Central Limit Theorems, Large Deviations and the Renormalization Group

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with Ivan Balog, Bertrand Delamotte, Félix Rose and Sankarshan Sahu

I. Balog, AR, B. Delamotte, PRL 129, 210602 (2022)
I. Balog, B. Delamotte, AR, arXiv:2409.01250 , arXiv:????.????
S. Sahu, B. Delamotte, AR, arXiv:2407.12603
F. Rose, I. Balog, AR, arXiv????.????

Université de Lille

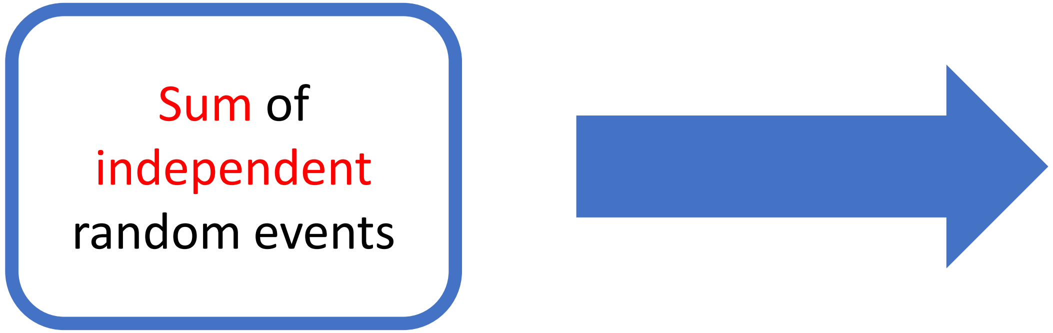
PhLAM Physique des Lasers Atomes et Molécules

Introduction

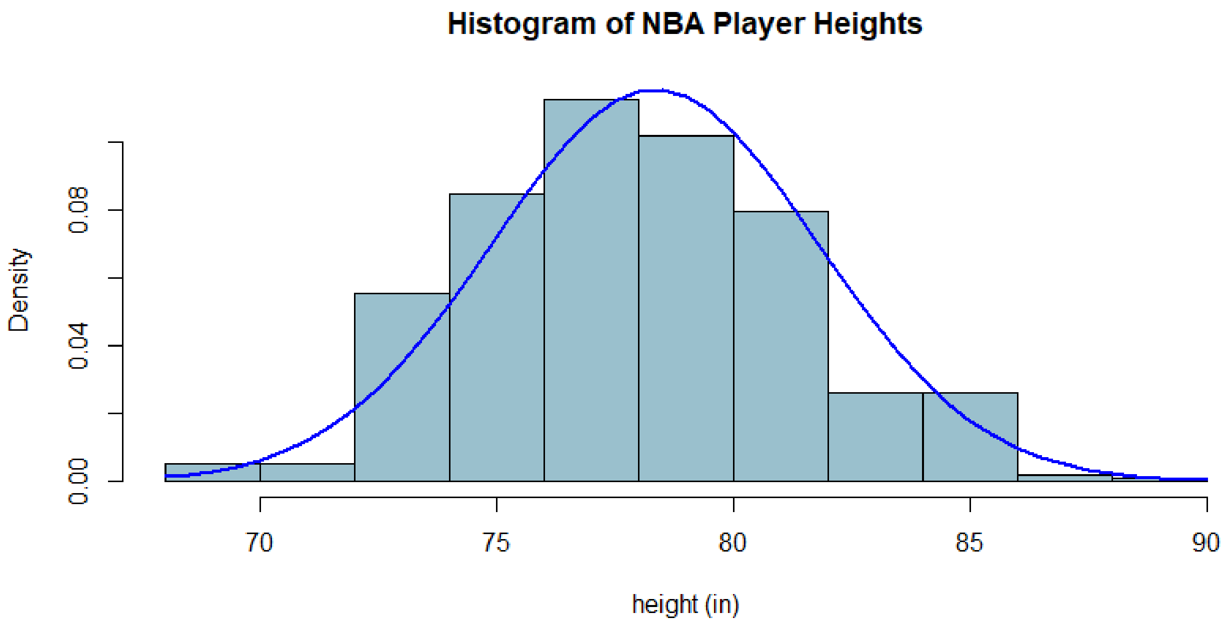
Statistics/probability needed for complex systems (many degrees of freedom, correlations, etc.)

Ex: biology, demographics, engineering, finance, statistical physics, …

When d.o.f independent or weak correlations: Central Limit Theorem (CLT) and generalization (Levy distribution)



Normal distribution



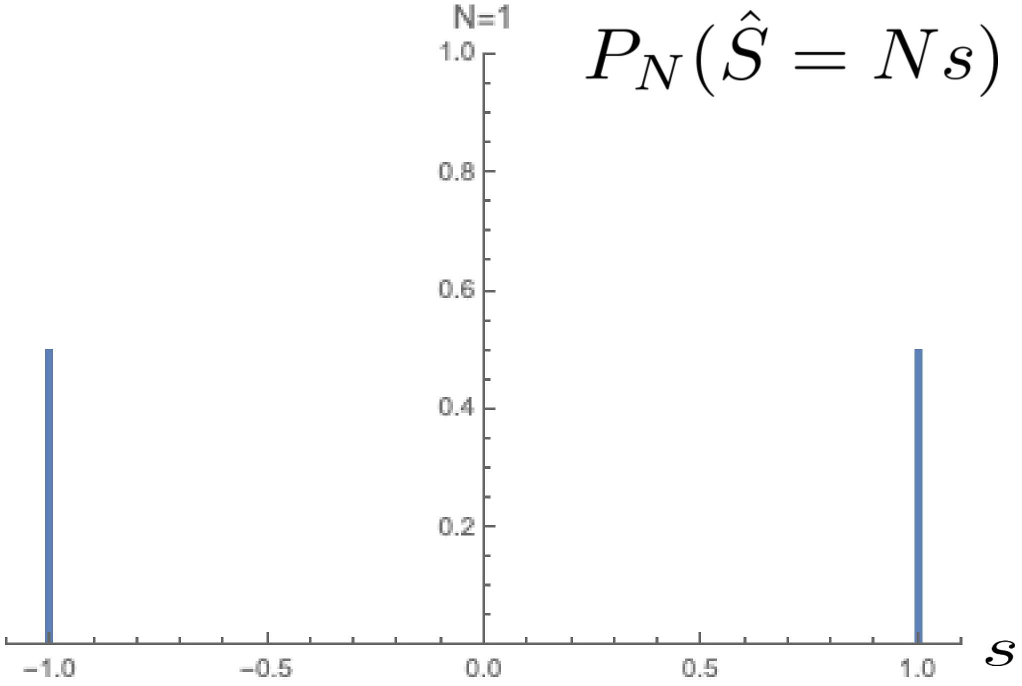
Central Limit Theorem

Take N random i.i.d variables $\hat{s}_i = \pm 1$ (coin flips, spins, Brownian motion, Ising at T=∞)

Distribution of the sum for large N, $\hat{S} = \sum_i \hat{s}_i$

Law of large number

$$P_N(\hat{S} = Ns)$$

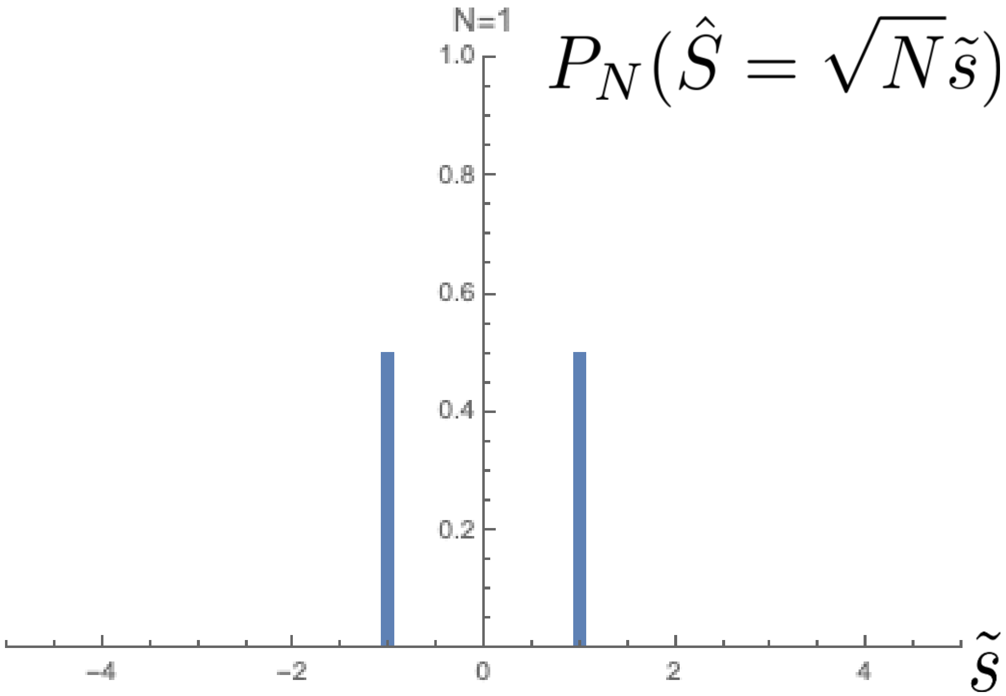


$$\langle s^2 \rangle = \frac{1}{N}$$

$s = \frac{1}{N} \sum_i s_i$

Central Limit Theorem (CLT)

$$P_N(\hat{S} = \sqrt{N}\tilde{s})$$

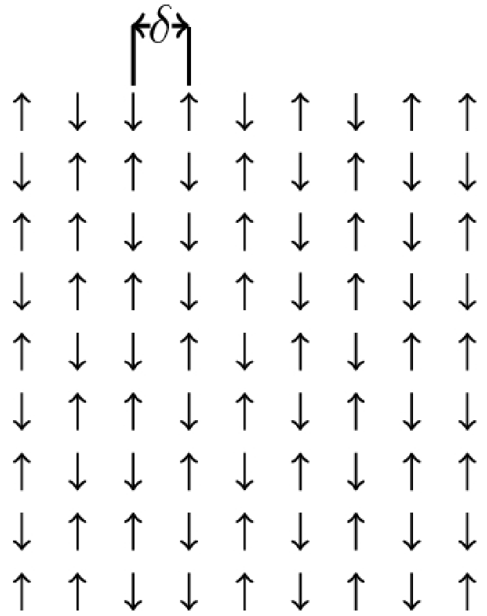


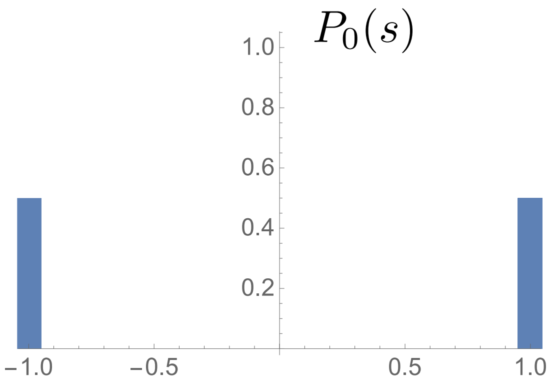
$$\langle \tilde{s}^2 \rangle = 1$$

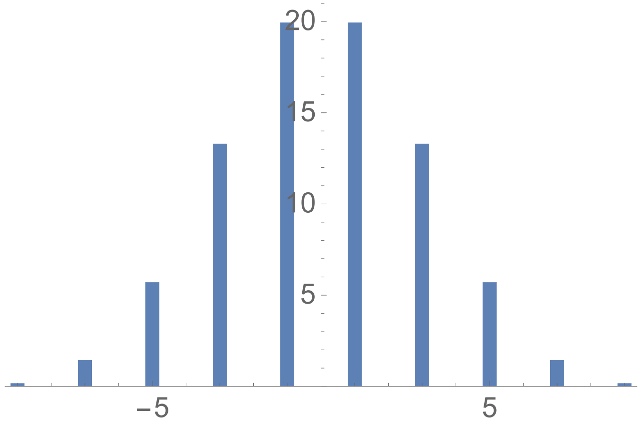
$\tilde{s} = \frac{1}{\sqrt{N}} \sum_i s_i$

CLT from RG (block spins)

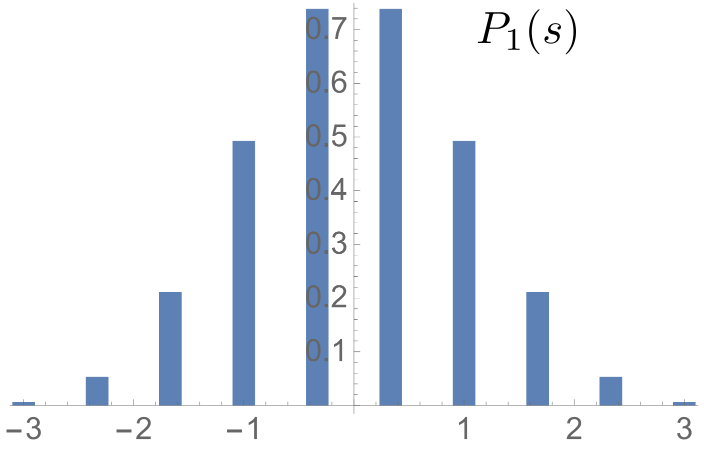
For independent variables







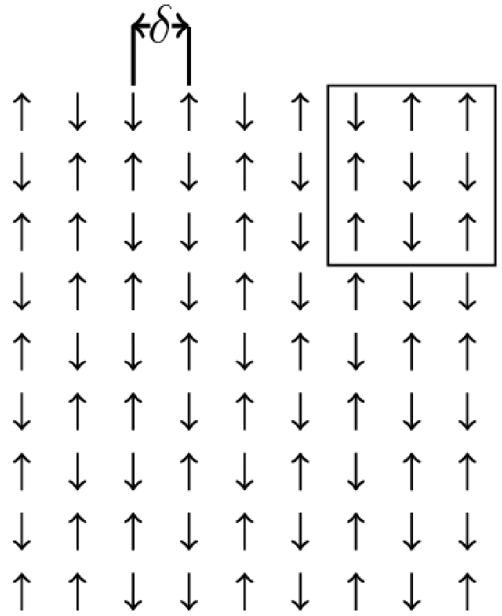
$$S_I = \sum_{i \in I} s_i$$

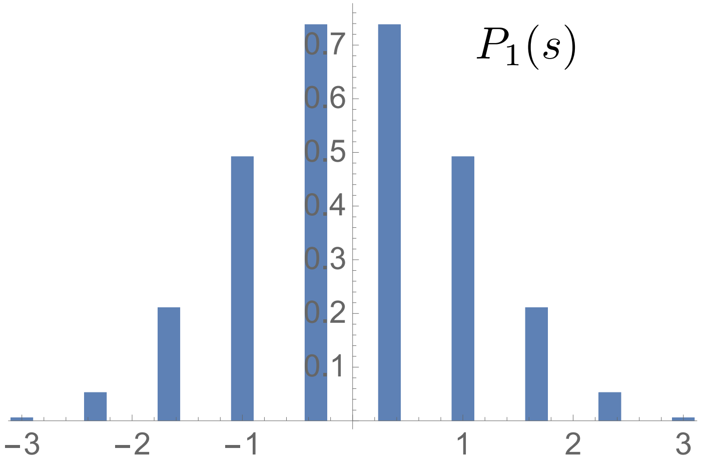


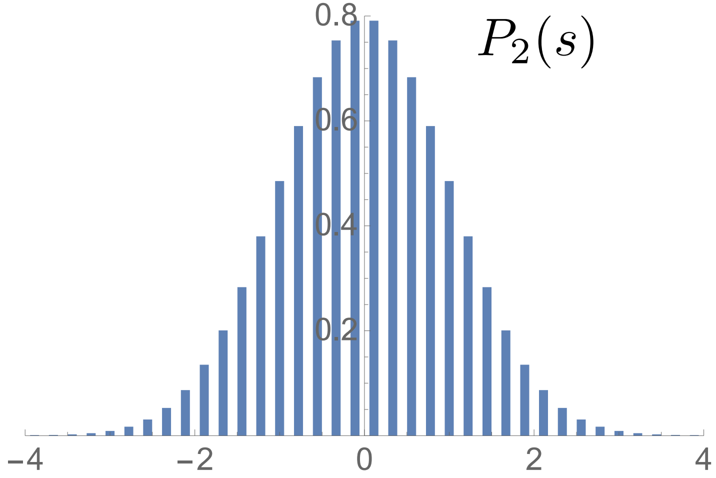
$$s_I = S_I / \sqrt{9}$$

CLT from RG (block spins)

For independent variables

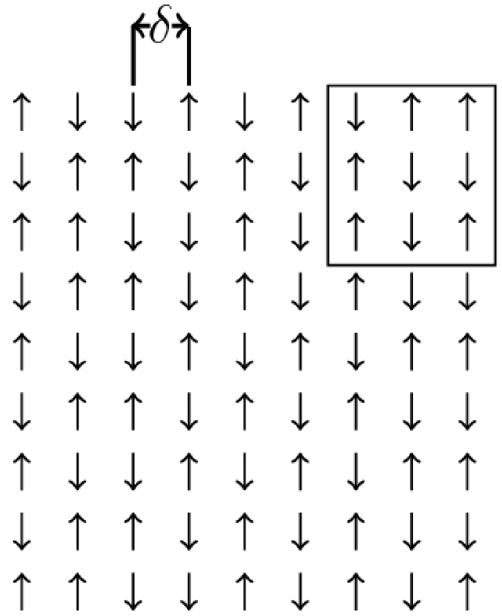


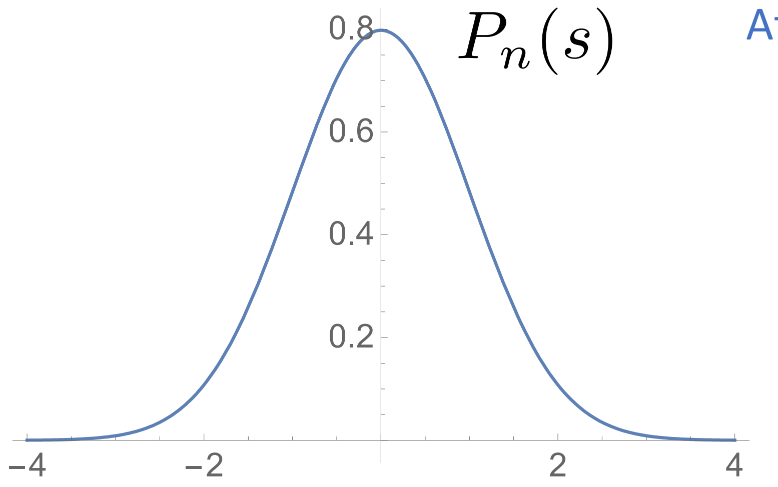




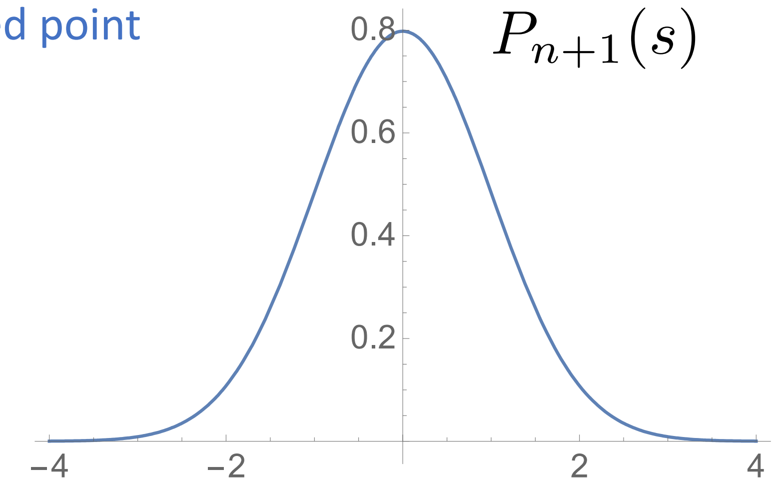
Example: CLT from blocking

For independent variables





After many iterations: reaches fixed point



RG and CLT

Koralov & Sinai, "Theory of Probability and Random Processes"

- Gaussian distribution is the fixed point of a "coarse-graining" transformation

- Gaussian FP has: 2 relevant perturbations (normalisation and mean)
  1 marignal perturbation (variance aka normalisation of variable)
  ∞-many irrelevant perturbations

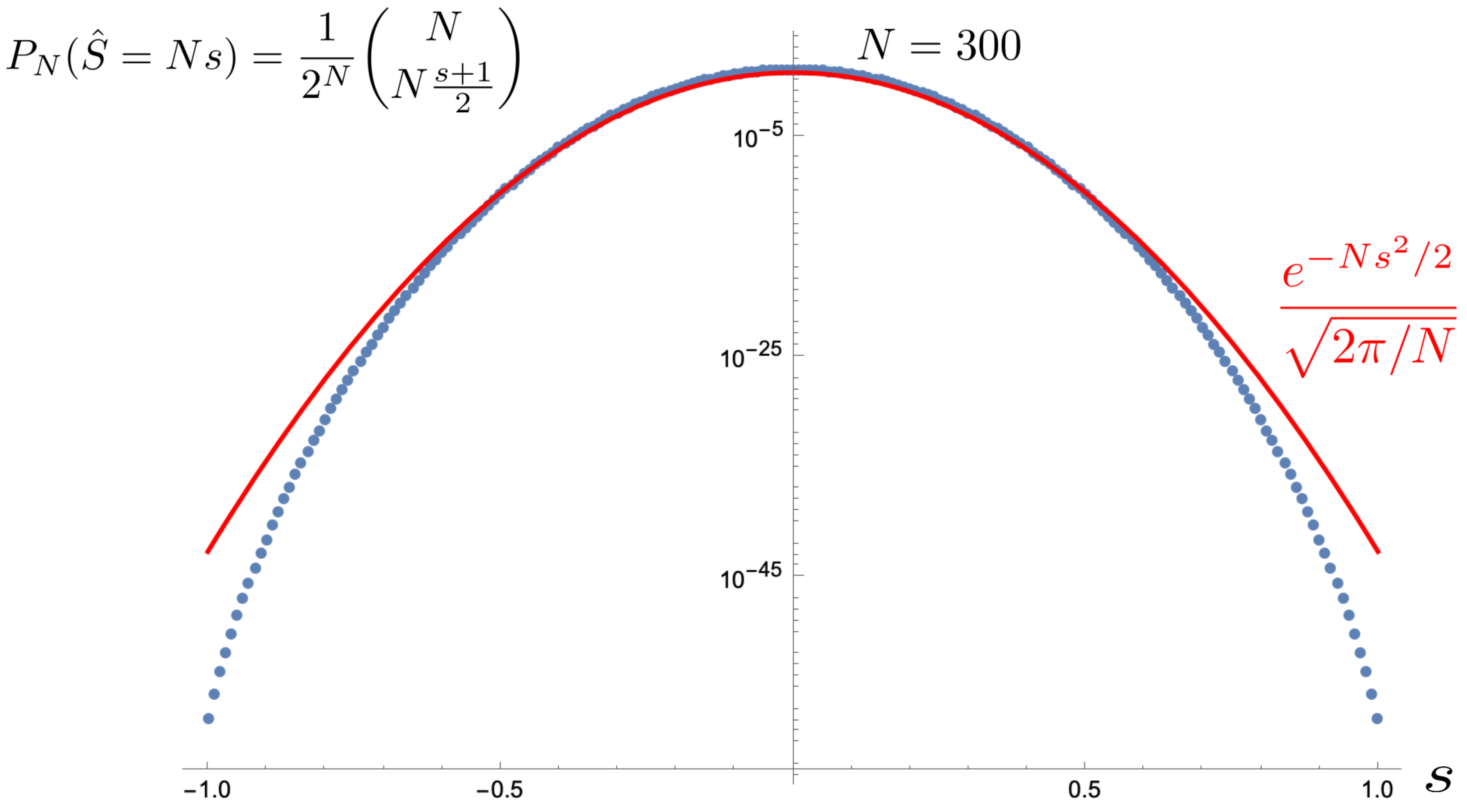
- Eigenperturbations are hermite functions

- Very large bassin of attraction (all finite variance probabilities)

What about rare events ?

Large Deviation Principle

Large deviations = rare events beyond the CLT



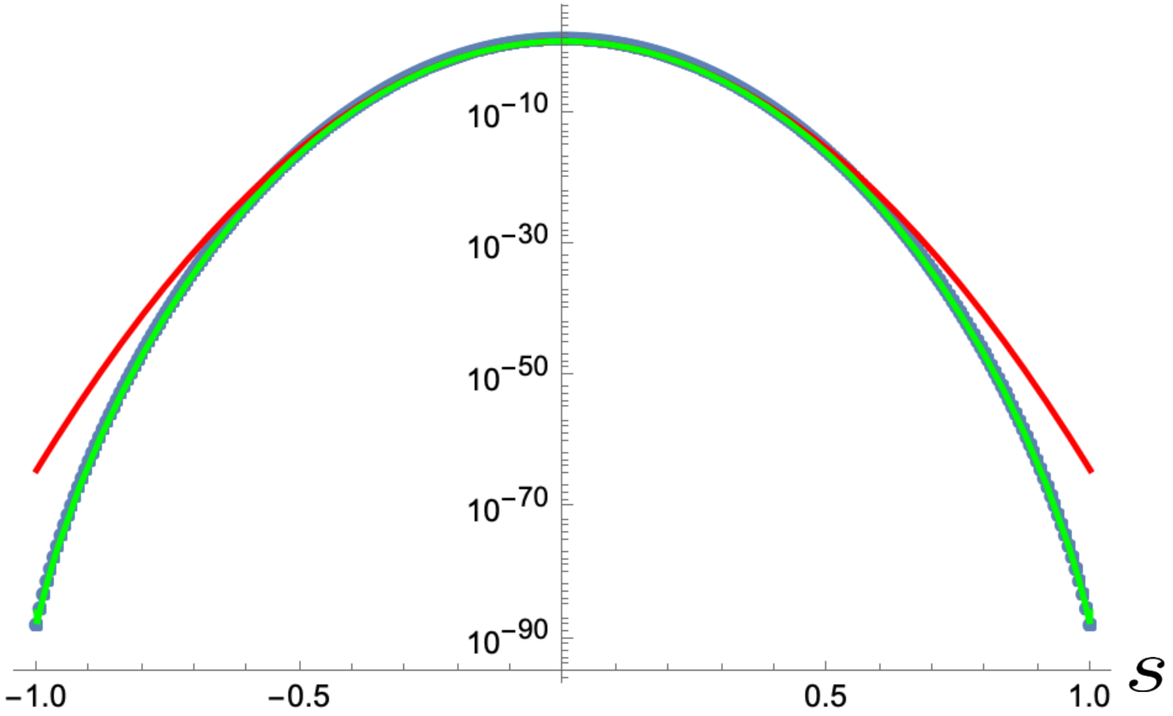
Rate function

Large Deviation Principle (LDP) $$P_N(\hat{S} = Ns) \simeq \sqrt{\frac{NI''(s)}{2\pi}} \exp\left(-NI(s)\right)$$

$$P_N(\hat{S} = Ns) = \frac{1}{2^N}\binom{N}{N\frac{s+1}{2}}$$

Rate function

$$I(s) = \frac{1+s}{2}\log\left(\frac{1+s}{2}\right) + \frac{1-s}{2}\log\left(\frac{1-s}{2}\right) + \log(2)$$



LDP to CLT (Cramer's series)

Recovering the CLT from the LDP $P(\hat{S} = \sqrt{N}\tilde{s}) \simeq \dfrac{e^{-I''(0)\tilde{s}^2/2}}{\sqrt{2\pi/I''(0)}}$ for $\tilde{s} = O(N^0)$

Universal (gaussian) law with one non-universal amplitude

"Finite size corrections" $$P(\hat{S} = \sqrt{N}\tilde{s}) \simeq \frac{e^{-I''(0)\tilde{s}^2/2}}{\sqrt{2\pi I''(0)}} e^{\frac{\tilde{s}^3}{\sqrt{N}}\lambda(\tilde{s}/\sqrt{N})}$$ $$\tilde{s} = o(\sqrt{N})$$

Carmer's series $$\lambda(z) = \sum_{k=0}^{\infty} a_k z^k$$

Universal power of system size

non-universal amplitude

Rate function and Legendre transform (Cramer's theorem)

$$P(\hat{S} = Ns) = \langle \delta(\hat{S} - Ns) \rangle$$
$$= \int_{a-i\infty}^{a+i\infty} \frac{dh}{2i\pi} e^{-Nhs} \langle e^{h\hat{S}} \rangle$$

$$\langle e^{h\hat{S}} \rangle = e^{-Nw(h)}$$

$$P(\hat{S} = Ns) = \int_{a-i\infty}^{a+i\infty} \frac{dh}{2i\pi} e^{-N(hs - w(h))},$$
$$\simeq \sqrt{\frac{N}{2\pi w''(h^*)}} e^{-N(h^* s - w(h^*))}$$

saddle-point

$\sup_{h\in\mathbb{R}}(hs - w(h))$

Effective potential

$$U(m) = \sup_{h\in\mathbb{R}}(hm - w(h))$$

Always convex

$$I(s) = U(m = s)$$

Valid in regions where rate function convex !

Generalization of CLT, LDP and Cramer’s series

from

(Functional) Renormalization Group

NB: connection between RG and CLT undertood already in the 70’s (Jona-Lasino)

Generalized CLT for strongly correlated variables

Correlations: $\langle s_i s_j \rangle = G(|i-j|)$

$$\tilde{s} = \frac{1}{\sqrt{N}} \sum_i s_i$$

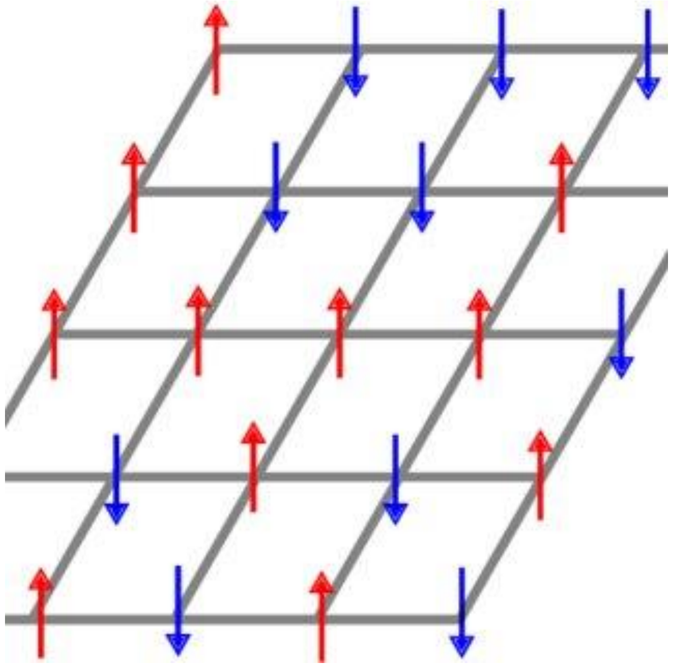
Weak correlations: $\langle \tilde{s}^2 \rangle \propto N^0$ Similar to i.i.d.'s: standard CLT

Strong correlations: $\lim\limits_{N \to \infty} \langle \tilde{s}^2 \rangle = \infty$

Example: Second order phase transitions

Standard model of statistical physics: Ising model

$$P(\{s_i\}) \propto e^{-H/T}$$



Energy of configuration: $H = -J \sum\limits_{\langle i,j \rangle} s_i s_j$

Number of spins: $N = L^d$

Basics of the Ising model

Phase transition from paramagnet at high T ($\langle s_i \rangle = 0$) to a ferromagnet at low T ($\langle s_i \rangle \neq 0$)

High temperature (T>>T_c): finite correlation length $\langle s_i s_j \rangle \propto e^{-|i-j|/\xi}$

$\xi, L \to \infty$

Close to T_c, many emergent phenomena:

- scale invariance at T_c $\quad \langle s_i s_j \rangle \propto \dfrac{1}{|i-j|^{d-2+\eta}}$

- scaling $\xi \propto |T_c - T|^{-\nu}$

  $\langle s_i \rangle \propto (T_c - T)^{\beta}$ $\qquad \dfrac{\beta}{\nu} = \dfrac{d-2+\eta}{2}$

- universality (e.g. liquid-gas critical point, many ferromagnets)

- conformal invariance, fractal dimension of domains,…

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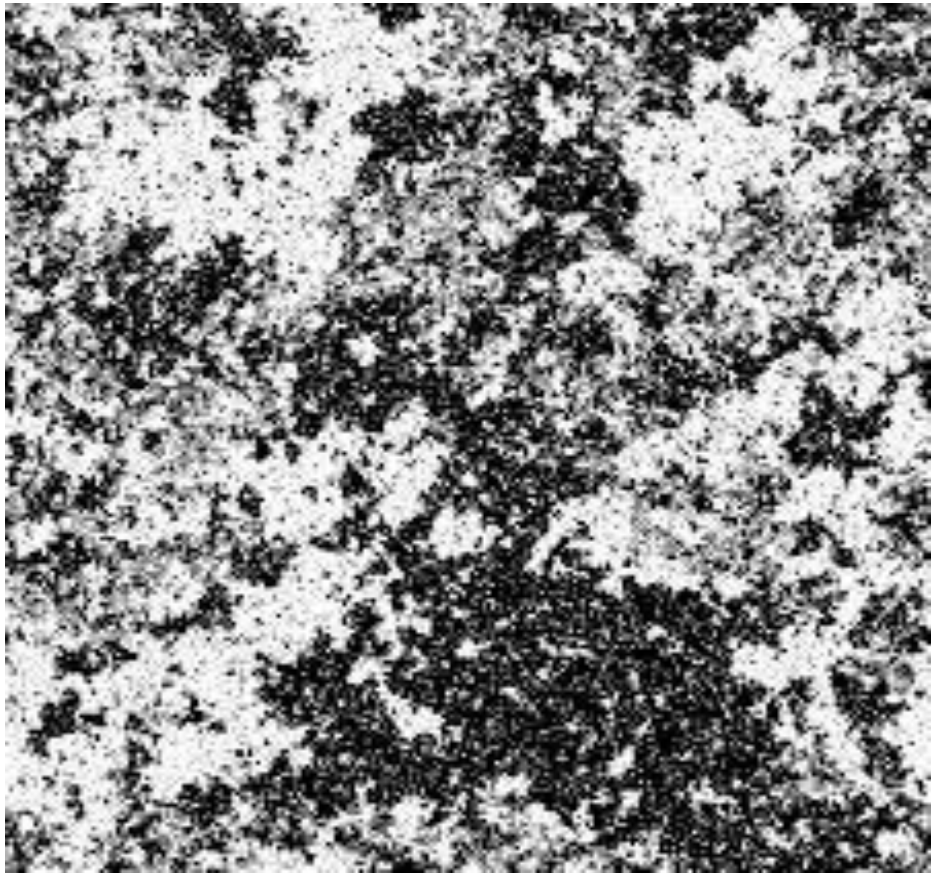


Distribution of the magnetization

High temperature phase: many independent blocks $\langle \tilde{s}^2 \rangle \propto \chi$

Weak correlations and gaussian distribution

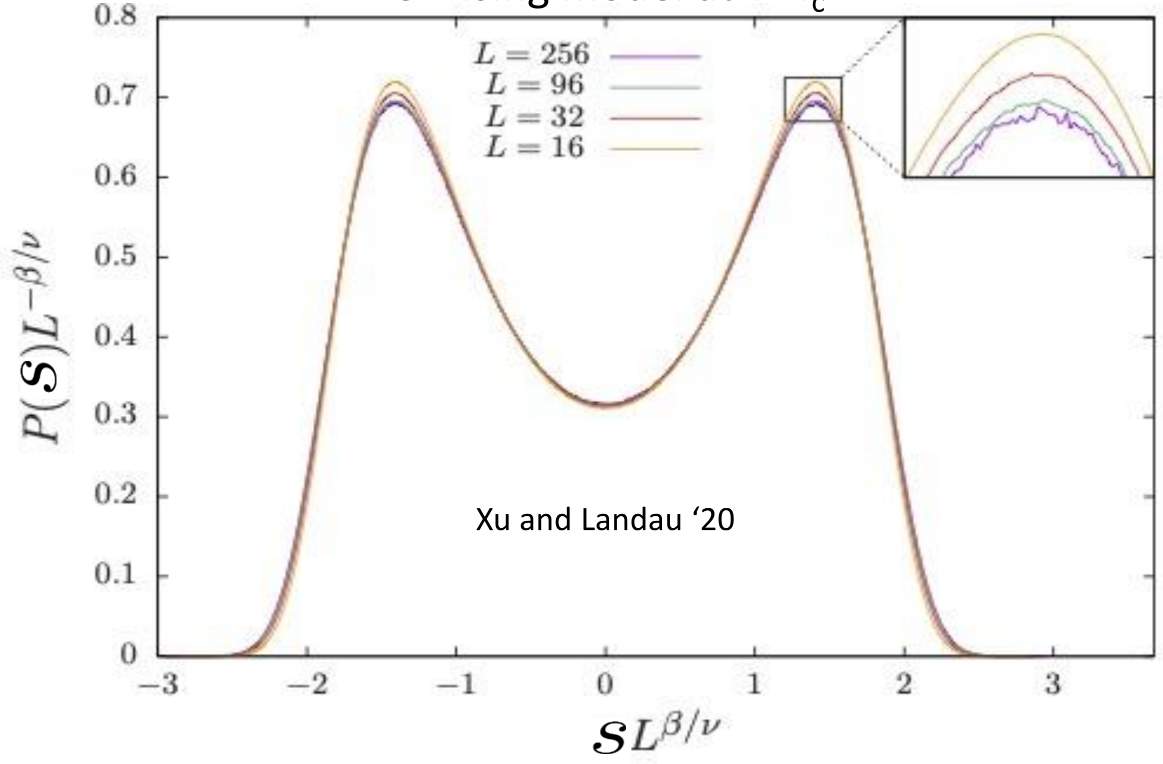
$$\tilde{s} = \frac{1}{\sqrt{N}} \sum_i s_i$$

Exactly at T_c : $\langle \tilde{s}^2 \rangle \propto L^{2-\eta}$ Strong correlations and non-Gaussian distribution

$$\langle s^2 \rangle \propto L^{-(d-2+\eta)} = L^{-2\beta/\nu}$$

$$s = \frac{1}{N} \sum_i s_i$$

3D Ising model at $T=T_c$



Stable law
for strongly correlated variables

$$P_L(s) \sim e^{-L^d I(s)} = e^{-\tilde{I}(sL^{\beta/\nu})}$$

standard CLT $P_N(s) \sim e^{-\frac{s^2}{2\sigma^2}}$

Understanding the scaling of typical magnetization fluctuations

$$\langle s^2 \rangle \propto L^{-(d-2+\eta)} = L^{-2\beta/\nu}$$

Number of "truly correlated" spins with a spin at position *i*: $N_{cor} \sim \sum_j \langle s_i s_j \rangle \sim L^{2-\eta}$

Assume that these correlated spins behave as a block-spin $S_I = \sum_{i \in I} s_i$

Number of block spins $N_{BS} \sim \frac{L^d}{N_{cor}} \sim L^{d-2+\eta}$

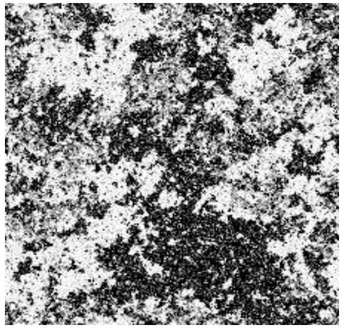
Total magnetization $\boldsymbol{S} = \frac{1}{L^d} \sum_i s_i = \frac{1}{N_{BS}} \sum_I S_I$

is thus expected to have fluctuations of order $\frac{1}{\sqrt{N_{BS}}} = L^{-\beta/\nu}$

Bouchaud & Georges '90

Understanding the scaling of typical magnetization fluctuations

Block-spins are not independent, or we would recover the standard CLT



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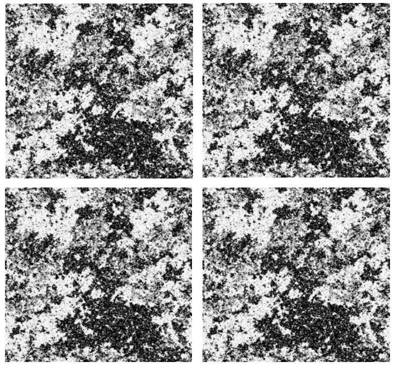
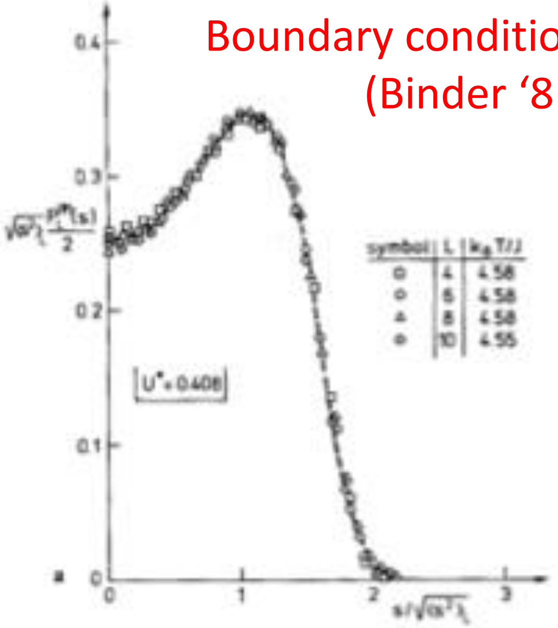
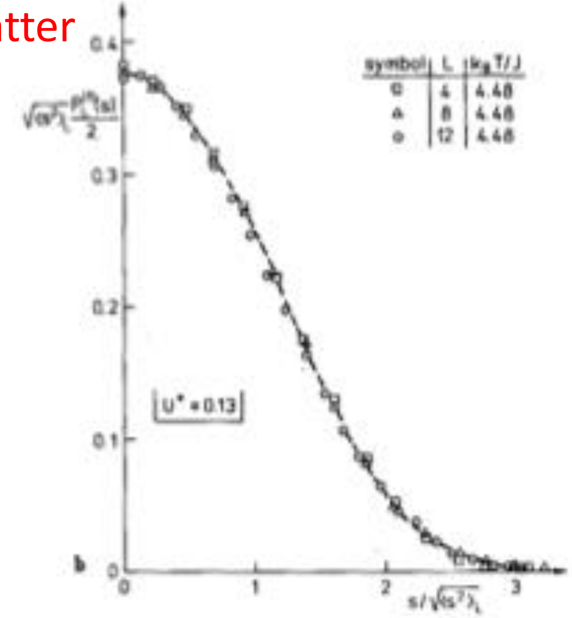


We can’t glue systems of size L/2 to get a system of size L because of long-range correlations

Boundary conditions matter
(Binder ‘81)

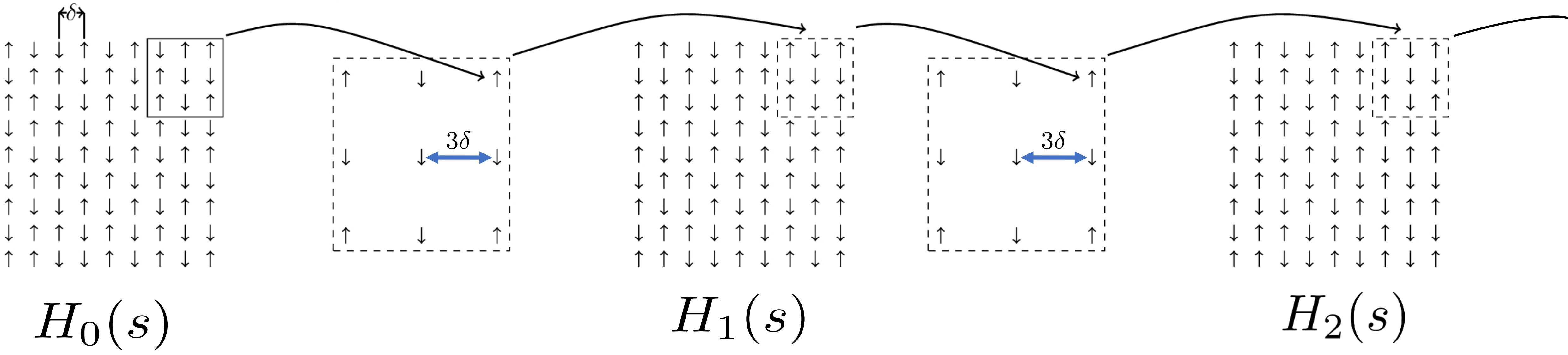




Can we still compute the stable law?

Renormalization group and CLT (bis)

Blocking procedure, again



At criticality, $H_n \to H_*$ fixed point Hamiltonian for block-spin variables

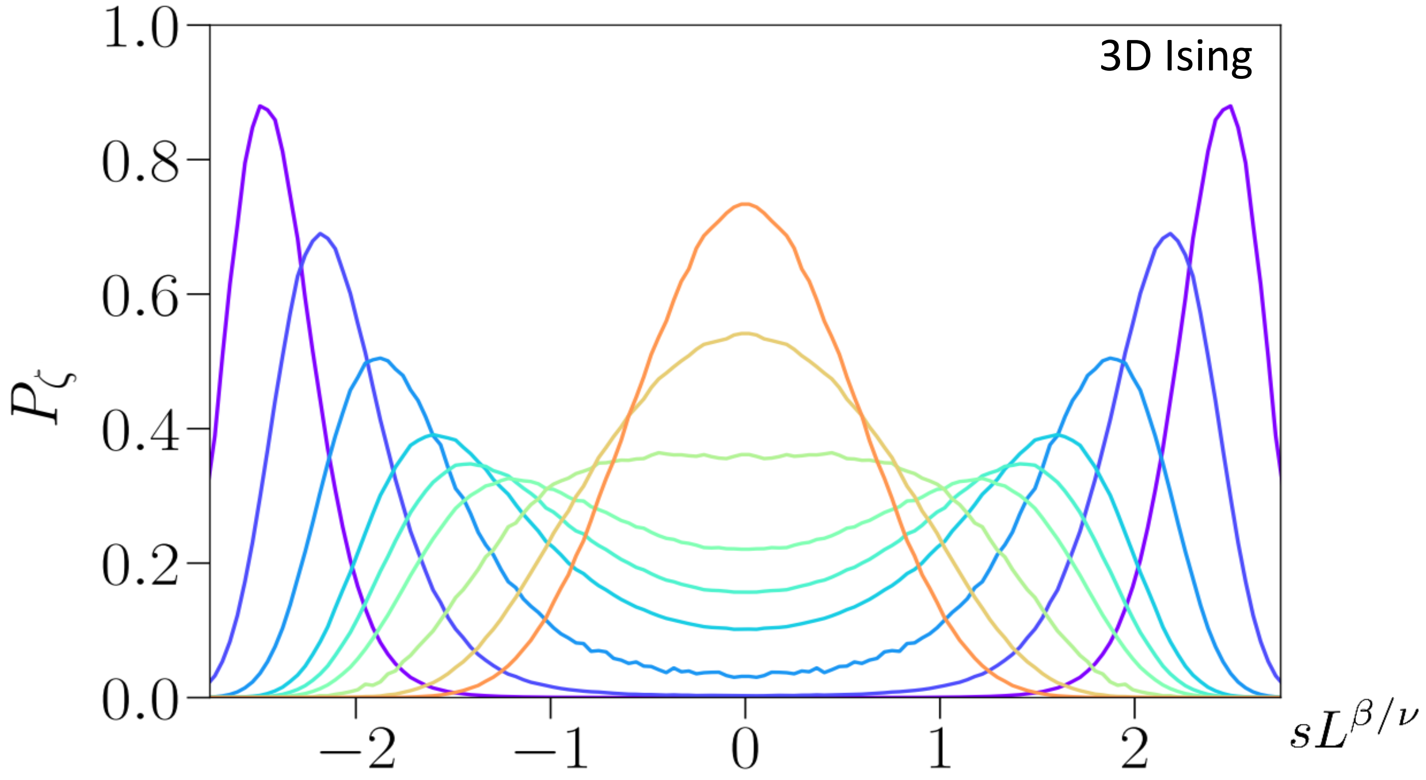
Remark: explicit rigorous calculations of non-trivial critical PDF for hierarchical model (Sinai, Bleher, etc '70s and '80s)

Generalized CLT from blocking?

Expectation $P_* \propto e^{-H_*}$ computed at 1-loop Bruce '81, Domb and Chen '96, Rudnik et al. '98, Sahu et al. '24

Problems: - H_* depends on blocking procedure (RG scheme)

- family of probability depending on $\zeta = L/\xi$, $L, \xi \to \infty$



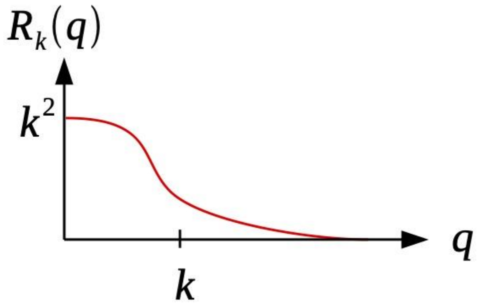
Functional RG

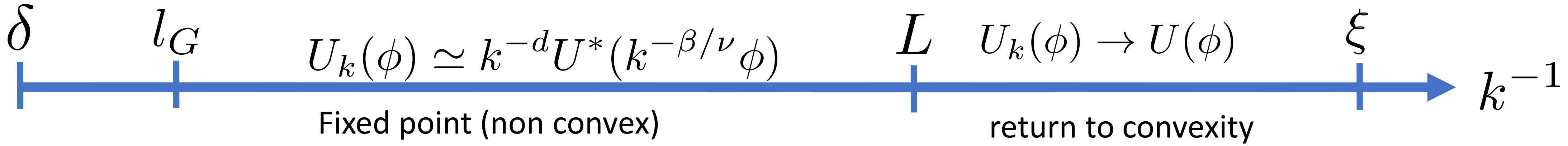
Scale-dependent Gibbs energy
(Wetterich '93)

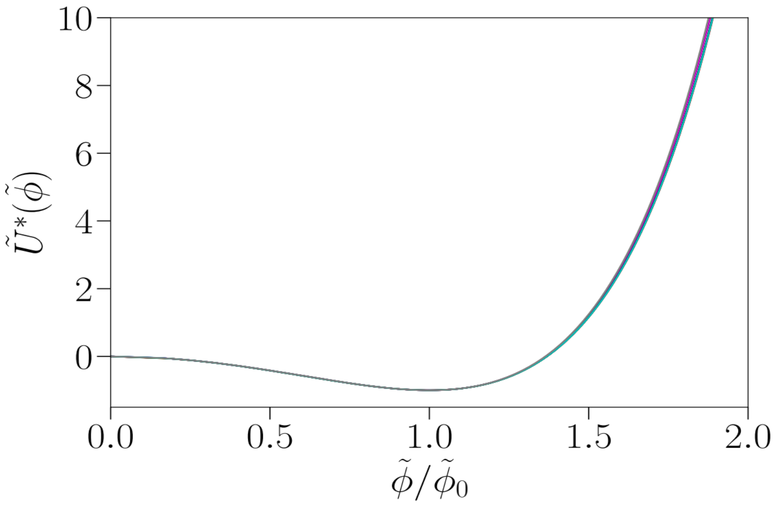
$$e^{-\Gamma_k[\phi]} = \int \mathcal{D}\varphi e^{-H[\varphi]-\frac{1}{2}(\phi-\varphi).R_k.(\phi-\varphi)+\frac{\delta\Gamma_k}{\delta\phi}.(\varphi-\phi)}$$

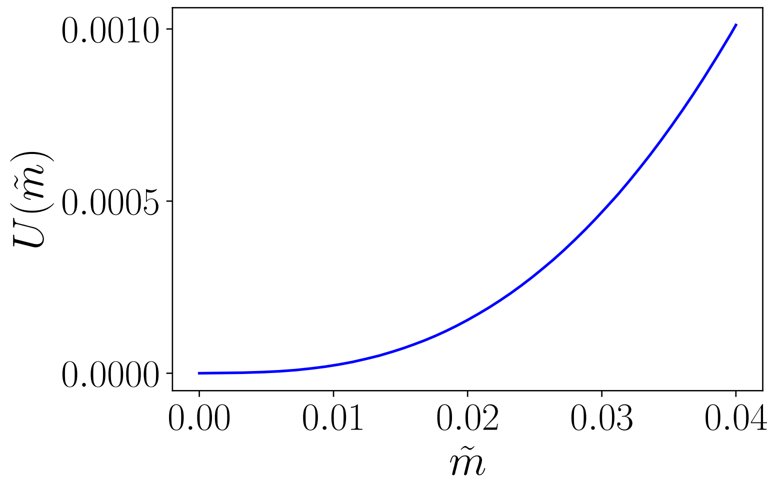
$$\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left(\partial_k R_k(\Gamma_k^{(2)}+R_k)^{-1}\right)$$

Generalizes trivially to finite size (periodic BC)
see also Fister & Pawlowski '15









Functional for the rate function

Define a "flowing" probability distribution and rate function

$$P_k(\boldsymbol{s}) = \exp(-L^d I_k(\boldsymbol{s})) = \int D\varphi\,\delta\left(\boldsymbol{s} - L^{-d}\int_x \varphi\right)\exp(-H(\varphi) - 1/2\varphi.R_k.\varphi)$$

Problem: flow of the rate function is not closed

Define a (scale-dependent) "constraint effective action"

$$\exp(-\hat{\Gamma}_k[\phi]) = \int D\varphi\exp\left(-H(\varphi) - 1/2(\phi-\varphi).\hat{R}_k.(\phi-\varphi) + \frac{\delta\hat{\Gamma}_k}{\delta\phi}.(\varphi-\phi)\right)$$

with $\hat{R}_k(q) = \begin{cases}\infty, & \text{if } q = 0,\\ R_k(q), & \text{if } q > 0.\end{cases}$

Flow equation of $\hat{\Gamma}_k[\phi]$ closed and $I_k(\boldsymbol{s}) = \hat{\Gamma}_k[\phi = \boldsymbol{s}]$

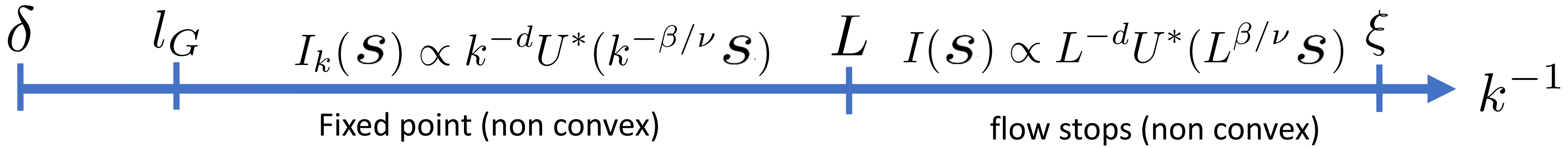
see Felix Rose's talk at 3:30, Parallel session B

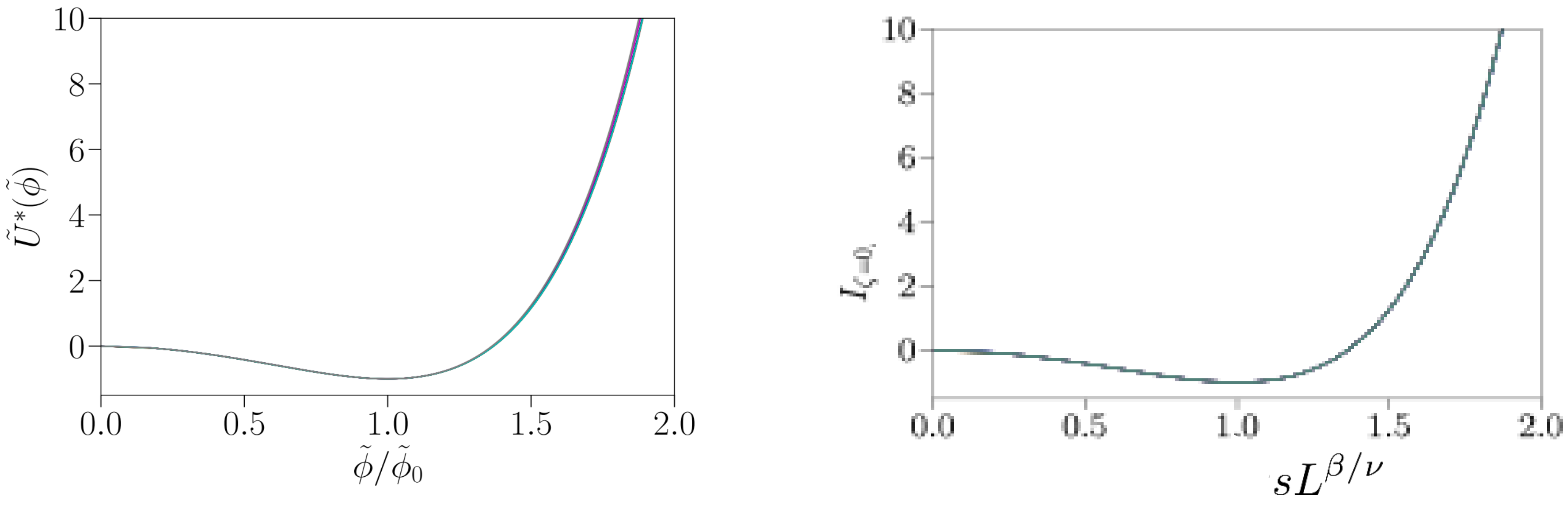
Balog et al '22

Flow equation for the rate function

Simple approximation: "Local Potential Approximation"

$$\partial_k I_k(\boldsymbol{s}) = \frac{1}{2L^d} \sum_{q\neq 0} \frac{\partial_k \hat{R}_k(q)}{q^2 + \hat{R}_k(q) + I_k''(\boldsymbol{s})}$$

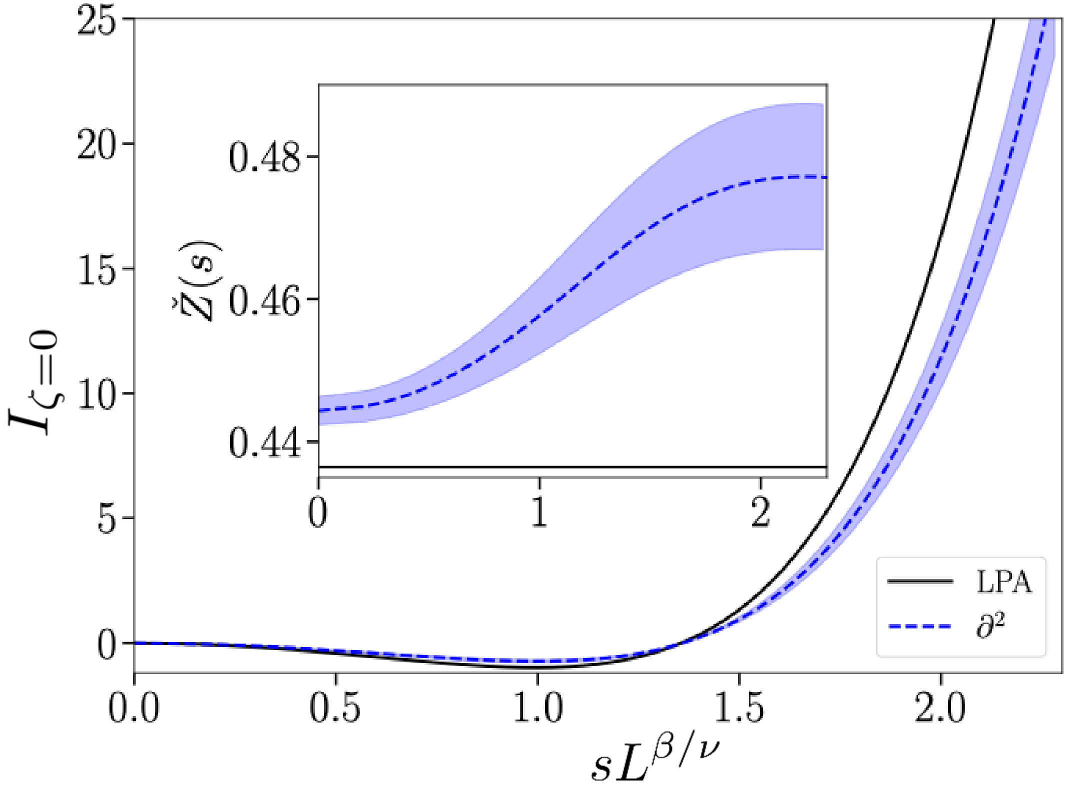




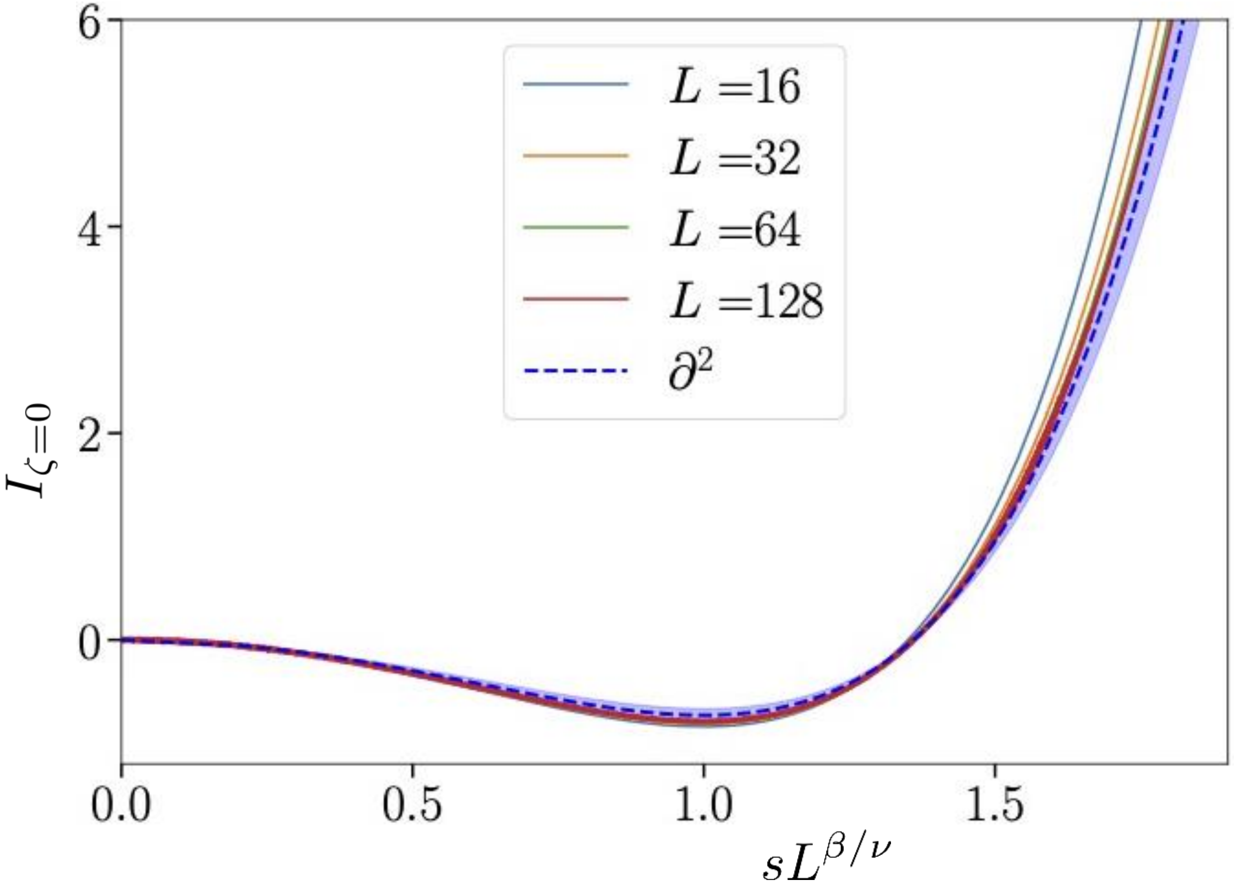
Balog et al '22

Convergence and comparison to Monte Carlo simulations

Possibility to do Derivative Expansion
(subtle, see F. Rose's talk)



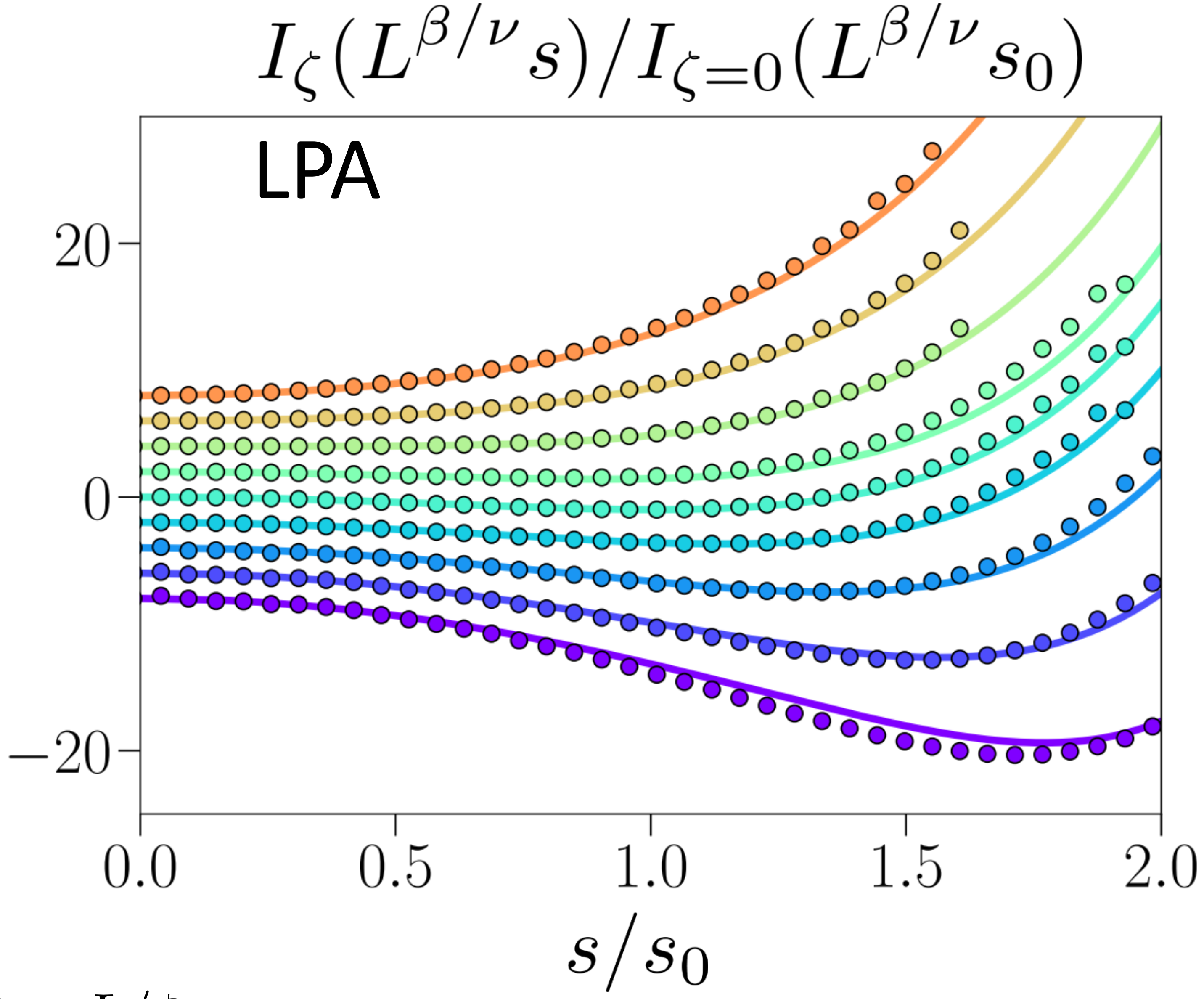
Comparison to MC (no fitting parameter)



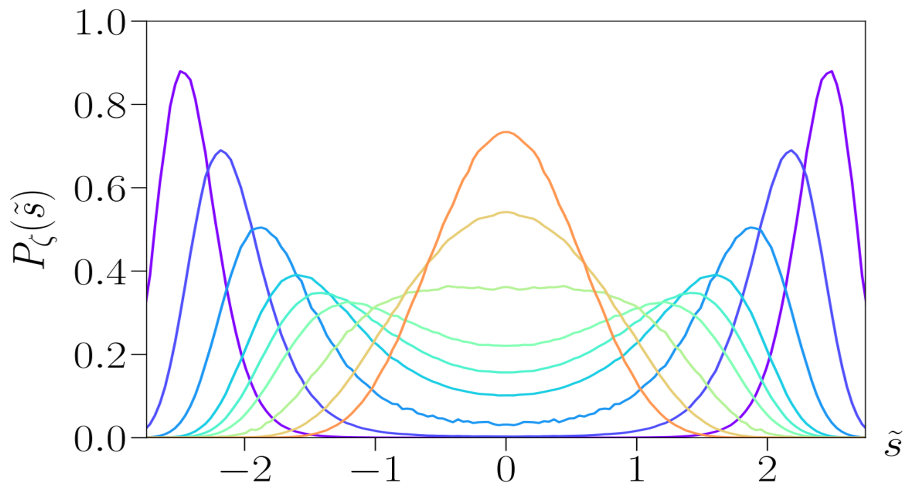
Error bar estimated following Balog et al '19 & De Polsi et al '20

Rose et al '24

Family of rate functions



$\zeta = L/\xi$



Shape of rate function at LPA correct for all $\boldsymbol{\zeta}$ up to global constant

(true also at 1-loop Sahu et al '24)

Balog et al '22

Critical rare event

Tail of distribution given by tail of rate function $I(s) \sim U(m = s) \sim s^{\delta+1}$

$$P_N(\hat{S} = Ns) \simeq \sqrt{\frac{NI''(s)}{2\pi}} \exp\left(-NI(s)\right) \quad \Rightarrow \quad P_L(s) \sim s^{\frac{\delta-1}{2}} e^{-L^d a\, s^{\delta+1}}$$

Fisher ‘66, Tsypin ’94, Bruce ‘95

Universal Large Deviations in the regime $L^{-\beta/\nu} \ll s \ll 1$

Can be generalized to $O(n)$ model $P_L(\mathbf{s}) \sim |\mathbf{s}|^{\psi} e^{-L^d a\, |\mathbf{s}|^{\delta+1}}$ $\qquad \psi = n\frac{\delta-1}{2}$

Power law prefactor observed: MC Ising 3D, hierarchical model, large n, FRG LPA (n=1,2,3)

Balog et al ‘24

Non-universal large deviations and finite size correction

For $s \sim 1$, proba has to be non-universal: transition from universal to non-universal rare events?

Correction to scaling induced by finite size (established explicitely in large n)

$$I(s) = L^{-d}\tilde{I}\left(sL^{\beta/\nu}\right) + \sum_k a_k L^{-d-\omega_k}\delta\tilde{I}_k\left(sL^{\beta/\nu}\right)$$

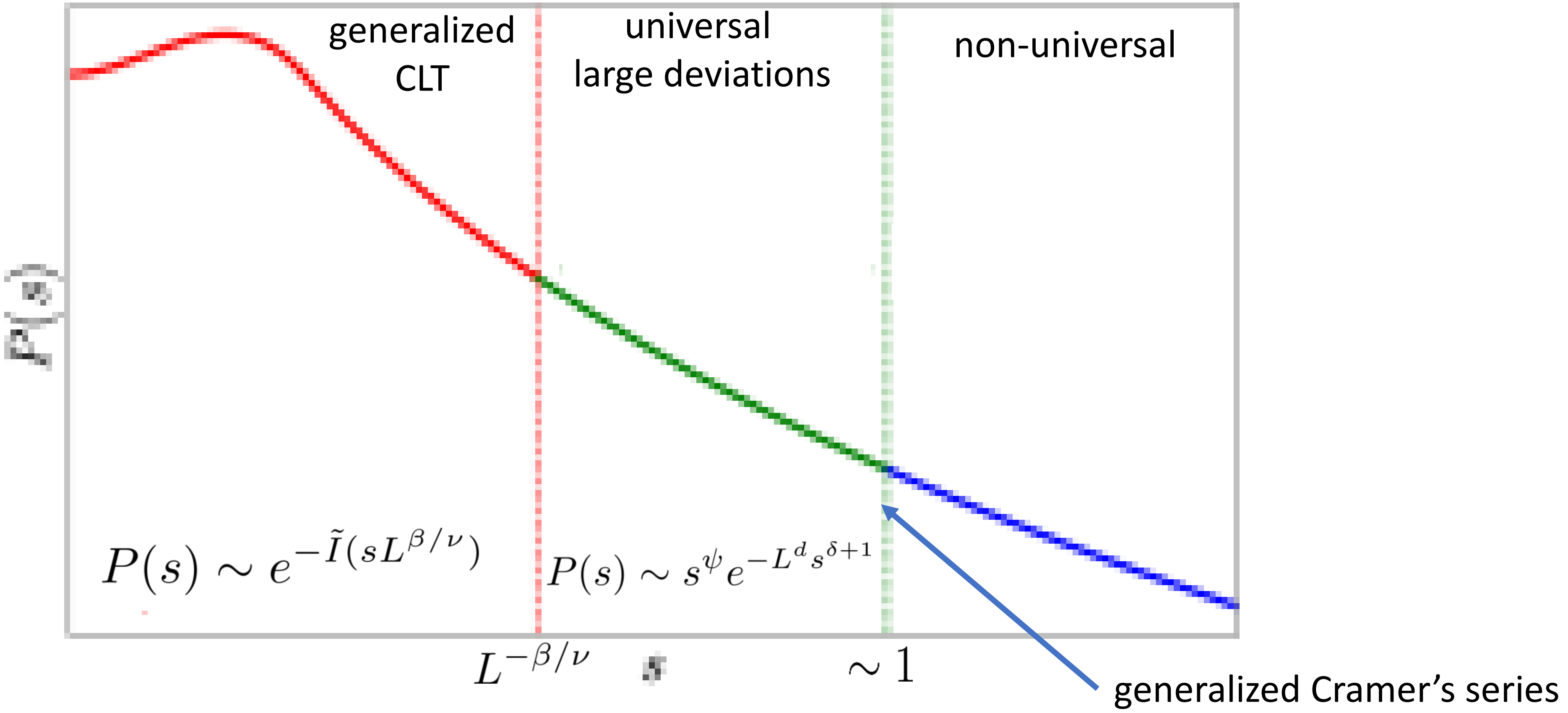
non-universal amplitudes

universal critical exponents (irrelevant)

universal irrelevant perturbations

Finite-size corrections-to-scaling generalizes Cramer's series!

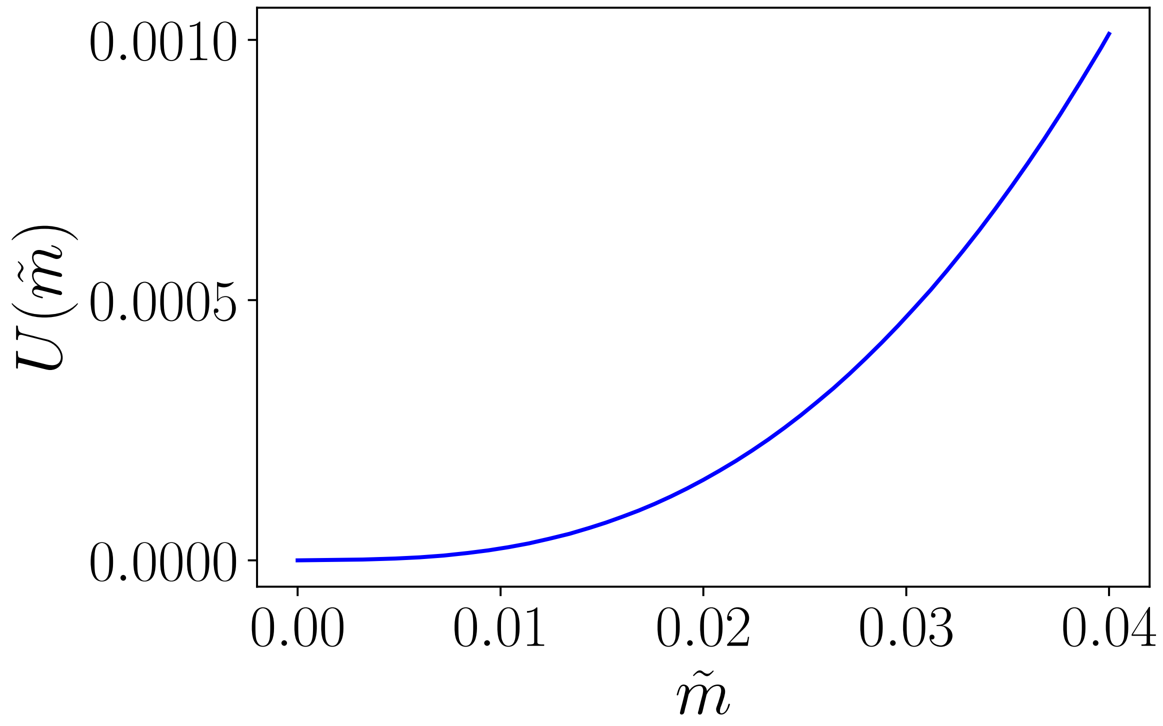
Conclusion



Perspectives: low-temperature phase, effects of domain walls, instantons, etc (cf Ivan Balog's talk at 3:00, parallel session B)

Template

Return to convexity

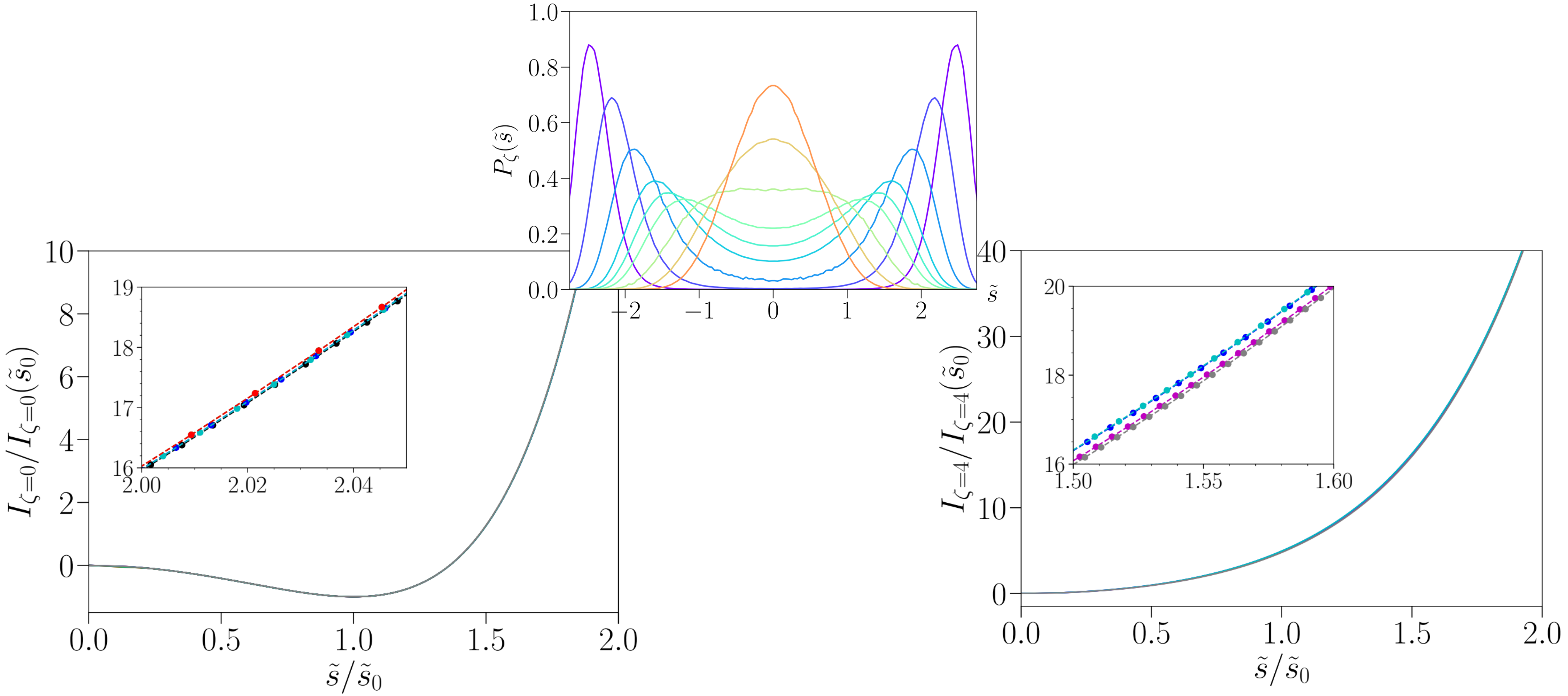


Universality of critical PDF: rate function

Choose $T(L) \to T_c$, such that $\zeta = L/\xi(T(L))$ is constant.

$$\tilde{m} = mL^{\beta/\nu}$$

$$P(m) \propto e^{-L^d I(m,\xi,L)} = e^{-I^*_\zeta(\tilde{m})}$$



Probability distribution and RG fixed points

Direct connection between the probability distribution and RG fixed point quantities we compute?

$$H(\varphi) = \int d^dx \, ((\nabla\varphi)^2 + r\varphi^2 + g\varphi^4)$$

Wilson's RG

$$H^*(\varphi) = \int d^dx \, ((\nabla\varphi)^2 + r^*\varphi^2 + g^*\varphi^4 + \cdots)$$

Fixed point coupling constants

Link between Wilson's RG and probability distribution:
Bruce '81, Domb and Chen '96, Rudnik et al. '98,…

$$P(s) \propto e^{-H^*(s)}$$???

Based on perturbative RG, inclusion of finite size ad hoc…

and fixed point couplings are scheme dependent!?!

Comparison fixed point vs rate function