2(2\pi)^d}\left[3(\partial_\phi G)^2 X + 4\partial_\phi(G^2)X^{(1)} + 2G^2X^{(2)}\right] \qquad G = \frac{1}{k^2+U^{(2)}}$$

- Field discretization



- First-order upwind scheme

$$\partial_t u_i + \frac{f_{i+1}-f_i}{\mathrm{d}\phi_i} = 0$$

- Alternating pattern

$$f_i^R = f\left(t, \frac{u_{i+1}-u_i}{\mathrm{d}\phi_i}\right), \quad f_i^L = f\left(t, \frac{u_i-u_{i-1}}{\mathrm{d}\phi_i}\right)$$

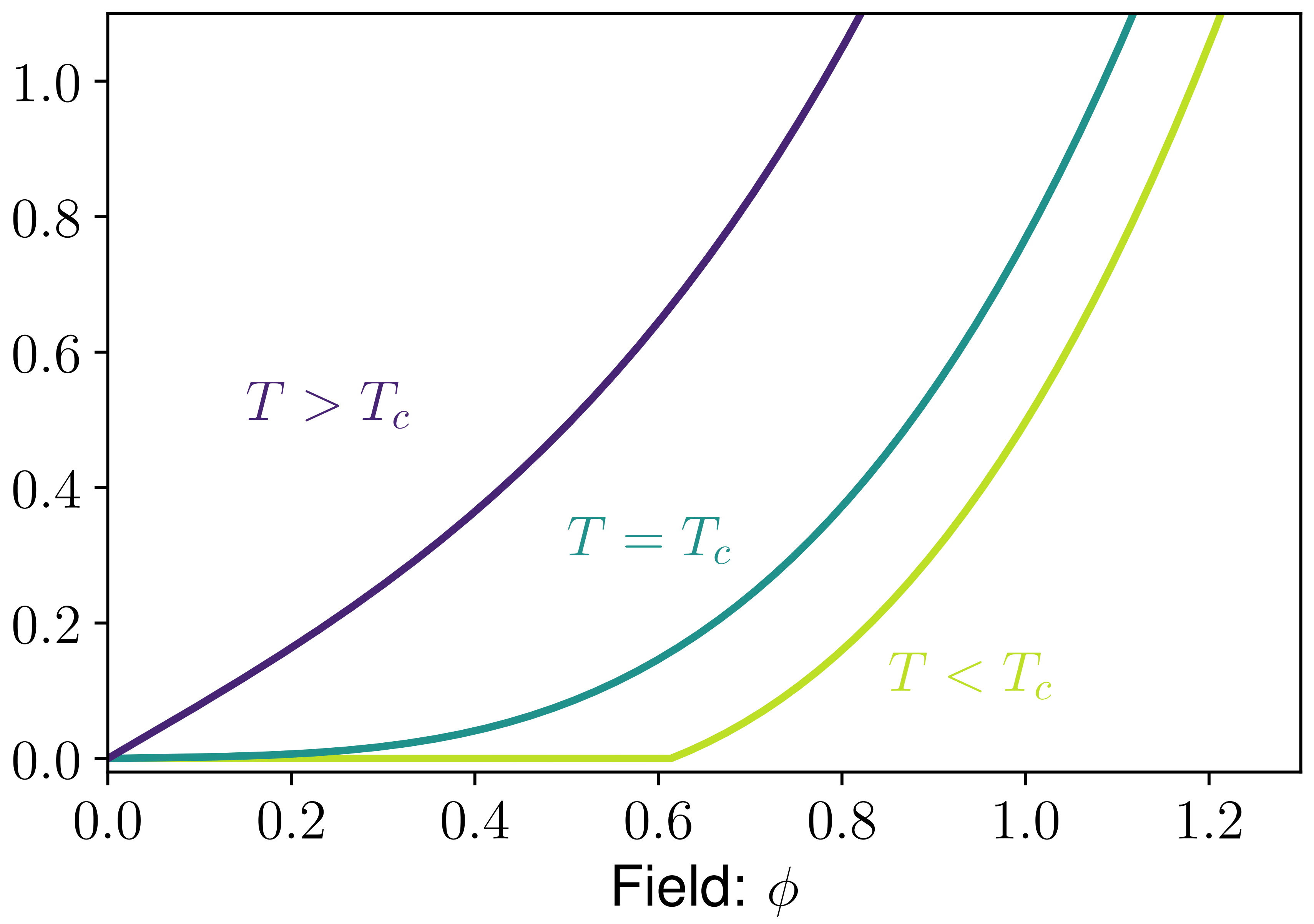
$$\longrightarrow \partial_t u_i = \frac{f_{i+1}^R - f_i^L}{\mathrm{d}\phi_i}.$$

- Similar for the relaxation rate equation

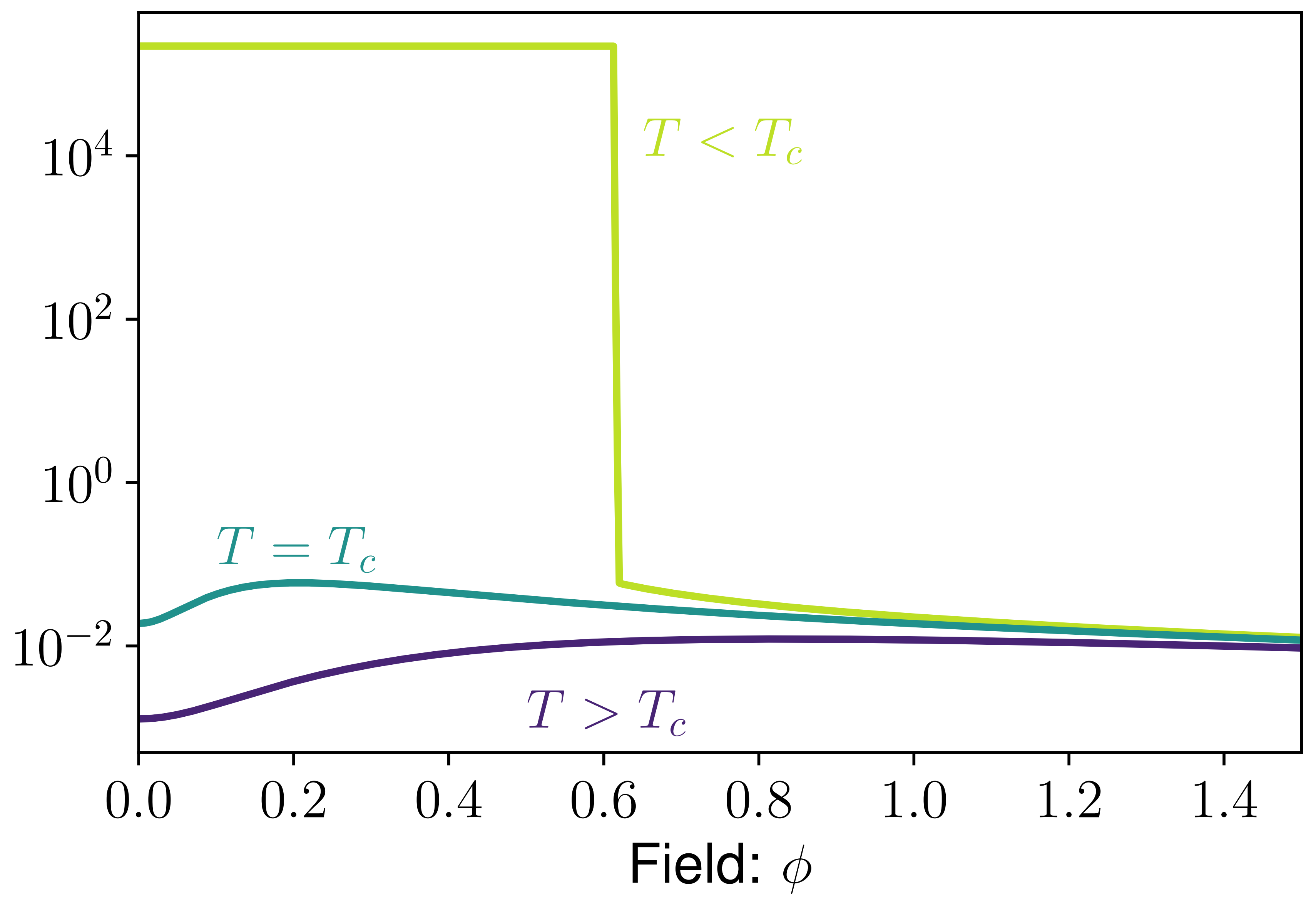
Potential and relaxation rate

L. Batini, E. G. and N. Wink, PRD 108 (2023)

Effective potential derivative $U^{(1)}(\phi)$



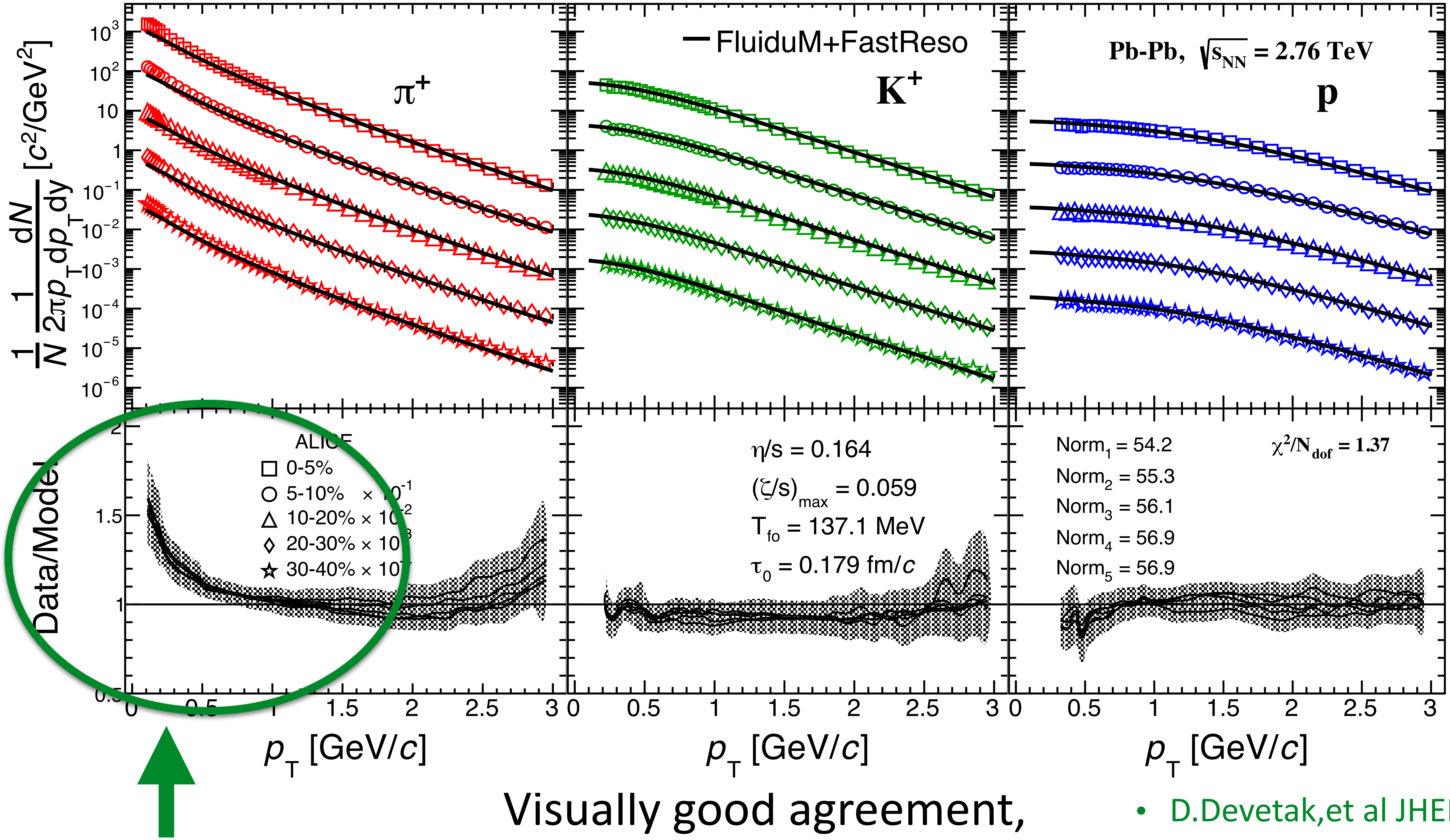
Relaxation rate $\log(X(\phi))$



Without the information from the field dipendent relaxation time solving the effective equations is rather questionable.

Why this is important

Fit the pt spectra of pions kaons and protons in the first five centralities



Visually good agreement,

but statistically significant deviations

The main discrepancy is for pions at low pt

- D.Devetak,et al JHEP (2020)

Outlook

- We deeply and intensively study the real time dynamics of model G
- The model can accomadate the prodution of goldstone bosons in real time
- We study the Kibble-Zurek protocol.
- Sudden quench into the broken phase induces pion pion correlation

More on phenomenology has to developed and studied