Renormalization of scalar Effective Field Theories from Geometry

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Based on [2308.06315] and [2310.19883]

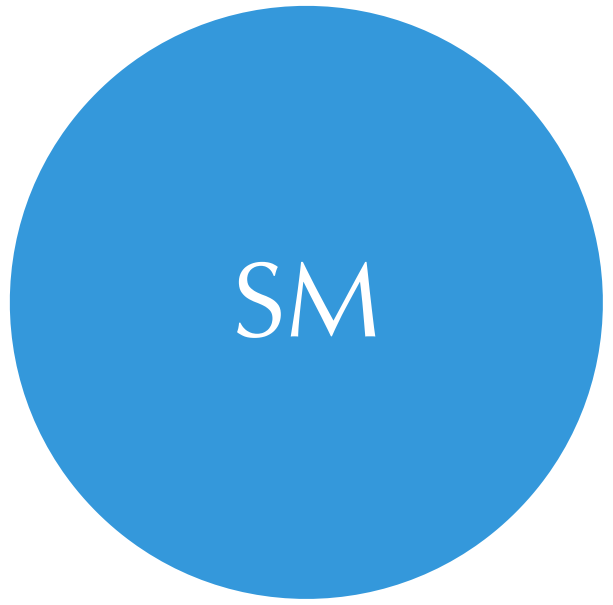
in collaboration with *Jenkins, Manohar* and *Naterop*

ERG 2024

Sep. 24

EFTs for New Physics

The Standard Model of Particle Physics



$$\mathscr{L}_{\text{SM}} = -\frac{1}{4}F^2_{\mu\nu} + \bar{\psi}_i i\not{D}\psi_i + \left(\bar{\psi}_{Li}Y_{ij}H^{(\dagger)}\psi_{Rj} + \text{h.c.}\right) + \mathscr{L}_{\text{Higgs}}$$

Symmetries
$SU(3)_c \times SU(2)_L \times U(1)_Y$
with gauge bosons

Matter content
$q_L \sim (3{,}2)_{1/6}$ $u_R \sim (3{,}1)_{2/3}$ $d_R \sim (3{,}1)_{-1/3}$
$\ell_L \sim (1{,}2)_{-1/2}$ $e_R \sim (1{,}1)_{-1}$

Higgs mechanism
electroweak symmetry breaking

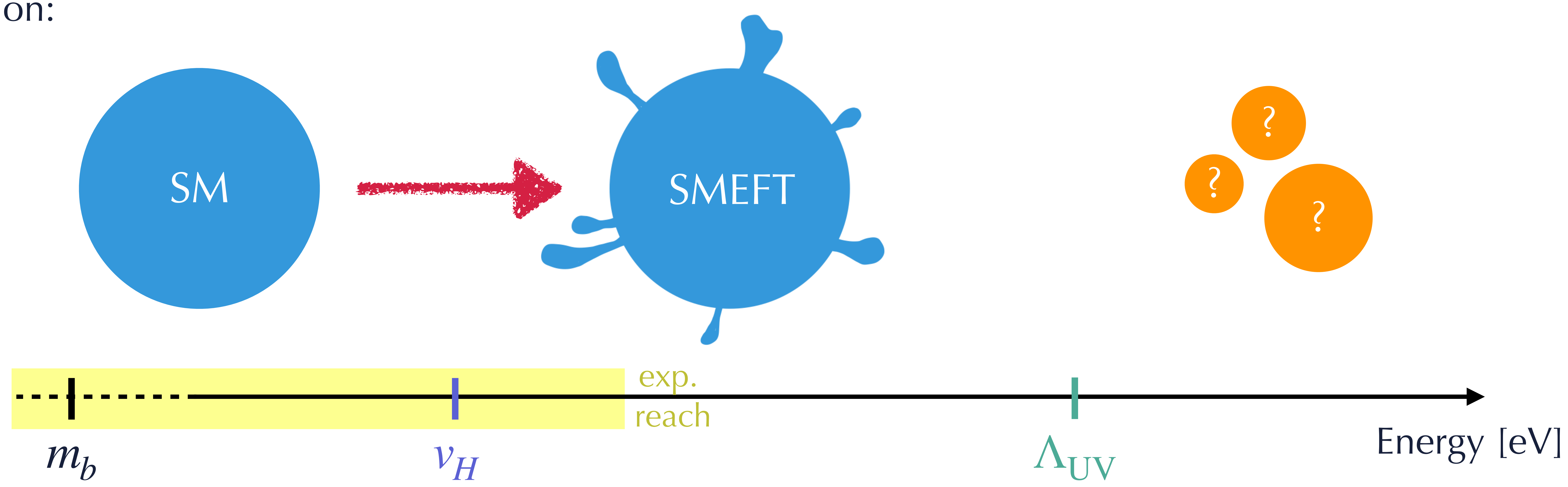
Extremely predictive theory, but still incomplete:

- neutrino masses
- matter-antimatter asymmetry
- dark matter

…

The pivotal role of (SM)EFT

For exploration:



Bottom-up EFT

Deform the SM to accommodate new effects observed in experiments

- "model-independent" correlations between observables
- indications on where to find the new physics scales where a new fundamental theory has to be formulated, e.g.

Fermi theory → m_W → SM

⇒ SMEFT = Extension of the SM

The EFT description

Starting from the SM, we can construct the SMEFT:

number of operators at dimension d

$$\boxed{\mathscr{L}_{\text{SMEFT}} = \mathscr{L}_{\text{SM}} + \sum_{d=5}^{d_{\max}} \frac{1}{\Lambda^{d-4}} \sum_{i=1}^{n_d} C_i^{[d]}\, O_i^{[d]}}$$

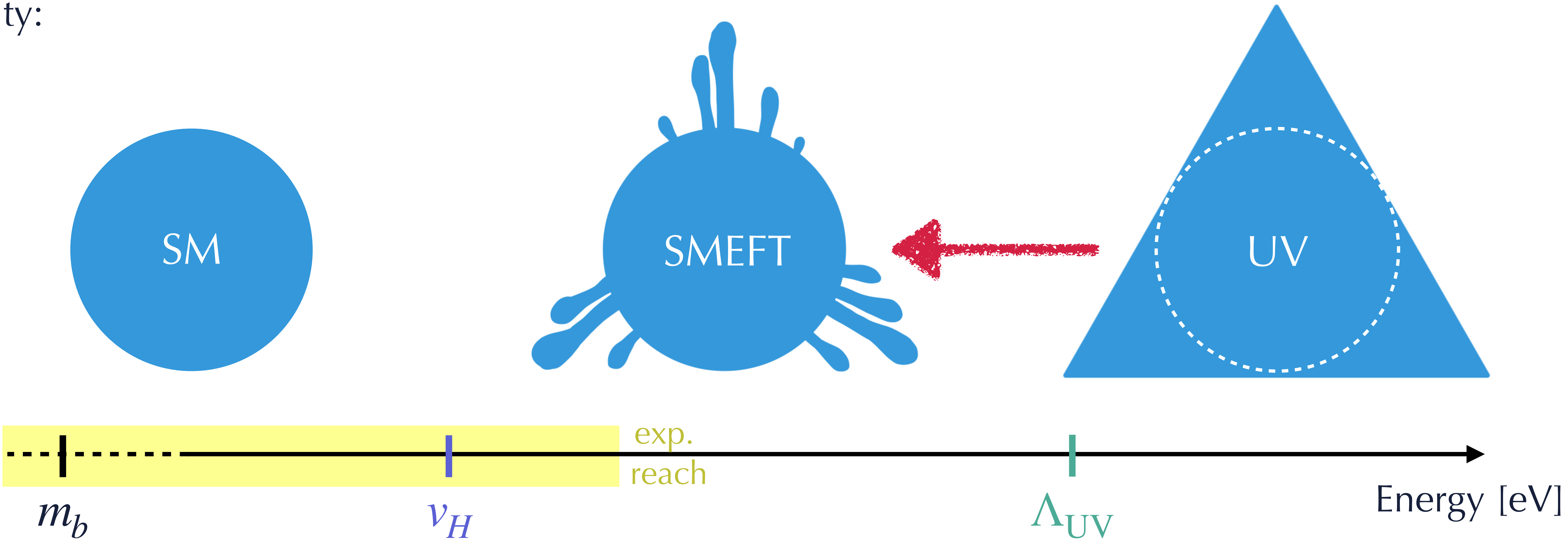
power counting parameter

- The operator basis $\{O_i^{[d]}\}$ is defined by all operators
    - made from the SM particle content
    - respecting the symmetries: Lorentz, gauge, (global)
    - up to the truncation order $d_{\max}$ (↔ precision required)

- The Wilson coefficients $\{C_i^{[d]}\}$ can be fitted to data ↔ encode the strength of the New Physics

The pivotal role of (SM)EFT

For universality:



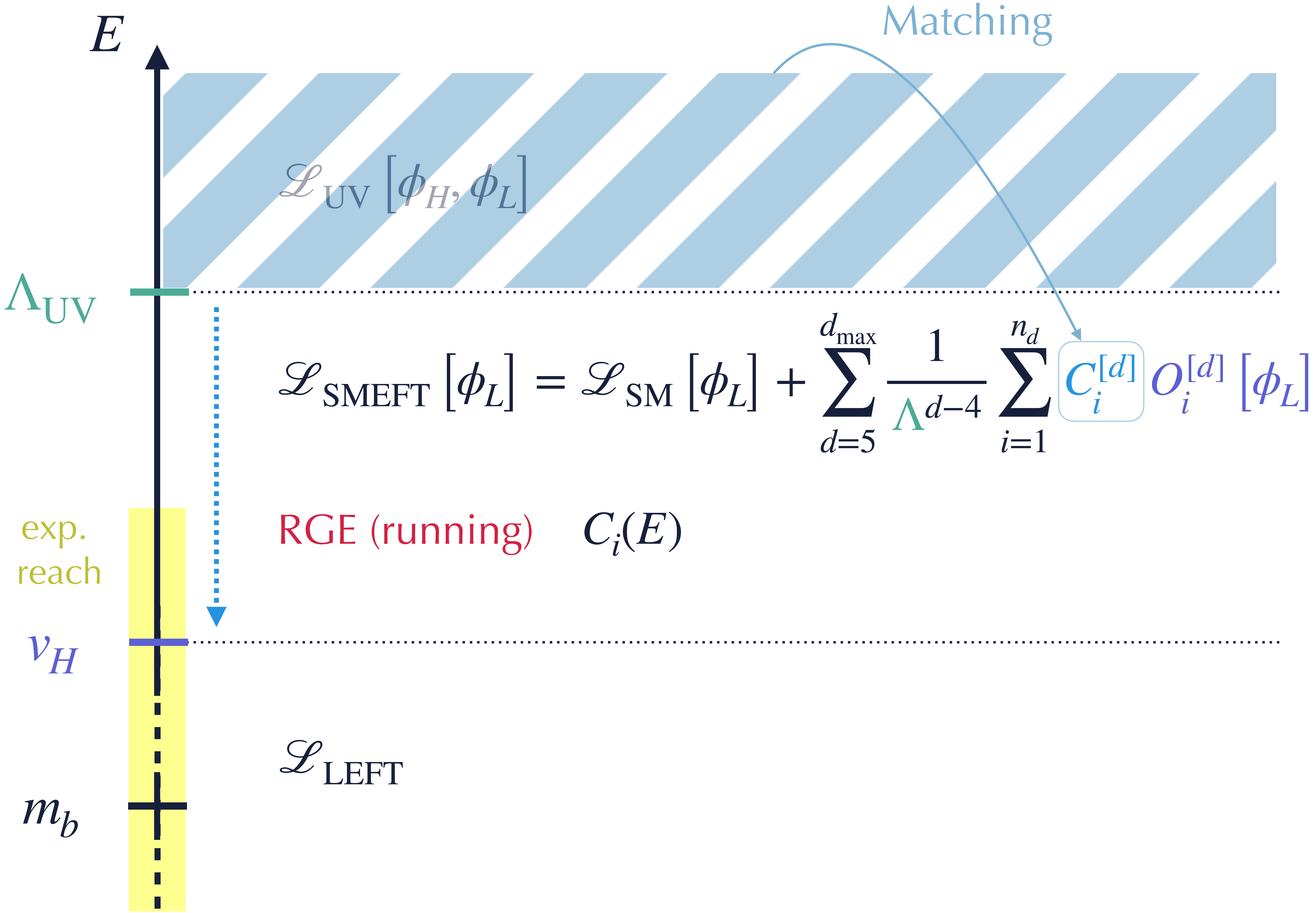
Top-down EFT

Starting from specific UV theory, the heavy modes can be integrated out providing:

- resummation of large logs (through RGE)
- a universal framework to compare with data (SMEFT)

$\Rightarrow$ SMEFT = UV theory approximation

Matching and running



$$\mathscr{L}_{\mathrm{SMEFT}}[\phi_L] = \mathscr{L}_{\mathrm{SM}}[\phi_L] + \sum_{d=5}^{d_{\max}} \frac{1}{\Lambda^{d-4}} \sum_{i=1}^{n_d} \boxed{C_i^{[d]}}\, O_i^{[d]}[\phi_L]$$

RGE (running) $C_i(E)$

$\mathscr{L}_{\mathrm{LEFT}}$

Matching = connect the UV theory to the EFT by deriving the relation between Wilson coefficients $\{C_i\}$ and UV couplings $\{\lambda_i\}$ such that

$$\mathscr{L}_{\mathrm{UV}}[\phi_H, \phi_L] \xrightarrow{E \ll \Lambda_{\mathrm{UV}}} \mathscr{L}_{\mathrm{EFT}}[\phi_L]$$

Automated at one-loop in:

MATCHETE

[Fuentes-Martín, König, JP, Thomsen, Wilsch, 2211.09144]

Two-loop running in the SMEFT is needed.

Geometry of EFTs

Geometric interpretation

A scalar field theory can be written as: [Alonso, Jenkins, Manohar, 1605.03602]

$$\mathscr{L}_{\mathrm{EFT}} = \frac{1}{2} g_{IJ}(\phi)\,(\partial_\mu \phi)^I (\partial^\mu \phi)^J - V(\phi) + \text{higher-derivative terms}$$

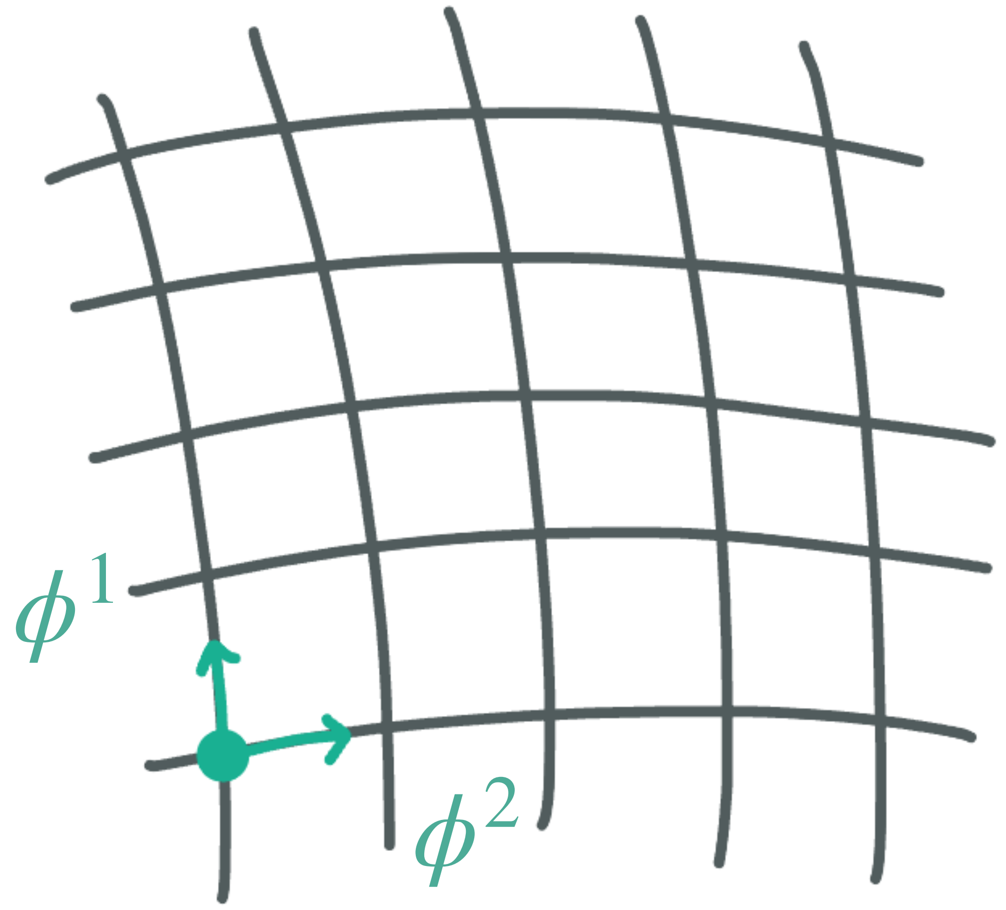
where

- field values = coordinates on a Riemannian manifold
- $g_{IJ}(\phi)$ = inner-product on the tangent space of the field manifold: metric

$$ds^2 \equiv g_{IJ}(\phi)\, d\phi^I\, d\phi^J$$

- potential $V(\phi)$ = function on the field manifold
- field redefinitions (without derivatives) = coordinate transformations

$$\phi^I \to \varphi^I(\phi)$$



SM scalar manifold is flat

Scalar geometry

Under a coordinate transformation,

$$\phi^I \to \varphi^I(\phi)$$

- the derivative of the scalar transforms as a vector

$$\partial_\mu \phi^I \to \left(\frac{\partial \varphi^I}{\partial \phi^J}\right) \partial_\mu \phi^J$$

- the metric transforms as a tensor

$$g_{IJ} \to \left(\frac{\partial \phi^K}{\partial \varphi^I}\right)\left(\frac{\partial \phi^L}{\partial \varphi^J}\right) g_{KL}$$

so $\mathscr{L}_{\text{kin}} = \dfrac{1}{2}\, g_{IJ}(\phi)\, (\partial_\mu \phi)^I (\partial^\mu \phi)^J$ is invariant.

From the metric we can define,

- Christoffel symbols

$$\Gamma^I_{JK} = \frac{1}{2} g^{IL} \left(g_{LJ,K} + g_{LK,J} - g_{JK,L}\right)$$

- Covariant derivatives

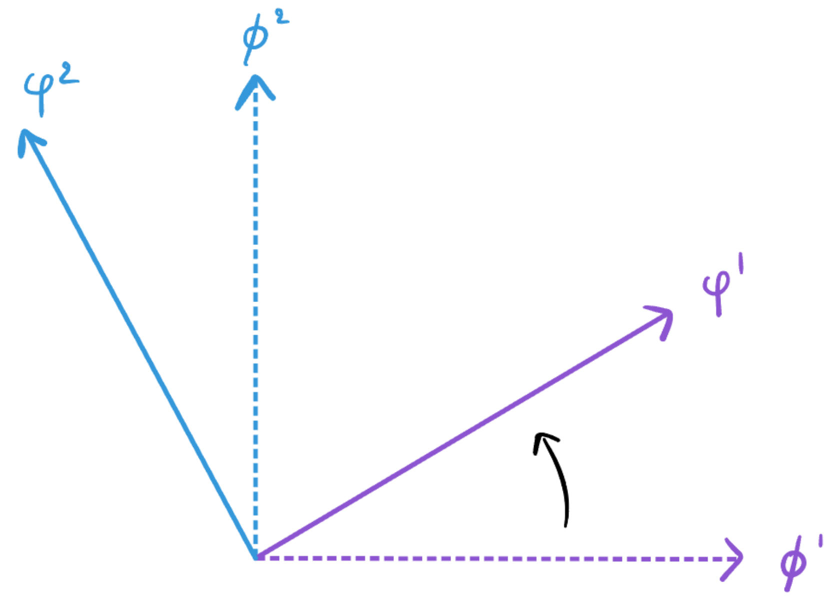
$$T_{J;I} \equiv \nabla_I T_J = \frac{\partial T_J}{\partial \phi^I} - \Gamma^K_{IJ}\, T_K$$

- Riemann curvature tensor

$$R^I_{JKL} = \partial_K \Gamma^I_{JL} + \Gamma^I_{KN} \Gamma^N_{JL} - (K \leftrightarrow L)$$

R and ∇ will appear in scattering amplitudes making them covariant.

$\Rightarrow$ field redefinition in-/covariance = coordinate in-/covariance



Algebraic RGE formulae

for renormalizable models

RGE from background field method

In MS schemes, renormalization group equations are given by the counterterms required to remove the divergences in loop graphs.

Compute the divergences with the background field method:

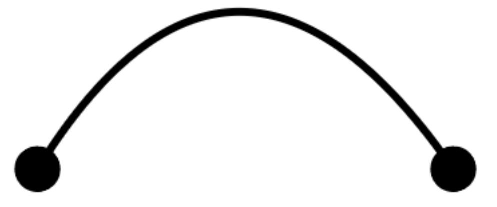
Split the field into background configuration $\hat{\phi}$ and quantum fluctuation η where and expand the Lagrangian in η (loops contain only quantum fields).

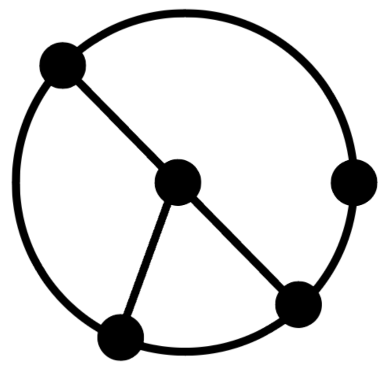
$$\left.\frac{\delta\mathscr{L}[\phi]}{\delta\phi}\right|_{\phi=\hat{\phi}} = 0$$

To which order in η for one-/two- loop graphs? → topological identity

for connected graphs

$$\overset{\text{\# vertices}}{V} - \underset{\text{\# internal lines}}{I} + \overset{\text{\# loops}}{L} = \underset{\text{Euler character}}{1} \qquad \text{and} \qquad \overset{\text{\# external fields}}{F} = \sum_{i=1}^{V} \underset{\text{\# fields at each vertex}}{F_i} - 2I$$





$$\Rightarrow \boxed{(F-2) + 2L = \sum_{i=1}^{V} (F_i - 2)}$$

No external quantum field: $F = 0$.

For L=1: only quadratic vertices → $\mathcal{O}(\eta^2)$,

For L=2: 2 cubic vertices or 1 quartic vertex + any number of quadratic vertices → $\mathcal{O}(\eta^4)$.

One-loop RGE — scalar

Scalar theory at $\mathcal{O}(\eta^2)$, $\phi \to \hat{\phi} + \eta$

$$\delta^2 \mathcal{L} = \frac{1}{2}(\partial_\mu \eta)^T(\partial^\mu \eta) + (\partial_\mu \eta)^T N^\mu(\hat{\phi})\eta + \frac{1}{2}\eta^T X(\hat{\phi})\eta$$

where N^μ is antisymmetric without loss of generality and X is symmetric.

With the covariant derivative $D_\mu \eta \equiv \partial_\mu \eta + N_\mu \eta$ and redefining X we have

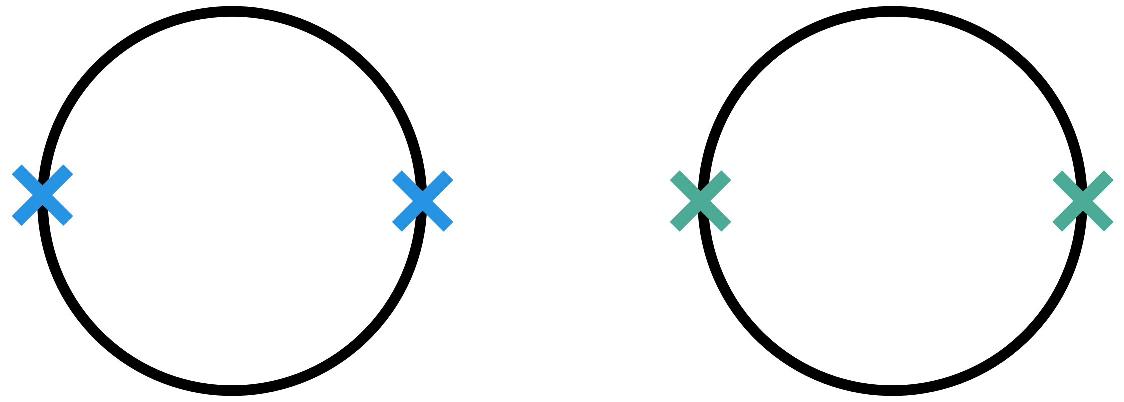
$$\delta^2 \mathcal{L} = \frac{1}{2}(D_\mu \eta)^T(D^\mu \eta) + \frac{1}{2}\eta^T X \eta$$

Using naive dimensional analysis, the 't Hooft formula for one-loop counterterms is ['t Hooft, Nucl.Phys.B 62 (1973)]

Mass dimension:
$[X] = 2$
$[Y_{\mu\nu}] = 2$

$$\boxed{\mathcal{L}^{(1)}_{\text{c.t.}} = \frac{1}{16\pi^2\epsilon}\text{Tr}\left[-\frac{1}{4}X^2 - \frac{1}{24}Y^2_{\mu\nu}\right]}$$

with $Y_{\mu\nu} = [D_\mu, D_\nu]$



Two-loop RGE — scalar

For two-loop:

$\mathcal{O}(\eta^3)$: $$\delta^3\mathcal{L} = A_{abc}\eta^a\eta^b\eta^c + A^{\mu}_{a|bc}(D_\mu\eta)^a\eta^b\eta^c + A^{\mu\nu}_{ab|c}(D_\mu\eta)^a(D_\nu\eta)^b\eta^c$$

$\mathcal{O}(\eta^4)$: $$\delta^4\mathcal{L} = B_{abcd}\eta^a\eta^b\eta^c\eta^d + B^{\mu}_{a|bcd}(D_\mu\eta)^a\eta^b\eta^c\eta^d + B^{\mu\nu}_{ab|cd}(D_\mu\eta)^a(D_\nu\eta)^b\eta^c\eta^d$$

where A and B are symmetric and the completely symmetric parts of A^μ and B^μ vanish.

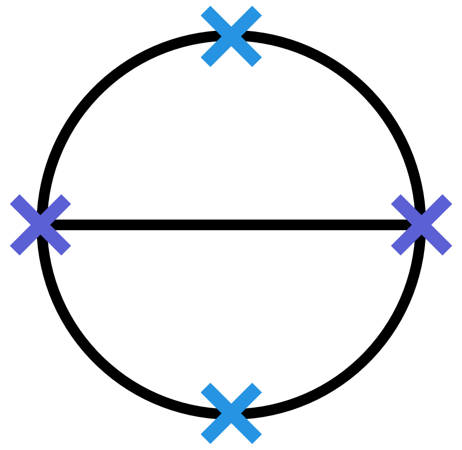
The graphs to compute to derive the two-loop algebraic formula are

Mass dimension:

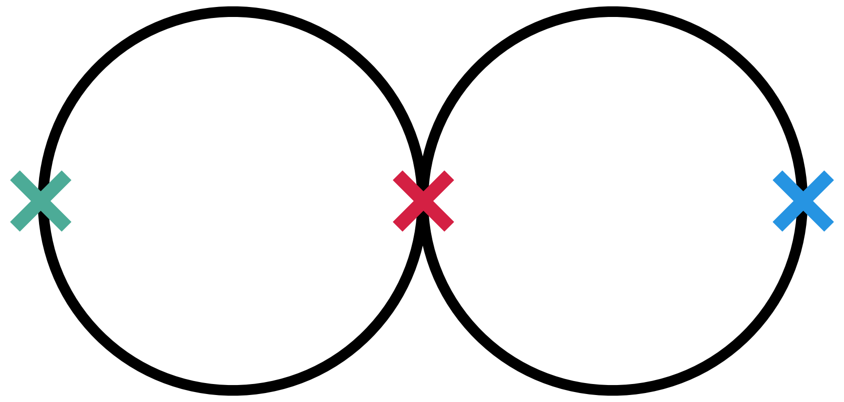
$[A] = 1$ $[B] = 0$

$[A^\mu] = 0$ $[B^\mu] = -1$

$[A^{\mu\nu}] = -1$ $[B^{\mu\nu}] = -2$



with 0, 1 or 2 insertions of X / $Y_{\mu\nu}$



with 2 or 3 insertions of X / $Y_{\mu\nu}$

Structures from NDA and symmetries

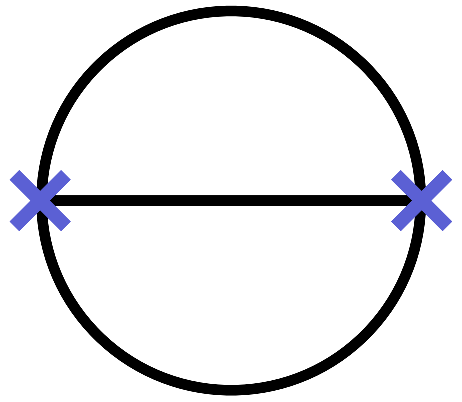
Mass dimension:

$[A] = 1$ $[B] = 0$

$[A^\mu] = 0$ $[B^\mu] = -1$

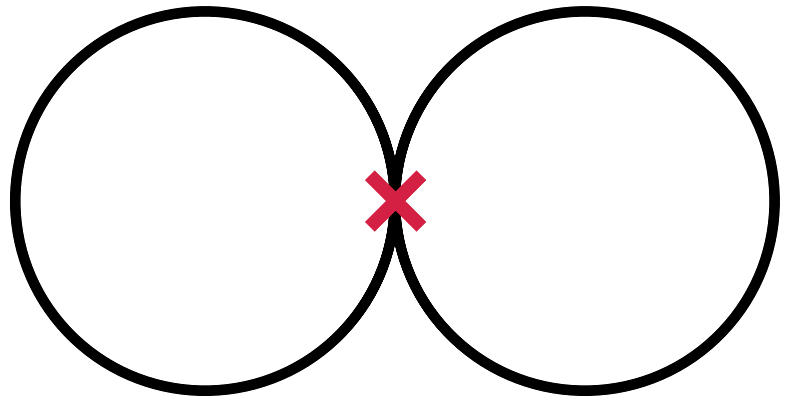
$[A^{\mu\nu}] = -1$ $[B^{\mu\nu}] = -2$

A-type counterterms



AA	$D^2,\ X,\ \cancel{Y}$
$A^\mu A$	$\cancel{D^3},\ XD,\ YD$
$A^\mu A^\mu$	$D^4,\ XD^2,\ YD^2,\ X^2,\ XY,\ Y^2$
$A^{\mu\nu}A$	$D^4,\ XD^2,\ YD^2,\ X^2,\ XY,\ Y^2$
$A^{\mu\nu}A^\mu$	$D^5,\ XD^3,\ YD^3,\ X^2D,\ XYD,\ Y^2D$
$A^{\mu\nu}A^{\mu\nu}$	$D^6,\ XD^4,\ YD^4,\ X^2D^2,\ XYD^2,\ Y^2D^2,\ X^3,\ X^2Y,\ XY^2,\ Y^3$

B-type counterterms

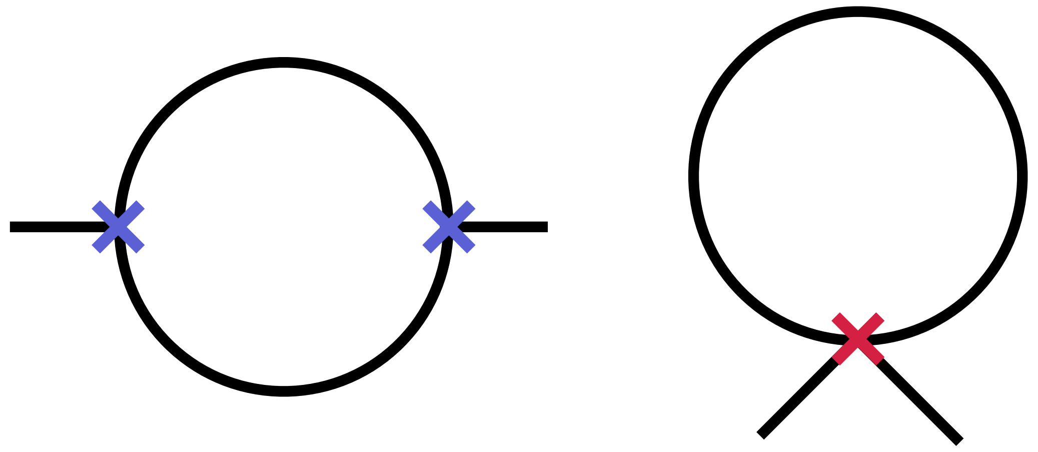


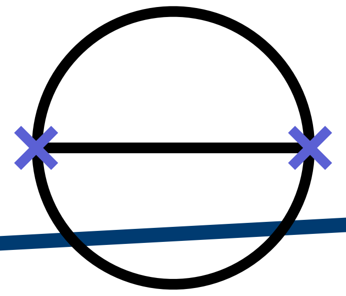
B	$\cancel{D^4},\ \cancel{XD^2},\ \cancel{YD^2},\ X^2, \cancel{XY},\ \cancel{Y^2}$
B^μ	$\cancel{D^5},\ \cancel{XD^3},\ \cancel{YD^3}\ X^2D,\ XYD,\ \cancel{Y^2D}$
$B^{\mu\nu}$	$\cancel{D^6},\ X^2D^2,\ XYD^2,\ Y^2D^2,\ X^3,\ X^2Y,\ XY^2,\ Y^3$

Some graph vanish by symmetry (Lorentz, flavor).

Compute all the remaining graphs + subtract one-loop subdivergences

Full computation steps in [Jenkins, Manohar, Naterop, JP, 2308.06315]



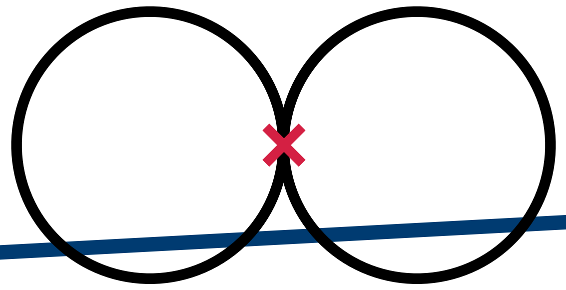


A-type counterterms

$$
\begin{aligned}
\mathcal{L}_{\text{c.t.}}^{(A,2)} = \frac{1}{(16\pi^2)^2}\Big[& a_{1,1} D_\mu A_{abc} D_\mu A_{abc} + a_{2,1} A_{abc} X_{cd} A_{abd} \\
& + a_{3,1} D_\mu A^\mu_{a|bc} A_{abd} X_{cd} + a_{3,2} A^\mu_{a|bc} D_\mu A_{abd} X_{cd} + a_{4,1} D_\nu A^\mu_{a|bc} A_{abd} Y^{\mu\nu}_{cd} + a_{4,2} A^\mu_{a|bc} D_\nu A_{abd} Y^{\mu\nu}_{cd} \\
& + a_{5,1} D^2 A^\mu_{a|bc} D^2 A^\mu_{a|bc} + a_{5,2} D_\alpha D_\mu A^\mu_{a|bc} D_\alpha D_\nu A^\nu_{a|bc} \\
& + a_{6,1} D^2 A^\mu_{a|bc} A^\mu_{a|bd} X_{cd} + a_{6,2} D^2 A^\mu_{c|ab} A^\mu_{d|ab} X_{cd} + a_{6,3} D_\alpha A^\mu_{a|bc} D_\alpha A^\mu_{a|bd} X_{cd} + a_{6,4} D_\alpha A^\mu_{c|ab} D_\alpha A^\mu_{d|ab} X_{cd} \\
& + a_{6,5} D_\mu A^\mu_{a|bc} D_\nu A^\nu_{a|bd} X_{cd} + a_{6,6} D_\mu A^\mu_{c|ab} D_\nu A^\nu_{d|ab} X_{cd} + a_{6,7} D_\nu A^\mu_{a|bc} D_\mu A^\nu_{a|bd} X_{cd} \\
& + a_{6,8} D_\nu A^\mu_{c|ab} D_\mu A^\nu_{d|ab} X_{cd} + a_{6,9} D_\nu D_\mu A^\mu_{a|bc} A^\nu_{a|bd} X_{cd} + a_{6,10} D_\nu D_\mu A^\mu_{c|ab} A^\nu_{d|ab} X_{cd} \\
& + a_{7,1} D_\alpha A^\mu_{a|bc} D_\alpha A^\nu_{a|bd} Y^{\mu\nu}_{cd} + a_{7,2} D_\alpha A^\mu_{c|ab} D_\alpha A^\nu_{d|ab} Y^{\mu\nu}_{cd} + a_{7,3} D_\mu A^\alpha_{a|bc} D_\nu A^\alpha_{a|bd} Y^{\mu\nu}_{cd} \\
& + a_{7,4} D_\mu A^\alpha_{c|ab} D_\nu A^\alpha_{d|ab} Y^{\mu\nu}_{cd} + a_{7,5} D_\mu A^\alpha_{a|bc} D_\nu A^\nu_{a|bd} Y^{\mu\alpha}_{cd} + a_{7,6} D_\mu A^\alpha_{c|ab} D_\nu A^\nu_{d|ab} Y^{\mu\alpha}_{cd} \\
& + a_{7,7} D_\nu A^\alpha_{a|bc} D_\mu A^\nu_{a|bd} Y^{\mu\alpha}_{cd} + a_{7,8} D_\nu A^\alpha_{c|ab} D_\mu A^\nu_{d|ab} Y^{\mu\alpha}_{cd} + a_{7,9} A^\alpha_{a|bc} D_\mu D_\nu A^\nu_{a|bd} Y^{\mu\alpha}_{cd} \\
& + a_{7,10} A^\alpha_{c|ab} D_\mu D_\nu A^\nu_{d|ab} Y^{\mu\alpha}_{cd} + a_{7,11} D_\mu D_\nu A^\alpha_{a|bc} A^\nu_{a|bd} Y^{\mu\alpha}_{cd} + a_{7,12} D_\mu D_\nu A^\alpha_{c|ab} A^\nu_{d|ab} Y^{\mu\alpha}_{cd} \\
& + a_{8,1} A^\mu_{c|ab} A^\mu_{d|ab} X_{ce} X_{ed} + a_{8,2} A^\mu_{a|bc} A^\mu_{a|bd} X_{ce} X_{ed} + a_{8,3} A^\mu_{a|bc} A^\mu_{e|bd} X_{ae} X_{cd} + a_{8,4} A^\mu_{a|bc} A^\mu_{a|de} X_{bd} X_{ce} \\
& + a_{9,1} A^\mu_{c|ab} A^\nu_{d|ab} (X_{ce} Y^{\mu\nu}_{ed} + Y^{\mu\nu}_{ce} X_{ed}) + a_{9,2} A^\mu_{a|bc} A^\nu_{a|bd} (X_{ce} Y^{\mu\nu}_{ed} + Y^{\mu\nu}_{ce} X_{ed}) \\
& + a_{9,3} A^\mu_{a|bc} A^\nu_{e|bd} X_{ae} Y^{\mu\nu}_{cd} + a_{9,4} A^\mu_{a|bc} A^\nu_{a|de} X_{ce} Y^{\mu\nu}_{bd} + a_{9,5} A^\mu_{a|bc} A^\nu_{e|bd} X_{cd} Y^{\mu\nu}_{ae} \\
& + a_{10,1} A^\mu_{c|ab} A^\mu_{d|ab} Y^{\alpha\beta}_{ce} Y^{\alpha\beta}_{ed} + a_{10,2} A^\mu_{a|bc} A^\mu_{a|bd} Y^{\alpha\beta}_{ce} Y^{\alpha\beta}_{ed} + a_{10,3} A^\mu_{c|ab} A^\nu_{d|ab} Y^{\mu\alpha}_{ce} Y^{\nu\alpha}_{ed} \\
& + a_{10,4} A^\mu_{a|bc} A^\nu_{a|bd} Y^{\mu\alpha}_{ce} Y^{\nu\alpha}_{ed} + a_{10,5} A^\mu_{c|ab} A^\nu_{d|ab} Y^{\nu\alpha}_{ce} Y^{\mu\alpha}_{ed} + a_{10,6} A^\mu_{a|bc} A^\nu_{a|bd} Y^{\nu\alpha}_{ce} Y^{\mu\alpha}_{ed} \\
& + a_{10,7} A^\mu_{a|bc} A^\mu_{e|bd} Y^{\alpha\beta}_{ae} Y^{\alpha\beta}_{cd} + a_{10,8} A^\mu_{a|bc} A^\mu_{a|de} Y^{\alpha\beta}_{bd} Y^{\alpha\beta}_{ce} + a_{10,9} A^\mu_{a|bc} A^\nu_{e|bd} (Y^{\mu\alpha}_{ae} Y^{\nu\alpha}_{cd} + Y^{\nu\alpha}_{ae} Y^{\mu\alpha}_{cd}) \\
& + a_{10,10} A^\mu_{a|bc} A^\nu_{a|de} (Y^{\mu\alpha}_{bd} Y^{\nu\alpha}_{ce} + Y^{\nu\alpha}_{bd} Y^{\mu\alpha}_{ce}) + a_{10,11} A^\mu_{a|bc} A^\nu_{b|ed} (Y^{\mu\alpha}_{ae} Y^{\nu\alpha}_{cd} - Y^{\nu\alpha}_{ae} Y^{\mu\alpha}_{cd}) \Big].
\end{aligned}
$$

$a_{1,1} = -\frac{3}{4\epsilon}$,	$a_{2,1} = \frac{9}{2\epsilon^2} - \frac{9}{2\epsilon}$,		
$a_{3,1} = \frac{3}{2\epsilon^2} - \frac{15}{4\epsilon}$,	$a_{3,2} = \frac{9}{2\epsilon^2} - \frac{9}{4\epsilon}$,	$a_{4,1} = -\frac{3}{2\epsilon^2} + \frac{7}{4\epsilon}$,	$a_{4,2} = -\frac{3}{2\epsilon^2} - \frac{5}{4\epsilon}$,
$a_{5,1} = \frac{1}{64\epsilon}$,	$a_{5,2} = -\frac{1}{48\epsilon}$,		
$a_{6,1} = \frac{1}{36\epsilon^2} + \frac{25}{216\epsilon}$,	$a_{6,2} = \frac{13}{72\epsilon^2} - \frac{107}{432\epsilon}$,	$a_{6,3} = -\frac{5}{36\epsilon^2} + \frac{37}{216\epsilon}$,	$a_{6,4} = \frac{2}{9\epsilon^2} - \frac{2}{27\epsilon}$,
$a_{6,5} = \frac{1}{36\epsilon^2} - \frac{5}{216\epsilon}$,	$a_{6,6} = -\frac{5}{72\epsilon^2} - \frac{65}{432\epsilon}$,	$a_{6,7} = \frac{1}{36\epsilon^2} - \frac{5}{216\epsilon}$,	$a_{6,8} = \frac{13}{72\epsilon^2} - \frac{11}{432\epsilon}$,
$a_{6,9} = -\frac{1}{9\epsilon^2} + \frac{5}{54\epsilon}$,	$a_{6,10} = \frac{1}{36\epsilon^2} - \frac{59}{216\epsilon}$,		
$a_{7,1} = -\frac{1}{48\epsilon}$,	$a_{7,2} = -\frac{13}{96\epsilon}$,	$a_{7,3} = \frac{1}{18\epsilon^2} + \frac{1}{432\epsilon}$,	$a_{7,4} = -\frac{1}{72\epsilon^2} - \frac{41}{864\epsilon}$,
$a_{7,5} = -\frac{1}{36\epsilon^2} + \frac{13}{432\epsilon}$,	$a_{7,6} = \frac{5}{72\epsilon^2} - \frac{191}{864\epsilon}$,	$a_{7,7} = \frac{1}{36\epsilon^2} - \frac{13}{432\epsilon}$,	$a_{7,8} = \frac{13}{72\epsilon^2} - \frac{61}{864\epsilon}$,
$a_{7,9} = -\frac{1}{36\epsilon^2} - \frac{17}{432\epsilon}$,	$a_{7,10} = \frac{5}{72\epsilon^2} - \frac{149}{864\epsilon}$,	$a_{7,11} = \frac{1}{36\epsilon^2} - \frac{19}{432\epsilon}$,	$a_{7,12} = \frac{13}{72\epsilon^2} - \frac{139}{864\epsilon}$,
$a_{8,1} = -\frac{5}{16\epsilon^2} + \frac{19}{96\epsilon}$,	$a_{8,2} = \frac{1}{8\epsilon^2} - \frac{11}{48\epsilon}$,	$a_{8,3} = -\frac{1}{4\epsilon^2} + \frac{5}{8\epsilon}$,	$a_{8,4} = -\frac{1}{2\epsilon^2} + \frac{1}{8\epsilon}$,
$a_{9,1} = \frac{13}{72\epsilon^2} - \frac{11}{432\epsilon}$,	$a_{9,2} = \frac{1}{36\epsilon^2} - \frac{5}{216\epsilon}$,	$a_{9,3} = -\frac{19}{36\epsilon^2} + \frac{5}{216\epsilon}$,	$a_{9,4} = \frac{11}{36\epsilon^2} + \frac{17}{216\epsilon}$,
$a_{9,5} = \frac{11}{36\epsilon^2} - \frac{145}{216\epsilon}$,			
$a_{10,1} = \frac{35}{1152\epsilon} - \frac{5}{96\epsilon^2}$,	$a_{10,2} = \frac{1}{48\epsilon^2} - \frac{25}{576\epsilon}$,	$a_{10,3} = \frac{13}{144\epsilon^2} + \frac{251}{1728\epsilon}$	$a_{10,4} = \frac{1}{72\epsilon^2} + \frac{11}{864\epsilon}$,
$a_{10,5} = \frac{13}{144\epsilon^2} - \frac{217}{1728\epsilon}$,	$a_{10,6} = \frac{1}{72\epsilon^2} - \frac{25}{864\epsilon}$,	$a_{10,7} = \frac{1}{72\epsilon^2} - \frac{67}{864\epsilon}$,	$a_{10,8} = \frac{1}{36\epsilon^2} - \frac{25}{1728\epsilon}$,
$a_{10,9} = -\frac{29}{144\epsilon}$,	$a_{10,10} = \frac{19}{288\epsilon}$,	$a_{10,11} = -\frac{1}{8\epsilon}$	

50 graphs



B-type counterterms

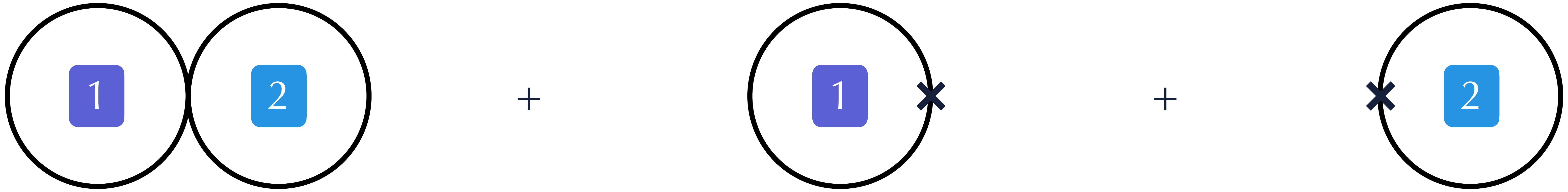
$$\begin{aligned}\mathcal{L}^{(B,2)}_{\text{c.t.}} = \frac{1}{(16\pi^2)^2\epsilon^2}\Bigg[&3B_{abcd}X_{ab}X_{cd} + \frac{3}{2}B^{\alpha}_{a|bcd}(D_\alpha X)_{ab}X_{cd} + \frac{1}{2}B^{\alpha}_{a|bcd}(D_\mu Y_{\mu\alpha})_{ab}X_{cd}\\ &+ \frac{1}{12}B^{\alpha\alpha}_{ab|cd}(D^2X)_{ab}X_{cd} + \frac{1}{12}B^{\mu\nu}_{ab|cd}(\{D_\mu, D_\nu\}X)_{ab}X_{cd} + \frac{1}{12}B^{\mu\nu}_{ab|cd}(D^2Y^{\mu\nu})_{ab}X_{cd}\\ &- \frac{1}{4}B^{\alpha\alpha}_{ab|cd}X_{ae}X_{eb}X_{cd} + \frac{1}{4}B^{\mu\nu}_{ab|cd}(X_{ae}Y^{\mu\nu}_{eb} + Y^{\mu\nu}_{ae}X_{eb})X_{cd}\\ &- \frac{1}{12}B^{\mu\nu}_{ab|cd}Y^{\mu\alpha}_{ae}Y^{\nu\alpha}_{eb}X_{cd} + \frac{1}{4}B^{\mu\nu}_{ab|cd}Y^{\nu\alpha}_{ae}Y^{\mu\alpha}_{eb}X_{cd} - \frac{1}{24}B^{\alpha\alpha}_{ab|cd}Y^{\mu\nu}_{ae}Y^{\mu\nu}_{eb}X_{cd}\\ &+ \frac{1}{2}B^{\mu\nu}_{ab|cd}(D_\mu X)_{ac}(D_\nu X)_{bd} + \frac{1}{18}B^{\mu\nu}_{ab|cd}(D_\alpha Y^{\alpha\mu})_{ac}(D_\beta Y^{\beta\nu})_{bd} + \frac{1}{6}B^{\mu\nu}_{ab|cd}(D_\mu X)_{ac}(D_\beta Y^{\beta\nu})_{bd}\Bigg]\end{aligned}$$

15 graphs

Notice: there is not $\frac{1}{\epsilon}$ B-type counterterm → factorizable topology

Factorizable topology

In MS schemes:



$$I_{\text{tot}} = \left[\frac{\overset{\text{divergence}}{I_{1\infty}}}{\epsilon} + \underset{\text{finite part}}{I_{1f}}\right]\left[\frac{I_{2\infty}}{\epsilon} + I_{2f}\right] + \left[\frac{I_{1\infty}}{\epsilon} + I_{1f}\right]\left[-\frac{I_{2\infty}}{\epsilon}\right] + \left[-\frac{I_{1\infty}}{\epsilon}\right]\left[\frac{I_{2\infty}}{\epsilon} + I_{2f}\right]$$

$$= -\frac{I_{1\infty}I_{2\infty}}{\epsilon^2} + I_{1f}I_{2f}$$

Generalizable to higher-loop graphs, lowest pole $= \dfrac{1}{\epsilon^{n_{\text{nf}}}}$ where n_{nf} is the number of non-factorizable parts.

$\Rightarrow$ Only fully non-factorizable graphs contribute to the RGE.*

* There is a subtlety with evanescent operators. Still true, but requires additional finite subtraction beyond MS.

~~RGE from Geometry~~

for EFTs

RGE from Geometry

What do we have?

- Algebraic RGE formulae for renormalizable theories ↔ flat field space.
- Geometric Lagrangians for bosonic EFTs with non-trivial metric on field space.

Next steps:

(1) Expand geometric Lagrangians to desired order in quantum fluctuation → use geodesic coordinates.

(2) Generalize our flat field space formulae to curved field space → use local orthonormal frame.

(3) Identify our covariant building blocks in the geometric Lagrangian expansions (match).
    a) at one loop: $Y_{\mu\nu}$, X, + b) at two loop: A, A^{μ}, $A^{\mu\nu}$, B, B^{μ}, $B^{\mu\nu}$

(4) Apply the generalized formulae to obtain covariant RGE results.

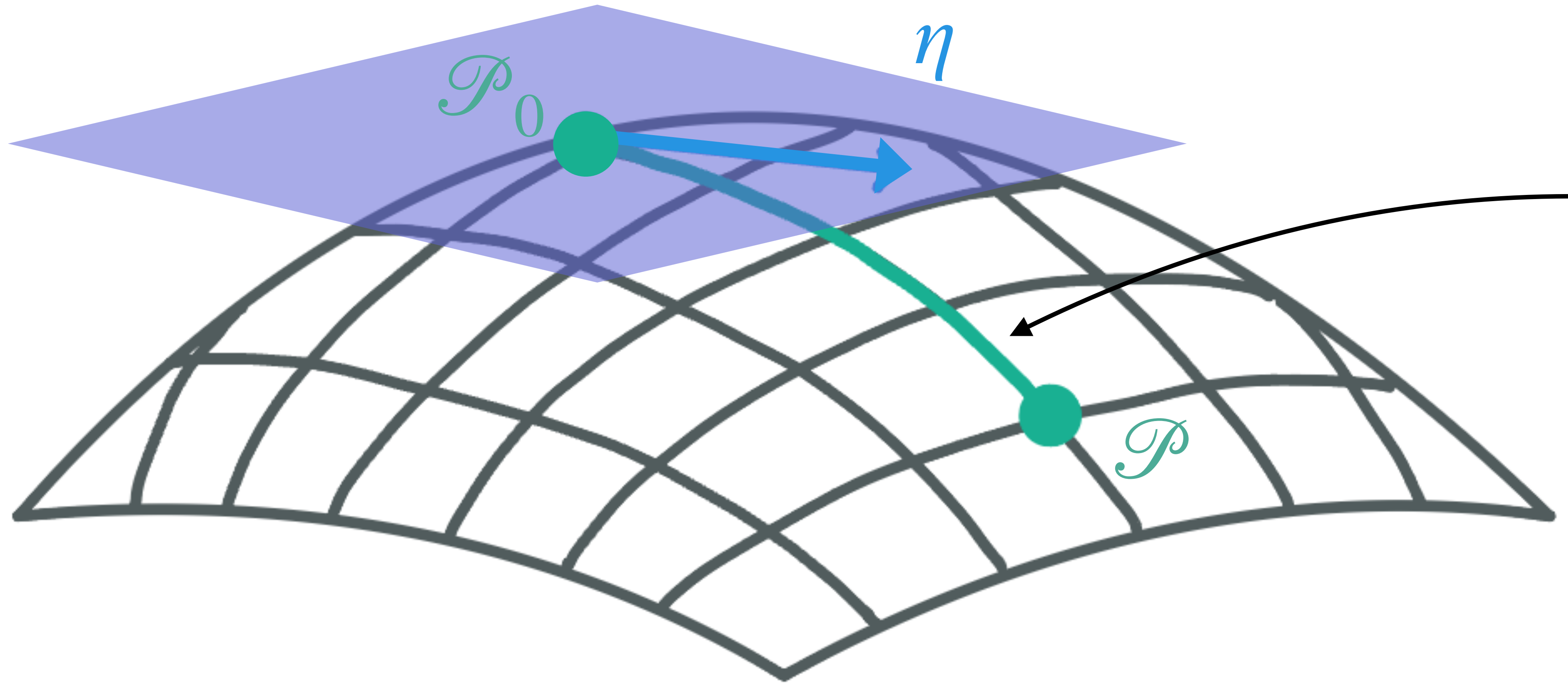
Geodesic coordinates

(1) Expand geometric Lagrangians to desired order in quantum fluctuation → use geodesic coordinates.
Using cartesian coordinates, we find that Lagrangian expansions are not covariant.
↪ Reason: ϕ is a coordinate $\phi^i \to \phi'^i$ and not a tensor… but tangent vectors are: $\eta^i \equiv \dfrac{\mathrm{d}\phi^i}{\mathrm{d}\lambda} \to \left(\dfrac{\partial \phi'^i}{\partial \phi^j}\right)\eta^j$.

Solution: use Riemann normal / geodesic coordinates (local coordinates obtained by applying the exponential map to the tangent space at $\mathscr{P}_0$) for the quantum fluctuation.

geodesic equation:

$$\frac{\mathrm{d}^2\phi^I}{\mathrm{d}\lambda^2} + \Gamma^I_{JK}(\phi)\frac{\mathrm{d}\phi^J}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi^K}{\mathrm{d}\lambda} = 0$$



geodesic starting at $\mathscr{P}_0$
with tangent vector $\eta(\lambda)$
ending at $\mathscr{P}$ in unit time

$$g_{IJ}(\mathscr{P}_0) = \delta_{IJ} \qquad \Gamma^I_{JK}(\mathscr{P}_0) = 0 \qquad g_{IJ}(\phi) = \delta_{IJ} - \frac{1}{3}R_{IKJL}(\mathscr{P}_0)\phi^K\phi^L + \mathcal{O}(\phi^3)$$

⇒ expand Lagrangian in

$$\phi^I \to \phi^I + \eta^I - \frac{1}{2}\Gamma^I_{JK}\eta^J\eta^K - \frac{1}{3!}\Gamma^I_{JKL}\eta^I\eta^J\eta^K - \frac{1}{4!}\Gamma^I_{JKLM}\eta^I\eta^J\eta^K\eta^M + \mathcal{O}(\eta^5)$$

Geodesic coordinates

(1) Expand geometric Lagrangians to desired order in quantum fluctuation → use geodesic coordinates.
The second variation of the scalar geometric Lagrangian

$$\mathscr{L} = \frac{1}{2}\, g_{IJ}(\phi)\, (\partial_\mu \phi)^I (\partial^\mu \phi)^J - V(\phi)$$

- With the shift $\phi^I \to \phi^I + \eta^I$

$$\delta^2 \mathscr{L} = \frac{1}{2}\left[g_{IJ}(\mathscr{D}_\mu \eta)^I (\mathscr{D}_\mu \eta)^J - R_{IJKL}(D_\mu \phi)^J (D_\mu \phi)^L \eta^I \eta^K - E_J \Gamma^J_{KL} \eta^K \eta^L - \nabla_J \nabla_I V \eta^I \eta^J \right]$$

non-covariant

with equation of motion $\delta \mathscr{L} = -\underbrace{\left(g_{IJ}(\mathscr{D}_\mu (D^\mu \phi))^I + \nabla_J V \right)}_{E_J} \eta^J$

- With the shift $\phi^I \to \phi^I + \eta^I - \frac{1}{2}\Gamma^I_{JK}\eta^J \eta^K + \mathscr{O}(\eta^3)$

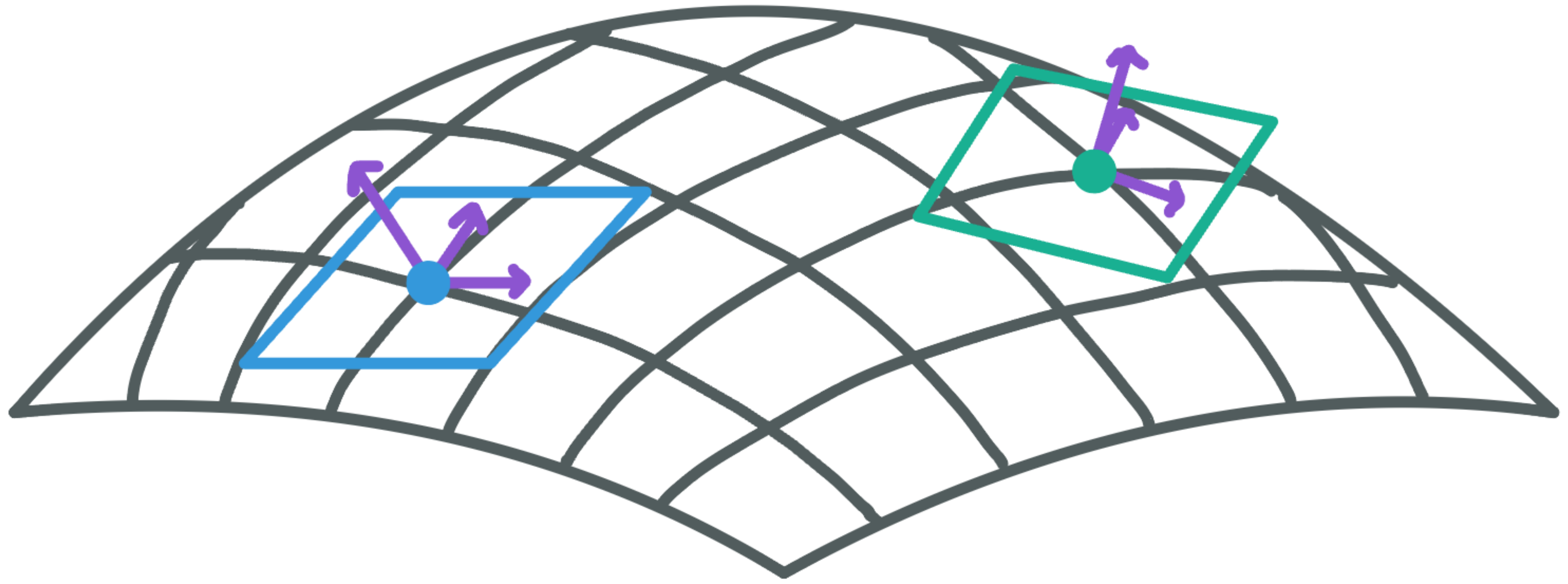
$$\delta^2 \mathscr{L} = \frac{1}{2}\left[g_{IJ}(\mathscr{D}_\mu \eta)^I (\mathscr{D}_\mu \eta)^J - R_{IJKL}(D_\mu \phi)^J (D_\mu \phi)^L \eta^I \eta^K - \nabla_J \nabla_I V \eta^I \eta^J \right]$$

Local orthonormal frame

(2) Generalize our flat field space formulae to curved field space → use local orthonormal frame.

Algebraic counterterm formulae were derived for renormalizable theories ⇔ for a flat field-space manifold.
They do not directly apply on the curved field-space manifold.

Solution: go to local orthonormal frames using vielbeins and apply formulae there.



$$g_{IJ}(\phi) = e^a{}_I(\phi) e^b{}_J(\phi) \delta_{ab} \qquad (\mathscr{D}_\mu \eta)^I = e^I{}_a (D_\mu \eta)^a \qquad R_{IJKL} = e^a{}_I e^b{}_J e^c{}_K e^d{}_L R_{abcd}$$

⇒ Since every indices are contracted, formulae are unchanged apart from uppercase ↔ lowercase indices.

Local orthonormal frame

(2) Generalize our flat field space formulae to curved field space → use local orthonormal frame.

For renormalizable theory, indices raised with δ^{ab}

$$\delta^2\mathscr{L} = \frac{1}{2}(D_\mu\eta)^T(D^\mu\eta) + \frac{1}{2}\eta^T X\eta$$

$$\mathscr{L}^{(1)}_{\text{c.t.}} = \frac{1}{16\pi^2\epsilon}\left[-\frac{1}{4}X_{ab}X^{ab} - \frac{1}{24}Y^{\mu\nu}_{ab}Y^{ab}_{\mu\nu}\right]$$

with $Y_{\mu\nu} = [D_\mu, D_\nu]$

For the geometric Lagrangian, indices raised with g^{IJ}

$$\delta^2\mathscr{L} = \frac{1}{2}\left[g_{IJ}(\mathscr{D}_\mu\eta)^I(\mathscr{D}_\mu\eta)^J - R_{IJKL}(D_\mu\phi)^J(D_\mu\phi)^L\,\eta^I\eta^K - \nabla_J\nabla_I V\,\eta^I\eta^J\right]$$

$$\mathscr{L}^{(1)}_{\text{c.t.}} = \frac{1}{16\pi^2\epsilon}\left[-\frac{1}{4}X_{IJ}X^{IJ} - \frac{1}{24}Y^{\mu\nu}_{IJ}Y^{IJ}_{\mu\nu}\right]$$

$$g^{IJ} = e^I{}_a e^J{}_b \delta^{ab}$$

$$(\mathscr{D}_\mu\eta)^I = e^I{}_a (D_\mu\eta)^a$$

$$R_{IJKL} = e^a{}_I e^b{}_J e^c{}_K e^d{}_L R_{abcd}$$

with

$\mathscr{O}(\eta^2)$

$$X_{IJ} = -R_{IKJL}(D_\mu\phi)^K(D^\mu\phi)^L - \nabla_J\nabla_I V$$

$$Y^{\mu\nu}_{IJ} = [\mathscr{D}^\mu, \mathscr{D}^\nu]_{IJ} = R_{IJKL}(D^\mu\phi)^K(D^\nu\phi)^L + F^{\mu\nu}_A\nabla_J t^A_I$$

RGE at one loop — fermion

(3) Identify our covariant building blocks in the geometric Lagrangian expansions (match).

a) at one loop: $Y_{\mu\nu}$, X

Linear expansion:

$$\delta^2\mathscr{L} = \frac{1}{2}(D_\mu\eta)^T(D^\mu\eta) + \frac{1}{2}\eta^T X\eta$$

Geodesic expansion:

$$\delta^2\mathscr{L} = \frac{1}{2}\left[g_{IJ}(\mathscr{D}_\mu\eta)^I(\mathscr{D}_\mu\eta)^J - R_{IJKL}(D_\mu\phi)^J(D_\mu\phi)^L\eta^I\eta^K - \nabla_J\nabla_I V\eta^I\eta^J\right]$$

Match to obtain

$$X_{IJ} = -R_{IKJL}(D_\mu\phi)^K(D^\mu\phi)^L - \nabla_J\nabla_I V$$

$$Y^{\mu\nu}_{IJ} = [\mathscr{D}^\mu, \mathscr{D}^\nu]_{IJ} = R_{IJKL}(D^\mu\phi)^K(D^\nu\phi)^L + F^{\mu\nu}_A\nabla_J t^A_I$$

RGE at two loop — scalar

(3) Identify our covariant building blocks in the geometric Lagrangian expansions (match).

b) at two loop: A, A^{μ}, $A^{\mu\nu}$, B, B^{μ}, $B^{\mu\nu}$

$\mathcal{O}(\eta^3)$

$$A_{abc} = -\frac{1}{6}\nabla_{(a}\nabla_{b}\nabla_{c)}V - \frac{1}{18}(\nabla_a R_{bdce} + \nabla_b R_{cdae} + \nabla_c R_{adbe})(D_\mu\phi)^d(D^\mu\phi)^e$$

$$A^{\mu}_{a|bc} = \frac{1}{3}(R_{abcd} + R_{acbd})(D^\mu\phi)^d$$

$$A^{\mu\nu}_{ab|c} = 0$$

$\mathcal{O}(\eta^4)$

$$B_{abcd} = -\frac{1}{24}\nabla_a\nabla_b\nabla_c\nabla_d V - \frac{1}{24}\nabla_a\nabla_d R_{becf}(D_\mu\phi)^e(D^\mu\phi)^f + \frac{1}{6}R_{eabf}R_{ecdg}(D_\mu\phi)^f(D^\mu\phi)^g \qquad \text{sym(bcd)}$$

$$B^{\mu}_{a|bcd} = \frac{1}{4}(\nabla_d R_{abce})(D^\mu\phi)^e \qquad \text{sym(bcd)}$$

$$B^{\mu\nu}_{ab|cd} = -\frac{1}{12}\eta^{\mu\nu}(R_{acbd} + R_{adbc})$$

(4) Apply the generalized formulae to obtain covariant RGE results.

Application

Example: O(N) EFT

Starting from the O(N) EFT in the basis

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi \cdot \partial^\mu \phi) - \frac{m^2}{2}(\phi \cdot \phi) - \frac{\lambda}{4}(\phi \cdot \phi)^2 + C_1(\phi \cdot \phi)^3 + C_E(\phi \cdot \phi)(\partial_\mu \phi \cdot \partial^\mu \phi)$$

where $C_1,\, C_E \ \sim \mathcal{O}\left(\Lambda^{-2}\right)$,

identify the geometric objects

$$g_{ij} = \delta_{ij}\left(1 + 2C_E(\phi \cdot \phi)\right)$$

$$\Gamma^i_{jk} = 2C_E\left(\delta^i_k \phi_j + \delta^i_j \phi_k - \delta_{jk}\phi^i\right) \qquad\qquad R_{ijkl} = 4C_E\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right)$$

and the potential

$$V = \frac{m^2}{2}(\phi \cdot \phi) + \frac{\lambda}{4}(\phi \cdot \phi)^2 - C_1(\phi \cdot \phi)^3$$

which define the building blocks	$Y_{\mu\nu}$,	X	and	A,	A^μ,	B,	B^μ,	$B^{\mu\nu}$
lowest order:	Λ^{-2}	Λ^2		1	Λ^{-2}	1	Λ^{-4}	Λ^{-2}

Example: O(N) EFT

To derive the counterterms

$$\begin{aligned}\mathscr{L} = \; & \frac{1}{2}Z_\phi(\partial_\mu\phi\cdot\partial^\mu\phi) - \frac{1}{2}\left(m^2 + m^2_{\text{c.t.}}\right)(\phi\cdot\phi) - \frac{1}{4}\mu^{2\epsilon}Z^2_\phi\left(\lambda+\lambda_{\text{c.t.}}\right)(\phi\cdot\phi)^2 \\ & +\mu^{4\epsilon}Z^3_\phi\left(C_1 + C_{1\text{c.t.}}\right)(\phi\cdot\phi)^3 + \mu^{2\epsilon}Z^2_\phi\left(C_E + C_{E\text{c.t.}}\right)(\phi\cdot\phi)(\partial_\mu\phi\cdot\partial^\mu\phi)\end{aligned}$$

at $\mathcal{O}(\Lambda^{-2})$ we can simply apply

$$\begin{aligned}\mathscr{L}_{\text{c.t.}} = & \left\{-\frac{1}{4\epsilon}\text{Tr}[X^2]\right\}_1 \\ & +\left\{-\frac{3}{4\epsilon}\mathscr{D}_\mu A_{ijk}\mathscr{D}^\mu A^{ijk} + \left(\frac{9}{2\epsilon^2}-\frac{9}{2\epsilon}\right)A_{ijk}X^k{}_l A^{ijl} + \left(\frac{3}{2\epsilon^2}-\frac{15}{4\epsilon}\right)\mathscr{D}_\mu A^\mu_{i|jk}X^k{}_l A^{ijl} + \left(\frac{9}{2\epsilon^2}-\frac{9}{4\epsilon}\right)A^\mu_{i|jk}X^k{}_l\mathscr{D}_\mu A^{ijl}\right. \\ & \left. + \frac{3}{\epsilon^2}B_{ijkl}X^{ij}X^{kl} + \frac{1}{8\epsilon^2}B^{\mu\mu}_{ij|kl}(\mathscr{D}^2X)^{ij}X^{kl} - \frac{1}{4\epsilon^2}B^{\mu\mu}_{ij|kl}X^i{}_m X^{mj}X^{kl} + \frac{1}{2\epsilon^2}B^{mu\nu}_{ij|kl}(\mathscr{D}_\mu X)^{ik}(\mathscr{D}_\nu X)^{jl}\right\}_2\end{aligned}$$

Example: O(N) EFT

The anomalous dimension is defined by

$$\dot{C}_i = -\epsilon(F_i - 2)C_i + \gamma_i$$

The counterterm can be organized into order of the pole k and power of loops L

$$C_i^{\text{bare}}\mu^{-(F_i-2)\epsilon} = C_i + \sum_{k=1}^{\infty}\sum_{L}\frac{a_i^{(k,L)}(\{C_j\})}{\epsilon^k}$$

Combining the two give the definition

$$\boxed{\gamma_i = 2\sum_L L a_i^{(1,L)}}$$

Only $1/\epsilon$ pole define the RGE.

$O(N)$ RGE at two loop:

$$\dot{m}^2 = \left\{2(n+2)\lambda m^2 - 8nm^4C_E\right\}_1 + \left\{-10(n+2)\lambda^2m^2 + \frac{80}{3}(n+2)\lambda m^4C_E\right\}_2$$

$$\dot{\lambda} = \left\{2(n+8)\lambda^2 - 16(n+3)\lambda m^2C_E - 24(n+4)m^2C_1\right\}_1$$

$$+\left\{-12(3n+14)\lambda^3 + \frac{32}{3}(22n+113)\lambda^2m^2C_E + 480(n+4)\lambda m^2C_1\right\}_2$$

$$\dot{C}_E = \left\{4(n+2)\lambda C_E\right\}_1 + \left\{-34(n+2)\lambda^2C_E\right\}_2$$

$$\dot{C}_1 = \left\{20\lambda^2C_E + 6(n+14)\lambda C_1\right\}_1 + \left\{-\frac{8}{3}(23n+259)\lambda^3C_E - 42(7n+54)\lambda^2C_1\right\}_2$$

RGE obtained from geometry

Using this technique RGE were computed for:

- up to one-loop order
  - SMEFT bosonic sector to dim 8 [Helset, Jenkins, Manohar, 2212.03253]
  - SMEFT bosonic operators from a fermion loop to dim 8 [Assi, Helset, Manohar, JP, Shen, 2307.03187]

→ agree with [Chala, Guedes, Ramos, Santiago, 2106.05291]
[Das Bakshi, Chala, Díaz-Carmona, Guedes, 2205.03301]

- up to two-loop order [Jenkins, Manohar, Naterop, JP, 2310.19883]
  - $O(N)$ scalar EFT to dim 6 → agree with [Cao, Herzog, Melia, Nepveu, 2105.12742]
  - SMEFT scalar sector to dim 6 → new!
  - χPT to $\mathcal{O}(p^6)$ → agree with [Bijnens, Colangelo, Ecker, hep-ph/9907333]

↪ directly usable for dim 8

Towards a complete geometric picture

- More RGEs
  - full one-loop RGE for SMEFT at dim 8
    - mixed scalar-fermion loops
    - four-fermion operators
    - contributions to fermionic operators
    - mixed vector-fermion loops

    [Assi, Helset, JP, Shen, w.i.p]
  - two-loop counterterm formula including fermions and gauge bosons
- More derivatives
  - operators with more than one derivative on each field
    - Lagrange spaces? [Craig, Lee, Lu, Sutherland, 2305.09722]
    - jet bundle geometry? [Alminawi, Brivio, Davighi, 2308.00017] [Craig, Lee, 2307.15742]
  - derivative field redefinition
    - on-shell covariance of amplitudes? [Cohen, Craig, Lu, Sutherland, 2202.06965] [Cohen, Lu, Sutherland, 2312.06748]
    - geometry-kinematics duality? [Cheung, Helset, and Parra-Martinez, 2202.06972]

Conclusion

Conclusion

- EFTs have a pivotal position between New Physics models and data interpretation.
- Field-space geometry offer an alternative, more basis-independent, description of EFTs.
- Algebraic formulae can be used to compute the Renormalization Group Equations.
  ↪ done at one loop for any spin, at two loop for scalars.
- RGE calculations with geometry become a pure algebraic exercise.
  ↪ applicable to any EFT order