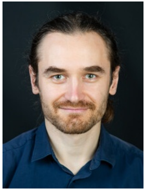
Efficient ~~quantum circuits~~ protocols for port-based teleportation via mixed Schur–Weyl duality

arXiv: 2312.03188, 2310.02252



Dmitry Grinko Adam Burchardt Māris Ozols

29 February 2024

QuSoft
Research Center for Quantum Software

University of Amsterdam

CWI
Research institute for mathematics & computer science in the Netherlands

Outline

1. Port-based teleportation

2. Overview of mixed Schur–Weyl duality

3. Gelfand–Tsetlin basis for partially transposed permutations

4. Mixed quantum Schur transform

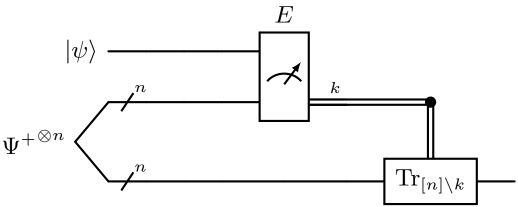
5. Efficient quantum circuits for port-based teleportation

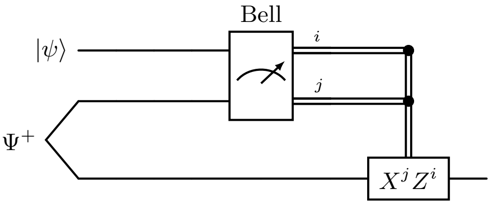
Credits

- Rene Allerstorfer
- Harry Buhrman
- Yanlin Chen
- Tudor Giurgica-Tiron
- Aram Harrow
- Hari Krovi
- Quynh Nguyen
- Florian Speelman
- Philip Verduyn Lunel
- John van de Wetering
- Adam Wills

Port-based teleportation

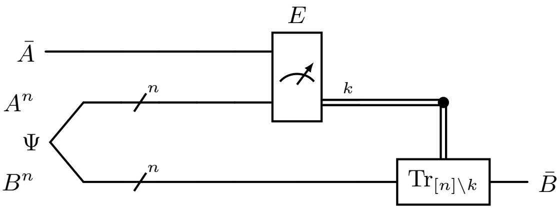
Port-based teleportation





- Introduced by Ishizaka and Hiroshima in 2008
- Bob does not need to apply a correction operation
- An example of *approximate universal quantum processor*
- Prior work:
  - Ishizaka, Hiroshima '08, '09
  - Beigi, König '11
  - Mozrzymas, Studziński, Strelchuk, Horodecki '17, '18
  - Christandl, Leditzky, Majenz, Smith, Speelman, Walter '18
  - Leditzky '20

Entanglement fidelity and success probability



- Terminology:
  - Ψ is a *resource state*
  - n *ports*
  - d is the *local dimension*

$$\mathcal{N}_{\bar{A}\to\bar{B}}(\rho) := \sum_{k=1}^{n} \mathrm{Tr}_{A^n \bar{A} B'_k}\Big[\Big((\sqrt{E_k})_{A^n\bar{A}} \otimes I_{B^n}\Big)(\Psi_{A^nB^n} \otimes \rho_{\bar{A}})\Big(\sqrt{E_k}_{A^n\bar{A}} \otimes I_{B^n}\Big)\Big],$$

$$F := \mathrm{Tr}\Big[\Phi^+_{\bar{B}R}(\mathcal{N}_{\bar{A}\to\bar{B}} \otimes I_R)\big[\Phi^+_{\bar{A}R}\big]\Big],$$

$$p_{\mathrm{succ}} := \mathrm{Tr}\big[\mathcal{N}_{\bar{A}\to\bar{B}}(I/d)\big]$$

Deterministic and Probabilistic PBT

Resource state	Protocol type	
	Deterministic inexact (dPBT)	Probabilistic exact (pPBT)
EPR	$F = 1 - O(1/n)$ $p_{\text{succ}} = 1$	$F/p_{\text{succ}} = 1$ $p_{\text{succ}} = 1 - O(1/\sqrt{n})$
Optimized	$F = 1 - O(1/n^2)$ $p_{\text{succ}} = 1$	$F/p_{\text{succ}} = 1$ $p_{\text{succ}} = 1 - O(1/n)$

- Optimal measurements are known. *Pretty good measurement* E (yellow) is the main ingredient.
- However, there were no known efficient implementations of these measurements prior to our work.
- Two ingredients are needed for a proper understanding:
  - *Mixed Schur–Weyl duality*
  - Representation theory of *partially transposed permutation matrix algebras*

Overview of mixed Schur–Weyl duality

Schur-Weyl duality

- $\mathcal{U}_n^d := \operatorname{span}_{\mathbb{C}}\{u^{\otimes n} : u \in \mathrm{U}_d\}$
- $\mathcal{A}_n^d := \psi(\mathbb{C}\mathrm{S}_n)$, where $\mathbb{C}\mathrm{S}_n$ is the *group algebra* of S_n and $\forall \sigma \in \mathrm{S}_n$:

$$\psi(\sigma)\big(|i_1\rangle \otimes \cdots \otimes |i_n\rangle\big) := \left|i_{\sigma^{-1}(1)}\right\rangle \otimes \cdots \otimes \left|i_{\sigma^{-1}(n)}\right\rangle .$$

- $\psi : \mathbb{C}\mathrm{S}_n \to \mathrm{End}((\mathbb{C}^d)^{\otimes n})$ is the *tensor representation* of $\mathbb{C}\mathrm{S}_n$
- $\mathcal{C}(\mathcal{A}, V) := \{B \in \mathrm{End}(V) : [A, B] = 0$ for every $A \in \mathcal{A}\}$

Theorem (Schur-Weyl duality)

- $\mathcal{U}_n^d$ *is the centraliser algebra of* $\mathcal{A}_n^d$ *in* $\mathrm{End}((\mathbb{C}^d)^{\otimes n})$ *and vice versa:*

$$\mathcal{U}_n^d = \mathcal{C}(\mathcal{A}_n^d, (\mathbb{C}^d)^{\otimes n}), \qquad \mathcal{A}_n^d = \mathcal{C}(\mathcal{U}_n^d, (\mathbb{C}^d)^{\otimes n}).$$

Moreover, when $d \geqslant n$ *the representation* ψ *is faithful, i.e.,* $\mathcal{A}_n^d \cong \mathbb{C}\mathrm{S}_n$.

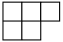
- $\exists$ *a Schur transform* U_{Sch} *such that for every* $\sigma \in \mathbb{C}\mathrm{S}_n$ *and* $u \in \mathrm{U}_d$*:*

$$U_{\mathrm{Sch}}\, \phi(u)\, U_{\mathrm{Sch}}^\dagger = \bigoplus_{\lambda \in \widehat{\mathcal{A}}_n^d} I_\lambda \otimes \phi_\lambda(u), \qquad U_{\mathrm{Sch}}\, \psi(\sigma)\, U_{\mathrm{Sch}}^\dagger = \bigoplus_{\lambda \in \widehat{\mathcal{A}}_n^d} \psi_\lambda(\sigma) \otimes I_\lambda$$

where $\widehat{\mathcal{A}}_n^d$ *is the set of irreducible representations of* $\mathcal{A}_n^d$.

Young diagrams. Notation

- $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition of n, written as $\lambda \vdash n$, if $\lambda_1 \geqslant \ldots \geqslant \lambda_k \geqslant 0$ and $\sum_{i=1}^{k} \lambda_i = n$.
- λ is represented by a Young diagram. For example, $\lambda = (3, 2, 0)$ is



- The *length* is defined as $\ell(\lambda) := \max\{k \mid \lambda_k > 0\}$.
- The set of irreducible representations (irreps) of $\mathbb{C}\mathrm{S}_n$ is indexed by Young diagrams:

$$\widehat{\mathbb{C}\mathrm{S}}_n = \{\lambda \vdash n\}$$

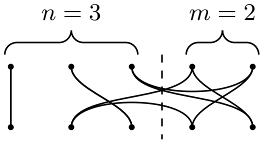
- The set of irreps of $\mathcal{A}_n^d = \psi(\mathbb{C}\mathrm{S}_n)$ is indexed by Young diagrams with bounded length:

$$\widehat{\mathcal{A}}_n^d = \{\lambda \vdash n \mid \ell(\lambda) \leqslant d\}$$

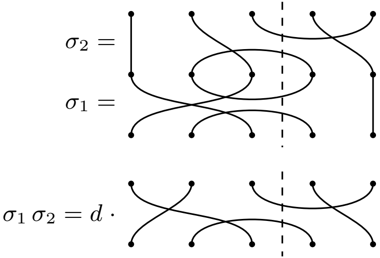
- We write $\lambda \vdash_d n$ to indicate that $\ell(\lambda) \leqslant d$.
- Set $\mathrm{AC}(\lambda)$ of *addable cells* a of λ: $\lambda \cup a$ is a valid partition.
- $\mathrm{AC}_d(\lambda) := \{a \in \mathrm{AC}(\lambda) \mid \ell(\lambda \cup a) \leqslant d\}$

Partially transposed permutations

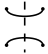
- *Walled Brauer algebra* $\mathcal{B}^d_{n,m}$



- Multiplication in $\mathcal{B}^d_{n,m}$:

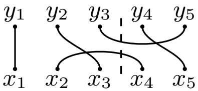


- Tensor representation $\psi : \mathcal{B}^d_{n,m} \to \mathrm{End}\big((\mathbb{C}^d)^{\otimes n+m}\big)$
- Transposition and contraction ($d = 2$):

$\psi\Big($  $\Big) = \begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}, \quad \psi\Big($  $\Big) = \begin{pmatrix} 1&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&1 \end{pmatrix}$

- General diagram:

$\langle y_1 \dots y_5 |\, \psi\Big($  $\Big)\, | x_1 \dots x_5 \rangle =$

$=$ 

$$= \delta_{x_1 y_1} \delta_{x_2 x_4} \delta_{x_3 y_2} \delta_{x_5 y_4} \delta_{y_3 y_5}$$

- Matrix algebra of *partially transposed permutations*:

$$\mathcal{A}^d_{n,m} := \psi(\mathcal{B}^d_{n,m})$$

- $\mathcal{A}^d_{n,m}$ is generated by transpositions $\sigma_i = (i, i+1)$, $i \neq n$ and the contraction σ_n.

Mixed Schur-Weyl duality

- Consider the map $\phi(u) := u^{\otimes n} \otimes \bar{u}^{\otimes m}$ for every $u \in \mathrm{U}_d$

Theorem (Koike 1989, Benkart et al. 1994)

∃ a mixed Schur transform $U_{\text{Sch}} \equiv U_{\text{Sch}}(n, m)$ such that for every $\sigma \in \mathcal{B}^d_{n,m}$ and $u \in \mathrm{U}_d$:

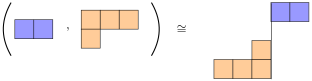
$$U_{\text{Sch}}\, \phi(u)\, U^{\dagger}_{\text{Sch}} = \bigoplus_{\lambda \in \widehat{\mathcal{A}}^d_{n,m}} I_\lambda \otimes \phi_\lambda(u), \qquad U_{\text{Sch}}\, \psi(\sigma)\, U^{\dagger}_{\text{Sch}} = \bigoplus_{\lambda \in \widehat{\mathcal{A}}^d_{n,m}} \psi_\lambda(\sigma) \otimes I_\lambda$$

where $\widehat{\mathcal{A}}^d_{n,m}$ is the set of irreducible representations of $\mathcal{A}^d_{n,m}$.
When $d \geqslant n + m$ the representation ψ is faithful, i.e., $\mathcal{A}^d_{n,m} \cong \mathcal{B}^d_{n,m}$.

- The irreps of $\mathcal{A}^d_{n,m}$ are labelled by pairs of Young diagrams (λ_l, λ_r). More formally:

$$\widehat{\mathcal{A}}^d_{n,m} := \Big\{\lambda = (\lambda_l, \lambda_r) : 0 \leqslant k \leqslant \min(n, m),\ \lambda_l \vdash n - k,\ \lambda_r \vdash m - k,\ \ell(\lambda_l) + \ell(\lambda_r) \leqslant d\Big\}.$$

- A pair $\lambda = (\lambda_l, \lambda_r)$ can be thought of as a *staircase* $\lambda = (\lambda_1, \dots, \lambda_d)$:



- We recover original Schur-Weyl duality when $n = 0$ or $m = 0$.
- What are $U_{\text{Sch}}(n, m)$ and $\psi_\lambda(\sigma)$?

Gelfand–Tsetlin basis for partially transposed permutations

Gelfand–Tsetlin basis

Definition

A family $(\mathcal{A}_0, \ldots, \mathcal{A}_n = \mathcal{A})$ of finite-dimensional semisimple algebras over $\mathbb{C}$ is *multiplicity-free* if:

(a) $\mathcal{A}_0 \cong \mathbb{C}$.

(b) For each k, there is a unity-preserving algebra embedding $\mathcal{A}_k \hookrightarrow \mathcal{A}_{k+1}$.

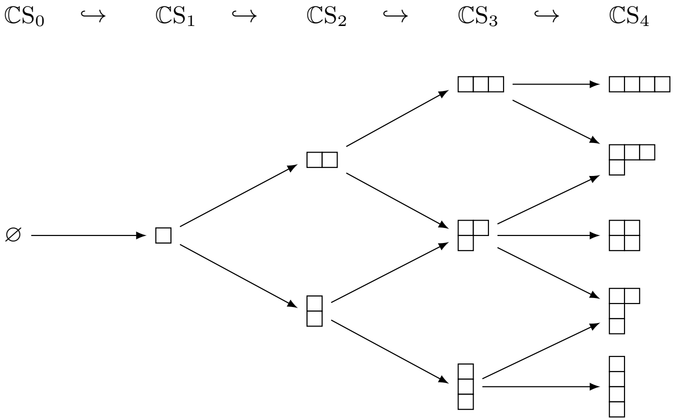
(c) The restriction of an $\mathcal{A}_k$ irrep to $\mathcal{A}_{k-1}$ is isomorphic to a direct sum of different $\mathcal{A}_{k-1}$ irreps.

- Repeated restriction produces a canonical *Gelfand–Tsetlin* basis of each $\mathcal{A}_n$ irrep V_λ:

$$\mathrm{Res}^{\mathcal{A}_1}_{\mathcal{A}_0} \ldots \mathrm{Res}^{\mathcal{A}_n}_{\mathcal{A}_{n-1}} V_\lambda = \bigoplus_{T \in \mathrm{Paths}(\lambda, \mathscr{B})} V_T,$$

- This basis is labeled by paths $T = (T^0, T^1, \ldots, T^n)$ in the *Bratteli diagram* $\mathscr{B}$.
- For $\mathbb{C}S_n$:
  - the Gelfand–Tsetlin basis is the Young–Yamanouchi basis,
  - the Bratteli diagram is the Young graph.

Example: Bratteli diagram for $\mathbb{C}S_n$ a.k.a. Young graph



- *Path* $\cong$ *standard Young tableau* $\cong$ *Yamanouchi word*. For example,

$$T = \left(\varnothing, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,2}\right) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} = (1,2,1,2)$$

- $d_\lambda := |\mathrm{Paths}(\lambda)|$.

Example: Gelfand–Tsetlin basis for $\mathbb{C}S_n$ a.k.a. Young-Yamanouchi basis

- The *content* of cell $u = (i, j)$ is $\mathrm{cont}(u) := j - i$.

0	1	2
−1	0	

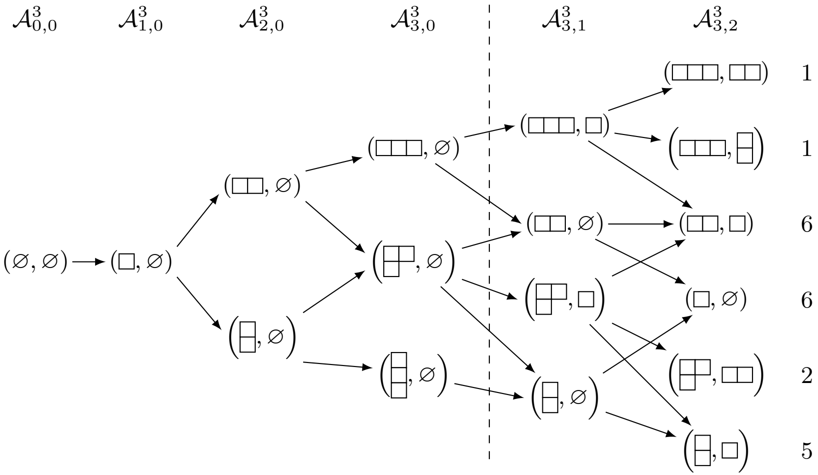
- *Content* of i in standard Young tableau T is defined as $\mathrm{cont}_i(T) := \mathrm{cont}(T^i \backslash T^{i-1})$.
- The *axial distance* between i and $i+1$ in T is $r_i(T) := \mathrm{cont}_{i+1}(T) - \mathrm{cont}_i(T)$.

Theorem (Young 1931, Yamanouchi 1936)

Given a generator σ_i of $\mathbb{C}S_n$, $i = 1, \dots, n-1$, the matrix $\psi_\lambda(\sigma_i)$ acts on the Gelfand–Tsetlin basis vectors $|T\rangle$, $T \in \mathrm{Paths}(\lambda, \mathscr{B})$ of an irrep $\lambda \in \widehat{\mathbb{C}S}_n$ as follows:

$$\psi_\lambda(\sigma_i)\,|T\rangle = \frac{1}{r_i(T)}\,|T\rangle + \sqrt{1 - \frac{1}{r_i(T)^2}}\,|\sigma_i T\rangle\,,$$

Example: Bratteli diagram for $\mathcal{A}^3_{3,2}$



Gelfand–Tsetlin basis for partially transposed permutations

Theorem (G., Burchardt, Ozols)

Given a generator σ_i *of* $\mathcal{A}^d_{n,m}$, $i = 1, \ldots, n+m-1$, *the matrix* $\psi_\lambda(\sigma_i)$ *acts on the Gelfand–Tsetlin basis vectors* $|T\rangle$ *with* $T \in \mathrm{Paths}(\lambda)$ *of an irrep* $\lambda \in \widehat{\mathcal{A}}^d_{n,m}$ *as follows:*

$$\psi_\lambda(\sigma_i)\,|T\rangle = \frac{1}{\tilde{r}_i(T)}\,|T\rangle + \sqrt{1 - \frac{1}{\tilde{r}_i(T)^2}}\,|\sigma_i T\rangle\,, \qquad \text{for } i \neq n,$$

$$\psi_\lambda(\sigma_n)\,|T\rangle = c(T)\,|v_T\rangle\,, \quad |v_T\rangle := \sum_{T' \in \mathcal{M}(T)} c(T')\,|T'\rangle\,, \qquad c(T) = \sqrt{\frac{m_{T^n}}{m_{T^{n-1}}}},$$

where m_{T^n} *is the dimension of unitary irrep* T^n.

- We recover Young-Yamanouchi basis when $n = 0$ or $m = 0$.

Example: $\mathcal{A}^{3}_{3,2}$

	σ_1	σ_2	σ_3	σ_4
$((3),(2))$	(1)	(1)	(0)	(1)
$((3),(1,1))$	(1)	(1)	(0)	(-1)
$((2),(1))$	$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} \frac12&0&\frac{\sqrt3}{2}&0&0&0\\ 0&\frac12&0&\frac{\sqrt3}{2}&0&0\\ \frac{\sqrt3}{2}&0&-\frac12&0&0&0\\ 0&\frac{\sqrt3}{2}&0&-\frac12&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&\frac43&0&\frac{2\sqrt5}{3}&0\\ 0&0&0&0&0&0\\ 0&0&\frac{2\sqrt5}{3}&0&\frac53&0\\ 0&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} \frac12&\frac{\sqrt3}{2}&0&0&0&0\\ \frac{\sqrt3}{2}&-\frac12&0&0&0&0\\ 0&0&\frac12&\frac{\sqrt3}{2}&0&0\\ 0&0&\frac{\sqrt3}{2}&-\frac12&0&0\\ 0&0&0&0&\frac15&\frac{2\sqrt6}{5}\\ 0&0&0&0&\frac{2\sqrt6}{5}&-\frac15 \end{pmatrix}$
$((1),\varnothing)$	$\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} -1&0&0&0&0&0\\ 0&\frac12&0&\frac{\sqrt3}{2}&0&0\\ 0&0&\frac12&0&\frac{\sqrt3}{2}&0\\ 0&\frac{\sqrt3}{2}&0&-\frac12&0&0\\ 0&0&\frac{\sqrt3}{2}&0&-\frac12&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} \frac13&\frac{2\sqrt2}{3}&0&0&0&0\\ \frac{2\sqrt2}{3}&\frac83&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&\frac43&\frac{2\sqrt5}{3}\\ 0&0&0&0&\frac{2\sqrt5}{3}&\frac53 \end{pmatrix}$	$\begin{pmatrix} -1&0&0&0&0&0\\ 0&\frac12&\frac{\sqrt3}{2}&0&0&0\\ 0&\frac{\sqrt3}{2}&-\frac12&0&0&0\\ 0&0&0&\frac12&\frac{\sqrt3}{2}&0\\ 0&0&0&\frac{\sqrt3}{2}&-\frac12&0\\ 0&0&0&0&0&1 \end{pmatrix}$
$((2,1),(2))$	$\begin{pmatrix} -1&0\\ 0&1 \end{pmatrix}$	$\begin{pmatrix} \frac12&\frac{\sqrt3}{2}\\ \frac{\sqrt3}{2}&-\frac12 \end{pmatrix}$	$\begin{pmatrix} 0&0\\ 0&0 \end{pmatrix}$	$\begin{pmatrix} 1&0\\ 0&1 \end{pmatrix}$
$((1,1),(1))$	$\begin{pmatrix} -1&0&0&0&0\\ 0&-1&0&0&0\\ 0&0&-1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} -1&0&0&0&0\\ 0&\frac12&0&\frac{\sqrt3}{2}&0\\ 0&0&\frac12&0&\frac{\sqrt3}{2}\\ 0&\frac{\sqrt3}{2}&0&-\frac12&0\\ 0&0&\frac{\sqrt3}{2}&0&-\frac12 \end{pmatrix}$	$\begin{pmatrix} \frac13&\frac{2\sqrt2}{3}&0&0&0\\ \frac{2\sqrt2}{3}&\frac83&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0\\ 0&\frac14&\frac{\sqrt{15}}{4}&0&0\\ 0&\frac{\sqrt{15}}{4}&-\frac14&0&0\\ 0&0&0&\frac14&\frac{\sqrt{15}}{4}\\ 0&0&0&\frac{\sqrt{15}}{4}&-\frac14 \end{pmatrix}$

Mixed quantum Schur transform

Our result

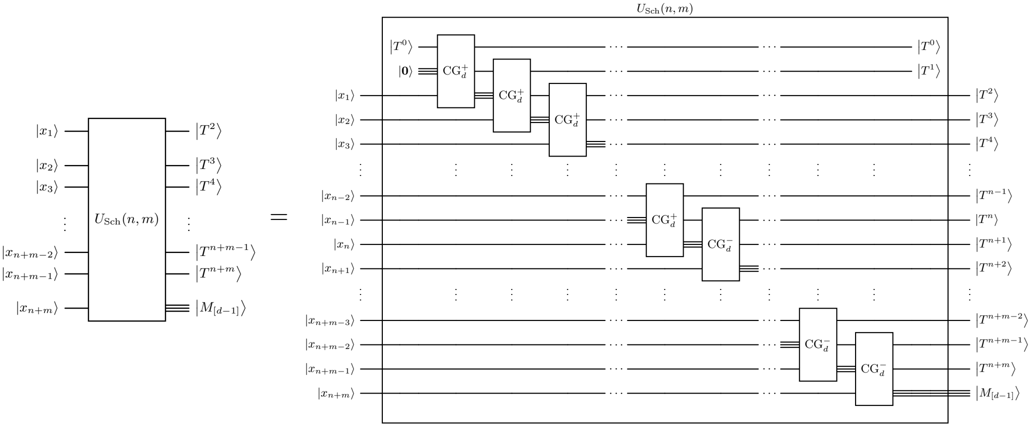
Theorem (G., Burchardt, Ozols '23; Nguyen '23)

The mixed quantum Schur transform has a quantum circuit with $\widetilde{O}\big((n+m)d^4\big)$ *gate and depth complexities, where* d *is the local dimension, and* n *and* m *are the parameters of* $\mathcal{A}^d_{n,m}$*.*
Two different encodings of the Gelfand–Tsetlin basis lead to the following space complexities:

- *standard encoding:* $\widetilde{O}((n+m+d)d\log(n+m))$,
- *Yamanouchi encoding:* $\widetilde{O}\big(d^2\log(n+m)\big)$.

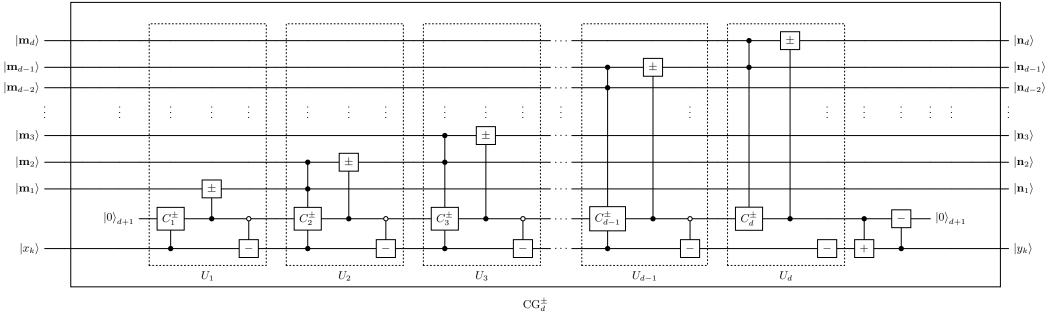
- Based on the original Schur transform from [Bacon, Chuang, Harrow 2005] by using *dual Clebsch–Gordan transforms*.

Mixed quantum Schur transform circuit



Recursion in $n+m$

Clebsch–Gordan transforms



Recursion in d

Efficient quantum circuits for port-based teleportation

Port-based teleportation. Main result

Theorem (G., Burchardt, Ozols '23)

The measurements for dPBT and pPBT protocols have gate complexities $\widetilde{O}(n^2 d^4)$ *and the following time and space complexities:*

1. *standard encoding:* $\widetilde{O}(nd^4)$ *time and* $\widetilde{O}((n+d)d\log(n))$ *space,*
2. *Yamanouchi encoding:* $\widetilde{O}(n^2 d^4)$ *time and* $\widetilde{O}(d^2\log(n))$ *space.*

- Independent work [Jiani Fei, Sydney Timmerman and Patrick Hayden 2023] describes a different approach to implementation of deterministic PBT via block encoding techniques
- Independent work [Adam Wills, Min-Hsiu Hsieh and Sergii Strelchuk 2023] describes qubit PBT constructions via block encoding techniques

Port-based teleportation

Resource state	Protocol type	
	Deterministic inexact (dPBT)	Probabilistic exact (pPBT)
EPR	$F = 1 - O(1/n)$ $p_{\text{succ}} = 1$	$F/p_{\text{succ}} = 1$ $p_{\text{succ}} = 1 - O(1/\sqrt{n})$
Optimized	$F = 1 - O(1/n^2)$ $p_{\text{succ}} = 1$	$F/p_{\text{succ}} = 1$ $p_{\text{succ}} = 1 - O(1/n)$

- Pretty good measurement $E = \{E_i\}_{i=0}^n$ (yellow) is given for every $k \in [n]$:

$$E_k := \rho^{-1/2} \rho_k \rho^{-1/2}, \quad \rho_k := \pi^k \sigma_n \pi^{-k}, \quad \rho := \sum_{k=1}^{n} \rho_k, \quad E_0 := I - \sum_{k=1}^{n} E_k,$$

  where $\pi \in \mathcal{A}_{n,1}^d$ is the cyclic shift on first n systems and $\sigma_n \in \mathcal{A}_{n,1}^d$ is the contraction generator.
- We can rewrite E in the Gelfand–Tsetlin basis and construct a Naimark dilation explicitly.

Naimark dilation

- The effect E_n in the Gelfand–Tsetlin basis of every irrep $(\lambda, \varnothing) \in \widehat{\mathcal{A}}^d_{n,1}$ for $\lambda \vdash_d n-1$ is

$$\psi_{(\lambda,\varnothing)}(E_n) = \sum_{S \in \mathrm{Paths}_{n-1}(\lambda,\mathscr{B})} |w_{S,\lambda}\rangle\langle w_{S,\lambda}|$$

$$|w_{S,\lambda}\rangle := \sum_{a \in \mathrm{AC}_d(\lambda)} \sqrt{\frac{d_{\lambda\cup a}}{n \cdot d_\lambda}}\, |S \circ (\lambda \cup a) \circ (\lambda, \varnothing)\rangle$$

$$\| |w_{S,\lambda}\rangle \|^2 = \sum_{a \in \mathrm{AC}_d(\lambda)} \frac{d_{\lambda\cup a}}{n \cdot d_\lambda}$$

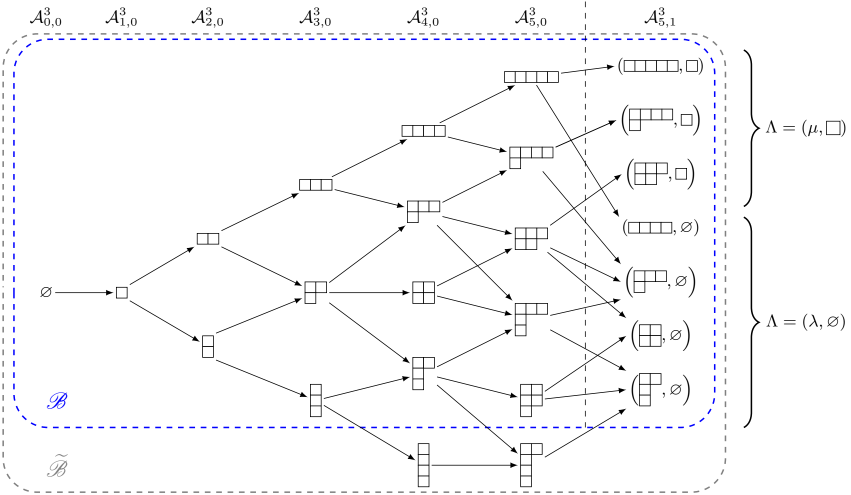
- Key fact: for every $\lambda \vdash n-1$ in the Young lattice the following relation holds:

$$n \cdot d_\lambda = \sum_{a \in \mathrm{AC}(\lambda)} d_{\lambda \cup a}$$

- Therefore:

$$\| |w_{S,\lambda}\rangle \|^2 = \begin{cases} 1 & \text{if } \ell(\lambda) < d \\ 1 - \frac{d_{\lambda\cup(d+1,1)}}{n \cdot d_\lambda} & \text{if } \ell(\lambda) = d \end{cases}$$

Naimark dilation



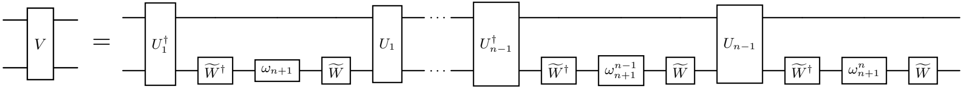
The Bratteli diagram $\mathscr{B}$ and the extended Bratteli diagram $\widetilde{\mathscr{B}}$ associated with the algebra $\mathcal{A}^3_{5,1}$

Implementation of the Naimark dilated PVM

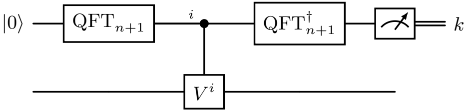
- POVM E is dilated to $\Pi = \{\Pi_k\}_{k=0}^n$:

$$\Pi_k = U_k \Pi_n U_k^\dagger \text{ for every } k \in \{1, \dots, n-1\}$$
$$\Pi_n = I \otimes \left(\widetilde{W} \left|0\right\rangle\!\left\langle 0\right| \widetilde{W}^\dagger \right)$$

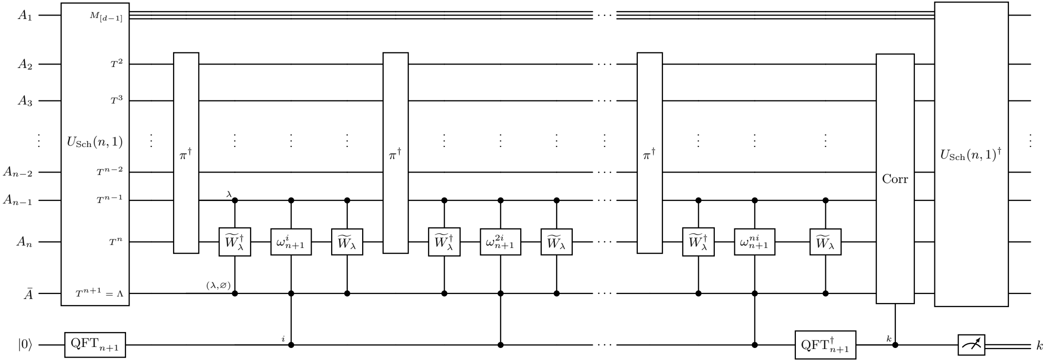
- U_k and $\widetilde{W}$ are easy-to-implement unitaries. In fact, $U_k = \pi^k$.
- Implementation of $V := \sum_{k=0}^n \omega_{n+1}^k \Pi_k$ is easy:



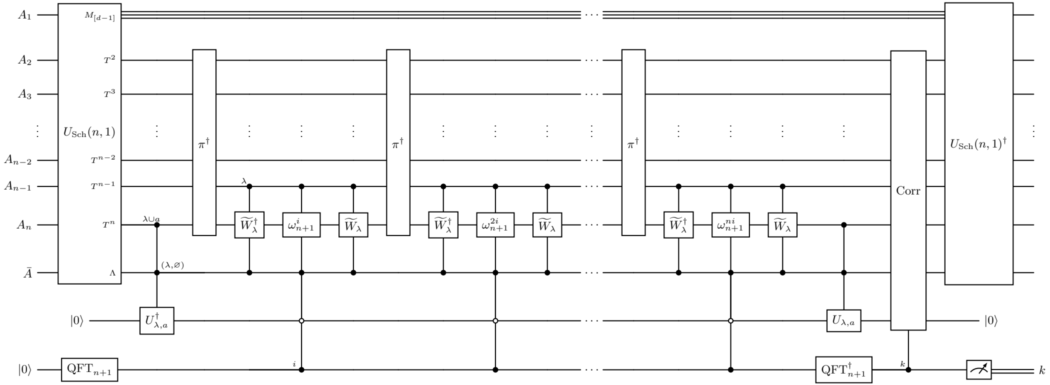
- Implementation of V^i is trivial. Now run the phase estimation circuit:



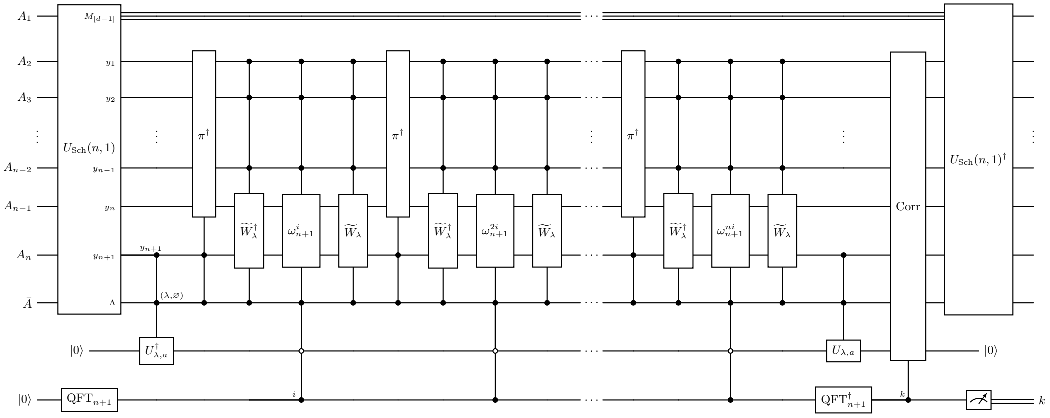
PGM circuit (standard encoding)



pPBT POVM circuit (standard encoding)



Yamanouchi encoding



Thanks for your attention!