Finite formulation of QFT, naturalness, and the effective action

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24 September 2024

SRS
SwissMAP Research
Station in Les Diablerets

12th International Conference on the Exact Renormalization Group 2024 (ERG2024)

Outline

- Motivation: Naturalness
- Finite approaches to QFT
- Callan-Symanzik method as a finite description of QFT
    - The method
    - Equation for effective action
    - Non-renormalisable theories
- Conclusions

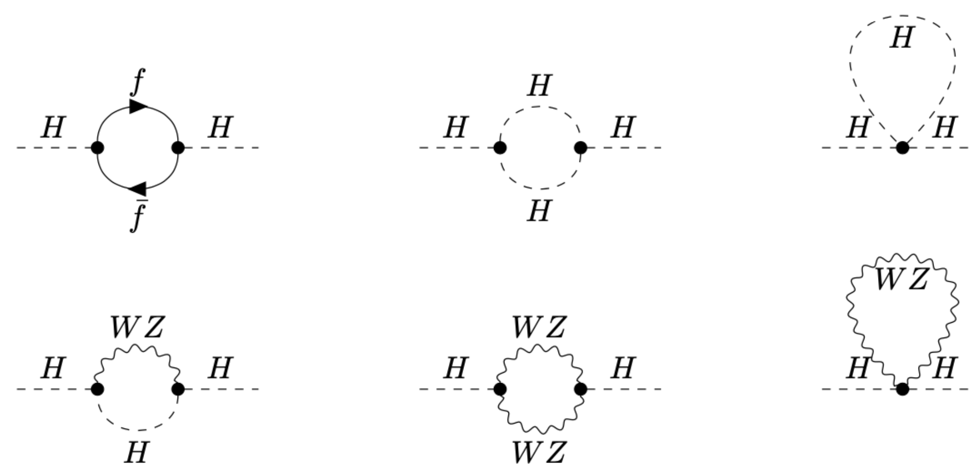
Based on: Sander Mooij and MS:
Nucl. Phys. B 990 (2023) 116172; B 990 (2023) 116176; B 1006 (2024) 11664;

Yulia Ageeva, Pavel Petrov and M. S: arXiv:2409.15036

Higgs mass fine-tuning

- The puzzle: take the Standard Model and consider radiative corrections to the Higgs mass. Quadratically divergent diagrams

$$\delta m_H^2 \propto f_t^2 \int \frac{d^4k}{k^2}$$



lead to the term $\delta m_H^2 \propto f_t^2 \Lambda^2$, f_t - top quark Yukawa coupling, Λ - the ultraviolet cutoff of the theory, i.e. the place where the Standard Model is substituted by the more fundamental theory of Nature. Since $m_H \ll \Lambda$, one has to fine-tune the tree Higgs mass M_{tree} to cancel the radiative correction(s). The amount of fine-tuning:

$$\epsilon_H = \frac{M_{\text{tree}}^2 - \delta m_H^2}{\Lambda^2} \sim \left(\frac{100 \text{ GeV}}{4\pi\Lambda}\right)^2 \ll 1$$

Cosmological constant fine-tuning

The similar logic can be applied to vacuum energy ϵ_{vac}:

$$\delta\epsilon_{\text{vac}} \propto \int d^4k$$

The radiative corrections are proportional to the fourth power of the cutoff scale, $\delta\epsilon_{\text{vac}} \propto \Lambda^4$ leading to even higher degree of fine-tuning

$$\epsilon_{\text{cc}} = \frac{\epsilon_{\text{vac}}^{\text{tree}} - \delta\epsilon_{\text{vac}}}{\Lambda^4} \sim \left(\frac{10^{-3}\ \text{eV}}{\Lambda}\right)^4 \ll\ll 1$$

Two problems

1. Why the physical values of the Higgs mass and of the cosmological constant are much smaller than the scale of new physics (cutoff Λ) ?

2. Why the tree values of these parameters are so fine-tuned to the radiative corrections?

Naturalness

- These fine tunings must be avoided at any price!

- The cutoff Λ must be of the order of the Fermi scale to screen the influence of high energy domain from low energy domain (SUSY, composite Higgs boson, large extra dimensions) ?

Natural theories

- SUSY: cancellation of quadratic divergences between bosons and fermions

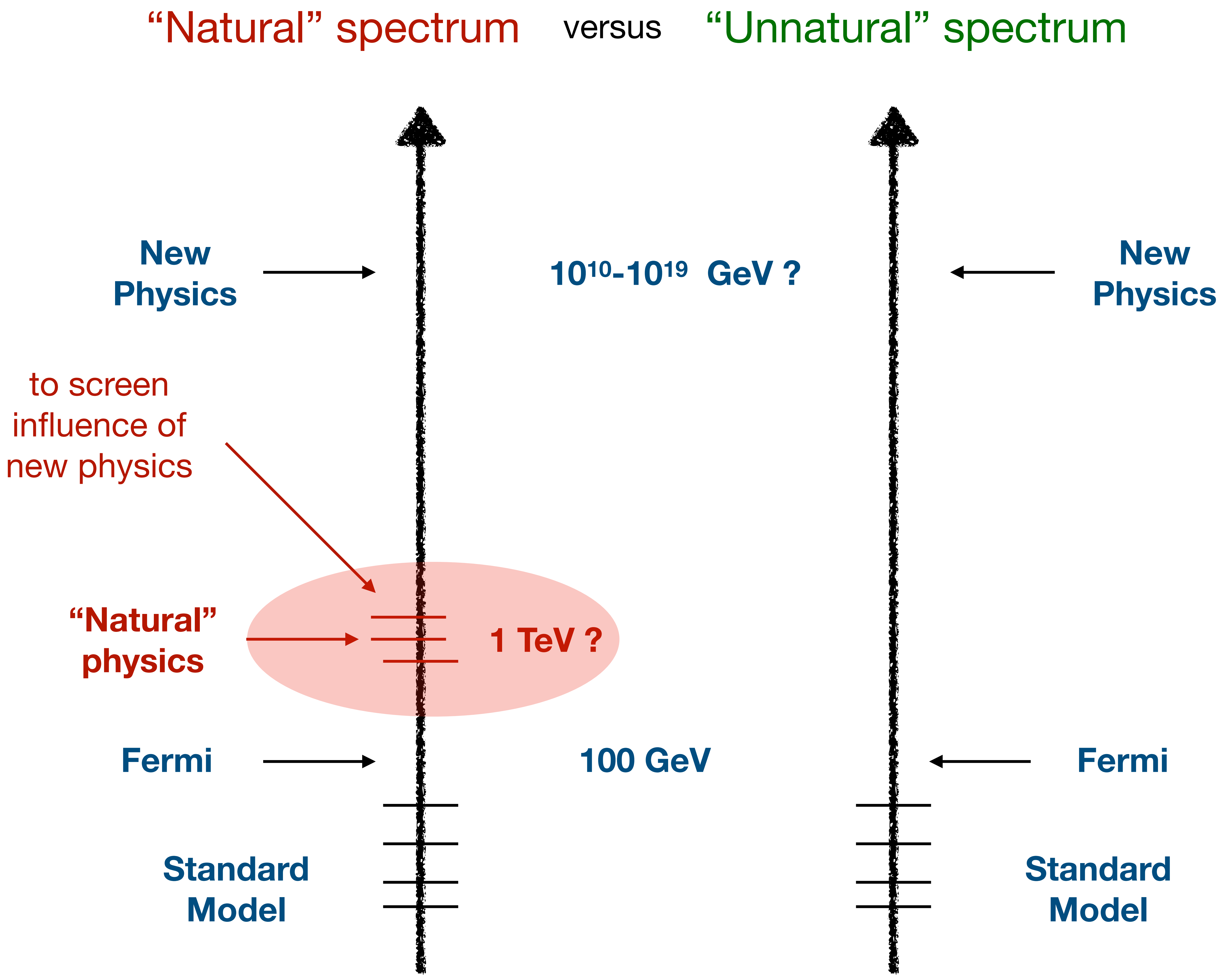
- Composite Higgs boson: no fundamental scalars

- Large extra dimensions: fundamental constant of gravity - Planck scale - is of the order of electroweak scale

- Cosmological evolution leading to $m_H \ll \Lambda$?

- Environmental selection leading to $m_H \ll \Lambda$?

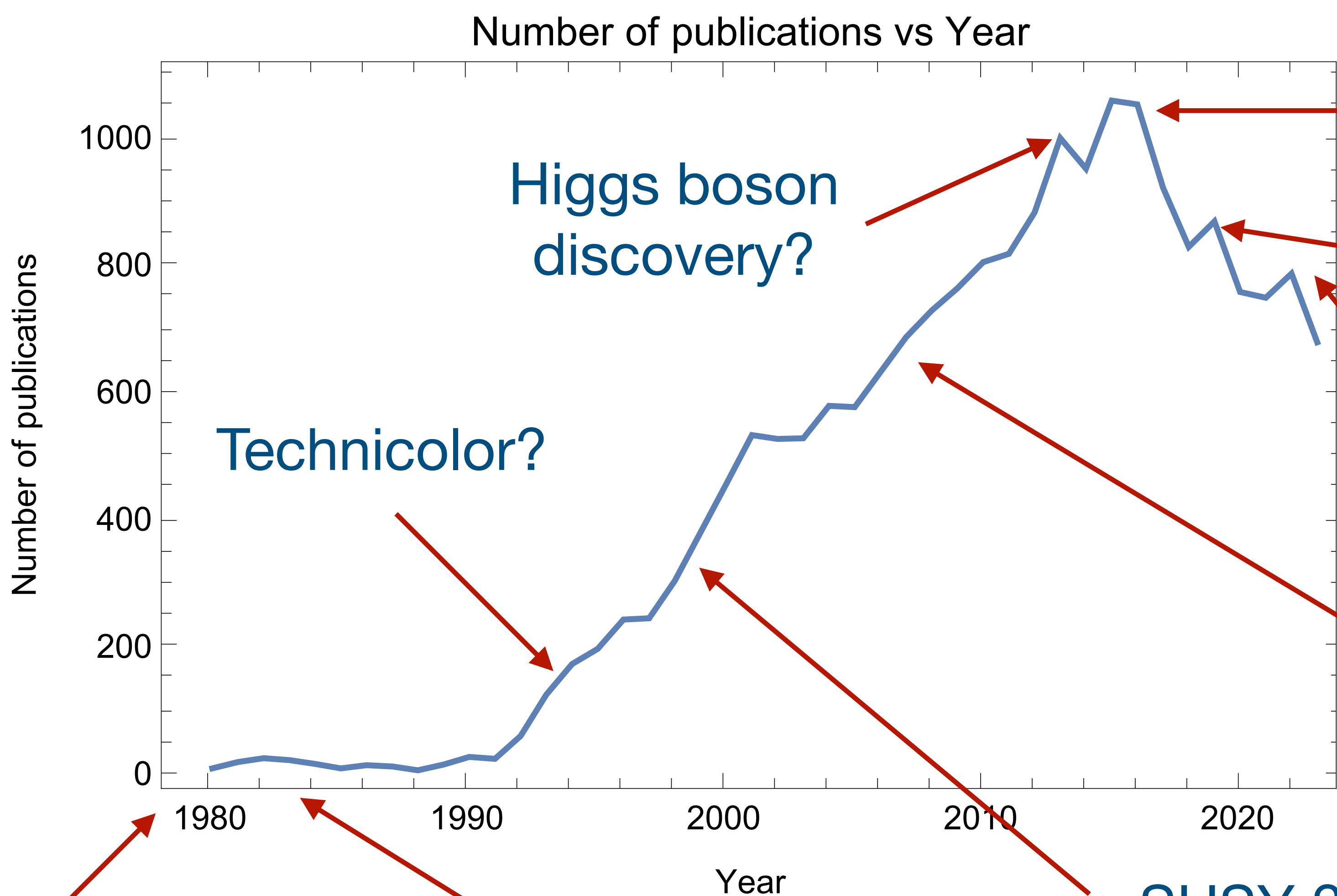
Generically, all these proposals lead to some kind of new physics right above the Fermi scale.


"Natural" spectrum versus "Unnatural" spectrum
New Physics
10^{10}-10^{19} GeV ?
New Physics
to screen influence of new physics
"Natural" physics
1 TeV ?
Fermi
100 GeV
Fermi
Standard Model
Standard Model

This problem attracted a lot of attention

The number of articles which mention
“hierarchy problem” or
“fine-tuning” or
“naturalness”
is very large:

Credit: Oleg Ruchayskiy
https://www.prophy.science



750 GeV resonance?

muon g-2?

flavour non-universality?

Large extra dimensions?

SUSY ?

Susskind ’84:
The Gauge Hierarchy Problem

Gildener ’76:
Gauge symmetry hierarchies

Origin of the fine-tunings

The core of the problem: quadratic (or quartic, if we talk about the cosmological constant) divergences, inevitably appearing in Feynman diagrams with loops in theories with fundamental scalar fields

Renormalisation:

- Regularise UV divergent expressions (cutoff, Pauli-Villars, dimreg,…)
- Subtract divergences (this is exactly where fine-tunings show up)
- Get finite values for physical observables

Renormalisable theory
Input:
several finite parameters of the theory

Multiplicative renormalisation:
infinities, regularisation,
counter-terms, lattice,
fine-tuned cancellations

Output:
Infinite number of physical observables:
finite values

Non-renormalisable theory
Input:
infinite number of finite parameters of the theory

The presence of ultra-violet divergences, even though they are cancelled by renormalization counterterms, means that in any process there are contributions from quantum fluctuations on every distance scale.

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RENORMALIZATION

An introduction to renormalization, the renormalization group, and the operator-product expansion

JOHN C.COLLINS
Illinois Institute of Technology

Renormalisable theory
Input:
several finite parameters of the theory

Finite formulation of QFT

Output:
Infinite number of physical observables:
finite values

Non-renormalisable theory
Input:
infinite number of finite parameters of the theory

Hierarchy problem in finite formulations of QFT?

No infinities (quartic, quadratic, log) in finite QFT - perhaps, no fine-tunings? Indeed, if all expressions are finite, the computation of low energy observables should not require the knowledge of the UV domain of the theory.

The existence of such a formalism without large cancellations would challenge the “naturalness” paradigm.

If just one particular formalism of computations in QFT without necessity of fine-tunings is found, it will provide a strong argument that the problem of quantum stability of the electroweak scale against radiative corrections is formalism dependent and thus unphysical.

Finite formulations of QFT

Bogolubov-Parasuk-Hepp-Zimmermann (BPHZ)

A certain procedure, called “R-operation” is applied to any Feynman graph before performing integrations over internal momenta) changing the integrand prescribed by the Feynman rules to another one. The resulting expression is then integrated, with no infinities encountered. The R-operation can be used in both renormalisable and non-renormalisable field theories.

INTRODUCTION
TO THE THEORY OF
QUANTIZED FIELDS

THIRD EDITION

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ISSN 1063-7796, Physics of Particles and Nuclei, 2020, Vol. 51, No. 4, pp. 503–507. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Author(s), 2020, published in Fizika Elementarnykh Chastits i Atomnogo Yadra, 2020, Vol. 51, No. 4.

The Bogolyubov ℛ-Operation in Nonrenormalizable Theories

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Received December 20, 2019; revised January 16, 2020; accepted January 29, 2020

Ввиду всего этого мы не придерживаемся «ренормализационной» терминологии, рассматривая процедуру введения контрчленов как *формальный прием*, обеспечивающий конечность результатов расчетов.

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Finite formulations of QFT

Callan-Symanzik - inspired finite renormalisation equations

Usually, CS equations are represented as a tool for the renormalisation group investigation of the high energy behaviour of the renormalised amplitudes. However, the same equations can be used for the construction of the divergence-free and thus completely finite perturbation theory.

Broken Scale Invariance in Scalar Field Theory*

Curtis G. Callan, Jr.†
California Institute of Technology, Pasadena, California 91109
and
Institute for Advanced Study, Princeton, New Jersey 08540

(Received 4 June 1970)

We use scalar-field perturbation theory as a laboratory to study broken scale invariance. We pay particular attention to scaling laws (Ward identities for the scale current) and find that they have unusual anomalies whose presence might have been guessed from renormalization-group arguments. The scaling laws also appear to provide a relatively simple way of computing the renormalized amplitudes of the theory, which sidesteps the overlapping-divergence problem.

Commun. math. Phys. 18, 227—246 (1970)

Small Distance Behaviour in Field Theory and Power Counting

K. Symanzik
Deutsches Elektronen-Synchrotron DESY, Hamburg

Received May 12, 1970

FIELD THEORY RENORMALIZATION USING THE CALLAN-SYMANZIK EQUATION

A.S. BLAER
Physics Department, Princeton University, Princeton, N.J. 08540, USA

K. YOUNG
Physics Department, The Chinese University of Hong Kong, Hong Kong

Received 18 June 1974

INTRODUCTION TO RENORMALIZATION THEORY

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Finite formulations of QFT

t'Hooft: Exact equations for irreducible two-, three -, and four-point vertices which do not contain any ultraviolet infinities. The idea is that any divergent n-point function can be rendered finite by subtracting the same n-point function evaluated at different values for the external momenta. This difference can be interpreted as a new irreducible Feynman diagram with n+1 external lines. Integrating these "difference diagrams" with respect to the external momenta yields renormalisation group equations. Potentially, these equations may result in a completely non-perturbative and divergence-free definition of the theory.

Lehmann, Symanzik and Zimmermann

Nishijima

ITP-UU-04/13
SPIN-04/07
hep-th/0405032

RENORMALIZATION WITHOUT INFINITIES *

Gerard 't Hooft

IL NUOVO CIMENTO — VOL. I, N. 1 — 1° Gennaio 1955

Zur Formulierung quantisierter Feldtheorien.

H. LEHMANN, K. SYMANZIK und W. ZIMMERMANN
Max-Planck-Institut für Physik - Göttingen (Deutschland)

(ricevuto il 22 Novembre 1954)

PHYSICAL REVIEW — VOLUME 119, NUMBER 1 — JULY 1, 1960

Asymptotic Conditions and Perturbation Theory

K. NISHIJIMA*
Department of Physics, Osaka City University, Osaka, Japan and Department of Physics, University of Illinois, Urbana, Illinois
(Received December 21, 1959)

Callan-Symanzik method as a finite approach to QFT

Idea of the method: θ-operation,

$$\Gamma_\theta^{(n)}(k^2) \equiv -i \times \frac{d}{dm_0^2}\,\Gamma^{(n)}(k^2)$$

- Graphically: cut propagator in two

$\left[-\Gamma^{(2)}\right]_\lambda$ $\left[-\Gamma_\theta^{(2)}\right]_\lambda$ $\left[-\Gamma_{\theta\theta}^{(2)}\right]_\lambda$

- Result: θ-operation renders diagrams more convergent (it reduces degree of divergence by two)

Callan-Symanzik method as a finite approach to QFT

Ingredients for the simplest scalar theory:

- Lagrangian:

Everything is finite!

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

- Equations for vertices with n legs $\bar{\Gamma}^{(n)}$ and new, θ-type vertices:

$$2im^2\,(1+\gamma)\cdot\bar{\Gamma}^{(n)}_\theta = \left[\left(m\,\frac{\partial}{\partial m} + \beta\,\frac{\partial}{\partial\lambda}\right) + n\cdot\gamma\right]\bar{\Gamma}^{(n)}$$

$$2im^2\,(1+\gamma)\cdot\bar{\Gamma}^{(n)}_{\theta\theta} = \left[\left(m\,\frac{\partial}{\partial m} + \beta\,\frac{\partial}{\partial\lambda}\right) + n\cdot\gamma + \gamma_\theta\right]\bar{\Gamma}^{(n)}_\theta$$

Callan-Symanzik method as a finite approach to QFT

Boundary conditions, valid in all orders of λ:

$$\left[\frac{d}{dk^2}\,\bar{\Gamma}^{(2)}(k^2)\right]_{k^2=0} = i, \qquad \bar{\Gamma}^{(2)}\left(k^2=0\right) = im^2, \qquad \bar{\Gamma}^{(4)}\left(k^2=0\right) = -\,i\lambda\,.$$

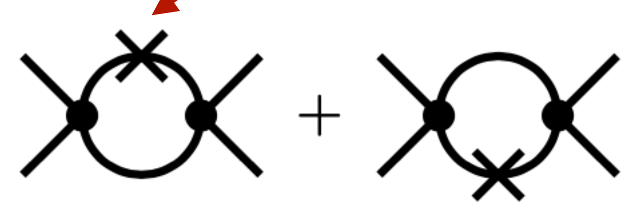
First order, tree approximations for 2 and 4 point functions, computed:

$$\left[\bar{\Gamma}^{(2)}\right]_{\lambda^0} = i\left(k^2+m^2\right)\ , \quad \left[\bar{\Gamma}^{(4)}\right]_{\lambda} = -\,i\lambda$$

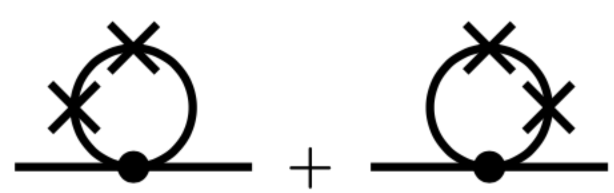
θ-operation: cuts the propagator in two

One-loop finite expressions, computed:

$$\left[\bar{\Gamma}^{(4)}_{\theta}\right]_{\lambda^2} = -\frac{\lambda^2}{32\pi^2}\sum_{3\ \mathrm{opt}}\int_0^1 dx\ \frac{1}{x(1-x)\kappa_1^2+m^2}$$



$$\left[\bar{\Gamma}^{(2)}_{\theta\theta}\right]_{\lambda} = -\frac{i\lambda}{32\pi^2}\,\frac{1}{m^2}$$



The unknown quantities β, γ, γ_θ , and the vertices are to be out by iterative procedure from these equations and boundary conditions. No infinities appear at any step of computation at any loop order.

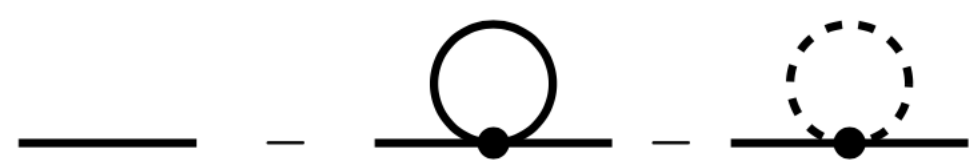
Fine-tunings with two mass scales in multiplicative renormalisation

Theory with two well separated physical mass scales, $M_{\rm phys} \gg m_{\rm phys}$

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\Phi\partial_\mu\Phi - \frac{m^2}{2}\phi^2 - \frac{M^2}{2}\Phi^2 - \frac{\lambda_\phi}{4!}\phi^4 - \frac{\lambda_{\phi\Phi}}{4}\phi^2\Phi^2 - \frac{\lambda_\Phi}{4!}\Phi^4$$

Standard approach, multiplicative renormalisation MS bar scheme, need to highly fine-tune the Lagrangian parameters m and M:

$\Gamma^{(2\phi)} =$



$+ \ \mathcal{O}\left(\lambda^2\right)$

$$\bar{\Gamma}^{(2\phi)} = i(k^2+m^2) - \frac{i\lambda_\phi m^2}{32\pi^2}\left(1+\ln\frac{\mu^2}{m^2}\right) - \frac{i\lambda_{\phi\Phi}M^2}{32\pi^2}\left(1+\ln\frac{\mu^2}{M^2}\right)$$

Absence of fine-tunings in finite QFT

- The same Lagrangian
- Equations for vertices $\bar{\Gamma}^{(n)}$ and new, θ-type vertices:

$$\bar{\Gamma}^{(n)}_{\theta,m}\,,\quad \bar{\Gamma}^{(n)}_{\theta,M}\,,\quad \bar{\Gamma}^{(n)}_{\theta\theta,mm}\,,\quad \bar{\Gamma}^{(n)}_{\theta\theta,MM}\,,\ \text{and}\quad \bar{\Gamma}^{(n)}_{\theta\theta,mM}$$

First CS equation: 2x1 matrix equation

$$i \cdot \mathcal{G} \cdot \begin{pmatrix} \bar{\Gamma}^{(n,N)}_{\theta,m} \\ \bar{\Gamma}^{(n,N)}_{\theta,M} \end{pmatrix} = \Bigg[\begin{pmatrix} \partial/\partial m^2 \\ \partial/\partial M^2 \end{pmatrix} + \sum_i \begin{pmatrix} \frac{1}{2m^2}\beta_{\lambda_{i,m}} \\ \frac{1}{2M^2}\beta_{\lambda_{i,M}} \end{pmatrix} \frac{\partial}{\partial \lambda_i}$$

$$+ \frac{n}{2} \begin{pmatrix} \frac{1}{m^2}\gamma_{\phi,m} \\ \frac{1}{M^2}\gamma_{\phi,M} \end{pmatrix} + \frac{N}{2} \begin{pmatrix} \frac{1}{m^2}\gamma_{\Phi,m} \\ \frac{1}{M^2}\gamma_{\Phi,M} \end{pmatrix} \Bigg] \bar{\Gamma}^{(n,N)}$$

$$\mathcal{G} = \begin{pmatrix} 1 + \gamma_{\phi,m} & \frac{M^2}{m^2}\gamma_{\Phi,m} \\ \frac{m^2}{M^2}\gamma_{\phi,M} & 1 + \gamma_{\Phi,M} \end{pmatrix}$$

Second CS equation: 2x2 matrix equation

$$
\begin{aligned}
& i \cdot \begin{pmatrix} 1+\gamma_{\phi,m} & \frac{M^2}{m^2}\gamma_{\Phi,m} \\ \frac{m^2}{M^2}\gamma_{\phi,M} & 1+\gamma_{\Phi,M} \end{pmatrix} \times \begin{pmatrix} \bar{\Gamma}_{\theta\theta,mm} & \bar{\Gamma}_{\theta\theta,mM} \\ \bar{\Gamma}_{\theta\theta,Mm} & \bar{\Gamma}_{\theta\theta,MM} \end{pmatrix} \\
& = \left[\begin{pmatrix} \partial/\partial m^2 \\ \partial/\partial M^2 \end{pmatrix} + \sum_i \begin{pmatrix} \beta_{\lambda_i,m}/2m^2 \\ \beta_{\lambda_i,M}/2M^2 \end{pmatrix} \frac{\partial}{\partial \lambda_i} + \frac{n}{2} \begin{pmatrix} \gamma_{\phi,m}/m^2 \\ \gamma_{\phi,M}/M^2 \end{pmatrix} + \frac{N}{2} \begin{pmatrix} \gamma_{\Phi,m}/m^2 \\ \gamma_{\Phi,M}/M^2 \end{pmatrix} \right] \begin{pmatrix} \bar{\Gamma}_{\theta,m} & \bar{\Gamma}_{\theta,M} \end{pmatrix} \\
& \quad + \left[\frac{1}{2m^2} \times \begin{pmatrix} \gamma_{\theta,mmm} & \gamma_{\theta,Mmm} \\ \gamma_{\theta,mmM} & \gamma_{\theta,MmM} \end{pmatrix} \bar{\Gamma}_{\theta,m} + \frac{1}{2M^2} \times \begin{pmatrix} \gamma_{\theta,mMm} & \gamma_{\theta,MMm} \\ \gamma_{\theta,mMM} & \gamma_{\theta,MMM} \end{pmatrix} \bar{\Gamma}_{\theta,M} \right],
\end{aligned}
$$

Absence of fine-tunings in finite QFT

- Boundary conditions, valid in all orders of λ:

$$\left[\frac{d}{dk^2}\,\bar{\Gamma}^{(2\phi)}(k^2)\right]_{k^2=0} = i\ , \qquad \bar{\Gamma}^{(2\phi)}\left(k^2=0\right) = im^2$$

$$\left[\frac{d}{dk^2}\,\bar{\Gamma}^{(2\Phi)}(k^2)\right]_{k^2=0} = i\ , \qquad \bar{\Gamma}^{(2\Phi)}\left(k^2=0\right) = iM^2$$

Absence of fine-tunings in finite QFT

- First order, tree approximations for 2 point functions, computed:

- $\left[\bar{\Gamma}^{(2\phi)}\right]_{\lambda^0} = i\left(k^2 + m^2\right)\ , \left[\bar{\Gamma}^{(2\Phi)}\right]_{\lambda^0} = i\left(k^2 + M^2\right)\ .$

- One-loop finite expressions, computed:

$$\left[\bar{\Gamma}^{(2\phi)}_{\theta\theta,mm}\right]_{\lambda} = -\frac{i\lambda_{\phi}}{32\pi^2}\frac{1}{m^2}\ , \left[\bar{\Gamma}^{(2\phi)}_{\theta\theta,mM}\right]_{\lambda} = 0\ , \left[\bar{\Gamma}^{(2\phi)}_{\theta\theta,MM}\right]_{\lambda} = -\frac{i\lambda_{\phi\Phi}}{32\pi^2}\frac{1}{M^2}$$

- Result, valid in all orders: no fine-tunings are needed, m and $m_{\rm phys}$ are small, M and $M_{\rm phys}$ are large.

Cosmological constant

The same consideration applies to the cosmological constant $\epsilon_{vac} \equiv \Lambda$, related to the zero point function $\bar{\Gamma}^{(0)}$ (Casimir effect, effective potential, etc). New object:

$$\bar{\Gamma}^{(0)}_{\theta\theta\theta} = 2 \times \frac{1}{2} \times (-1)^3 \int \frac{d^4 l}{(2\pi)^4} \left(\frac{-i}{l^2 + m^2} \right)^3 = \frac{1}{32\pi^2} \frac{1}{m^2}$$

Equations for $\bar{\Gamma}^{(0)}$:

$$i \cdot \bar{\Gamma}^{(0)}_{\theta} = \left(\frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} + \frac{\gamma_\Lambda}{2m^2} \frac{\partial}{\partial \Lambda} \right) \bar{\Gamma}^{(0)}$$

$$i \cdot \bar{\Gamma}^{(0)}_{\theta\theta} = \left(\frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} + \frac{\gamma_\Lambda}{2m^2} \frac{\partial}{\partial \Lambda} \right) \bar{\Gamma}^{(0)}_{\theta}$$

$$i \cdot \bar{\Gamma}^{(0)}_{\theta\theta\theta} = \left(\frac{\partial}{\partial m^2} + \beta \frac{\partial}{\partial \lambda} + \frac{\gamma_\Lambda}{2m^2} \frac{\partial}{\partial \Lambda} \right) \bar{\Gamma}^{(0)}_{\theta\theta}$$

lead to finite and tuning free computation of physical observables, such as the Casimir energy or effective potential.

PHYSICAL REVIEW D VOLUME 14, NUMBER 6 15 SEPTEMBER 1976

Gauge-symmetry hierarchies*

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(Received 15 June 1976)

It is shown that one cannot artifically establish a gauge hierarchy of any desired magnitude by arbitrarily adjusting the scalar-field parameters in the Lagrangian and using the tree approximation to the potential; radiative corrections will set an upper bound on such a hierarchy. If the gauge coupling constant is approximately equal to the electromagnetic coupling constant, the upper bound on the ratio of vector-meson masses is of the order of $\alpha^{-1/2}$, independent of the sclar-field masses and their self-couplings. In particular, the usual assumption that large scalar-field mass ratios in the Lagrangian can induce large vector-meson mass ratios is false. A thus far unsuccessful search for natural gauge hierarchies is briefly discussed. It is shown that if such a hierarchy occurred, it would have an upper bound of the order of $\alpha^{-1/2}$.

No problem in finite formulation of QFT!

Construct effective potential $V(\phi, \Phi)$ out of $\bar{\Gamma}$, $\bar{\Gamma}_{\theta}$, $\bar{\Gamma}_{\theta\theta}$, and $\bar{\Gamma}_{\theta\theta\theta}$. Use equations CS equations for effective potential with boundary conditions $dV/d\phi = v$, $dV/d\Phi = V$ with $V \gg v$. No fine-tunings between tree values and radiative corrections!

Equation for effective action

From equations for the Greens functions, ϕ^4 theory:

$$2im^2\ (1+\gamma)\cdot\bar{\Gamma}^{(n)}_{\theta} = \left[\left(m\ \frac{\partial}{\partial m} + \beta\ \frac{\partial}{\partial\lambda}\right) + n\cdot\gamma\right]\bar{\Gamma}^{(n)}$$

$$2im^2\ (1+\gamma)\cdot\bar{\Gamma}^{(n)}_{\theta\theta} = \left[\left(m\ \frac{\partial}{\partial m} + \beta\ \frac{\partial}{\partial\lambda}\right) + n\cdot\gamma + \gamma_\theta\right]\bar{\Gamma}^{(n)}_{\theta}$$

one can get equations for effective action

$$\Gamma_{\text{eff}\ \ldots\theta\ldots} = \sum_n \frac{1}{n!}\int d^4x_1\ldots d^4x_n\ \Gamma^{(n)}_{\ldots\theta\ldots}(x_1\ldots x_n)\phi(x_1)\ldots\phi(x_n)$$, namely

$$2im^2\ (1+\gamma)\cdot\Gamma_{\text{eff},\theta} = \left[\left(m\ \frac{\partial}{\partial m} + \beta\ \frac{\partial}{\partial\lambda}\right) + \gamma\frac{\delta}{\delta\phi}\right]\Gamma_{\text{eff}}$$

$$2im^2\ (1+\gamma)\cdot\Gamma_{\text{eff},\theta\theta} = \left[\left(m\ \frac{\partial}{\partial m} + \beta\ \frac{\partial}{\partial\lambda}\right) + \gamma_\theta + \gamma\frac{\delta}{\delta\phi}\right]\Gamma_{\text{eff},\theta}$$

Effective potential: consider space-time independent fields.

Solution in one loop, $\mathcal{O}(\hbar)$

- Feed in finite expression for

$$\Gamma_{\mathrm{eff},\theta\theta} = -\frac{1}{32\pi^2}\log\left[1+\frac{\lambda\phi^2}{2m^2}\right]$$

- Solve second order differential equation for $\Gamma_{\mathrm{eff}}[\phi/m]$.

- Find anomalous dimensions from requirement of regularity of $\Gamma_{\mathrm{eff}}[\phi/m]$ for $\phi \to 0$:

$$\gamma_\theta = \frac{\lambda}{16\pi^2},\ \beta_\lambda = \frac{3\lambda^2}{16\pi^2}$$

- Potential is determined up to two arbitrary constants, to be fixed by the physical mass and coupling constant.

All computations are explicitly finite, no fine-tunings are needed. The same is true for the theory with two scalar fields with hierarchical vevs.

Non-renormalisable theories

Most general action for scalar field, O_n is the full set of operators with mass dimension n, λ_n is the set of dimensionless constants:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \sum_{n=4}^{\infty}\frac{\lambda_n O_n}{M^{n-4}}$$

Relevant diagrams can be made finite by acting on it by θ-operation several times.
Equations for effective actions:

$$2im^2\,(1+\gamma)\cdot\Gamma_{\text{eff},\theta} = \left[\left(m\,\frac{\partial}{\partial m} + \sum_i \beta_i \frac{\partial}{\partial\lambda_i}\right) + \gamma\frac{\delta}{\delta\phi}\right]\Gamma_{\text{eff}}$$

$$2im^2\,(1+\gamma)\cdot\Gamma_{\text{eff},\theta\theta} = \left[\left(m\,\frac{\partial}{\partial m} + \sum_i \beta_i \frac{\partial}{\partial\lambda_i}\right) + \gamma_\theta + \gamma\frac{\delta}{\delta\phi}\right]\Gamma_{\text{eff},\theta}$$

$$\ldots\theta\theta\theta\ldots = \ldots\theta\theta, \quad \ldots\theta\theta\theta\theta\ldots = \ldots\theta\theta\theta, \ldots$$

Non-renormalisable theories

All equations can be combined in one by introducing a θ-dependent functional:

$$\left[2m^2\left(\frac{\partial}{\partial m^2} - i(1+\gamma)\frac{\partial}{\partial \theta}\right) + \gamma_\theta\, \theta\frac{\partial}{\partial \theta} + \gamma\, \phi\frac{\delta}{\delta \phi} + \sum_i \beta_i \frac{\partial}{\partial \lambda_i}\right]\Gamma_{\text{eff}}[\theta, \phi] = 0$$

Here

$$\Gamma_{\text{eff}}[\theta, \phi] = \sum_{n=0}^{\infty} \Gamma_{\text{eff},n\theta}[\phi]\frac{\theta^n}{n!}$$

Solution in one loop, $\mathcal{O}(\hbar), \mathcal{O}(1/M^2)$

Example of computation with 6-dimensional operators:

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 - \frac{g}{6!M^2}\phi^6 + \frac{\xi}{M^2}\phi(\Box^2\phi) + \frac{f}{3!M^2}\phi^3\Box\phi$$

Reparametrisation freedom:

$$\phi \to \phi + C_1\frac{\phi^3}{M^2} + \frac{C_2\Box\phi}{M^2}$$

allows to remove terms $\propto \xi$ and f, the only relevant operator is $\propto \phi^6$

Solution in one loop, $\mathcal{O}(\hbar), \mathcal{O}(1/M^2)$

- Feed in finite expression for

$$\Gamma_{\theta\theta} = -\frac{1}{32\pi^2} \ln\left[1 + \frac{\lambda\phi^2}{2m^2} + \frac{g}{4!M^2}\frac{\phi^4}{m^2}\right]$$

- Solve second order differential equation for $\Gamma_{\rm eff}[\phi/m]$.

- Find anomalous dimensions from requirement of regularity of $\Gamma_{\rm eff}[\phi/m]$ for $\phi \to 0$: $\gamma_\theta = \frac{\lambda}{16\pi^2}$, $\beta_\lambda = \frac{3\lambda^2}{16\pi^2} + \frac{gm^2}{16\pi^2 M^2}$, $\beta_g = \frac{15 g\lambda}{16\pi^2}$

- Potential is determined up to two arbitrary constants, to be fixed by the physical mass and coupling constant.

All computations are explicitly finite, no fine-tunings are needed. The procedure can be repeated in higher orders of $\hbar$ and $1/M^2$, $O_{8,1} = \phi^8$, $O_{8,2} = [(\partial_\mu\phi)^2]^2$

Conclusions

- Finite QFT formulation based on Callan-Symanzik equations does not require any fine-tunings in the theories with well separated mass scales, both in renormalisable and non-renormalisable QFTs.

- The so-called hierarchy problem (the sensitivity of low energy physics to high energy physics) depends on the formulation of quantum field theory, and is absent in finite formulation of QFT.

- The conclusions drawn about new physics in finite QFT approach are very different from those of the standard one: “naturalness” leads to the conjecture about the existence of new physics right above the Fermi scale, whereas the use of a finite formulation of QFT says that no such a conclusion can be made on physical grounds.

- Though the problem of the quantum stability of the Higgs mass and of cosmological constant can be resolved by finite QFT, the question about the origin of widely separated scales in Nature (such as vacuum energy, Fermi, GUT or Planck scale) remains.

Remarks

- At the technical side, the CS method as it stands cannot work for massless particles, such as gauge bosons or gravitons. However, this problem occurs in the infrared rather than in the UV. Therefore, we expect that a “gauge symmetry preserving generalisation” of the CS method does not change the hierarchy discussion. The ’t Hooft method does not seem to have this problem.

- The CS method is rooted to perturbation theory. Does it incorporate non-perturbative effects, such as instantons, etc?

- Relation of CS equations for effective action/potential to ERG?