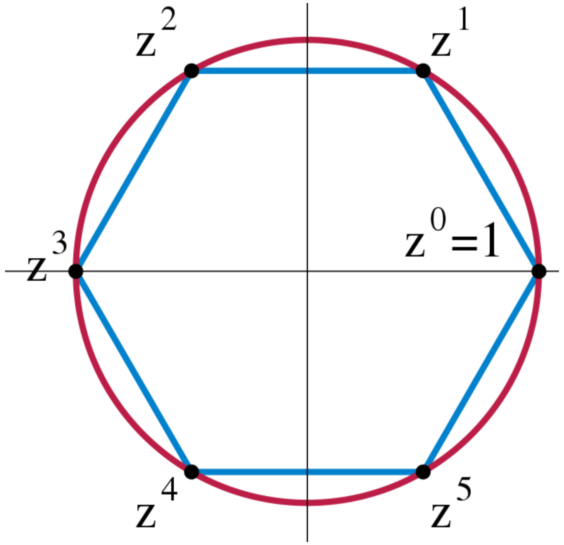
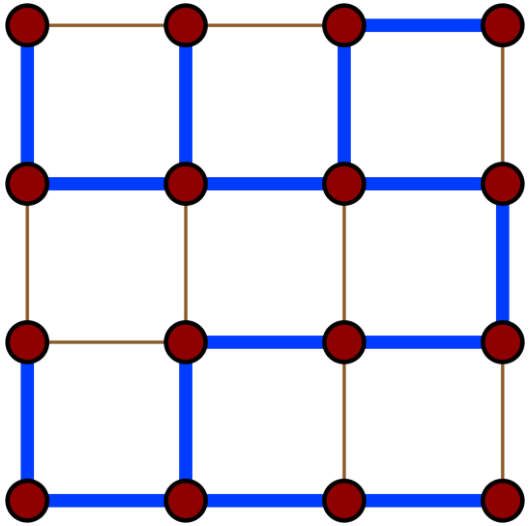
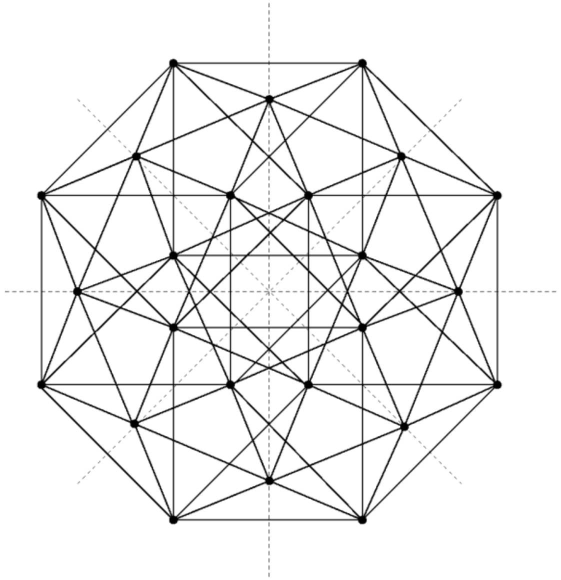
How many states are gauge-invariant?







Alessandro Mariani
University of Turin, Italy

Some sign problems are hard

Example: real-time dynamics:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

Interest in Hamiltonian methods:
tensor networks, quantum simulation, etc


Neural AI
Quantum
Processor
4K
Processore
Neural
Quantum 4K

Some sign problems are hard

Example: real-time dynamics:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$$

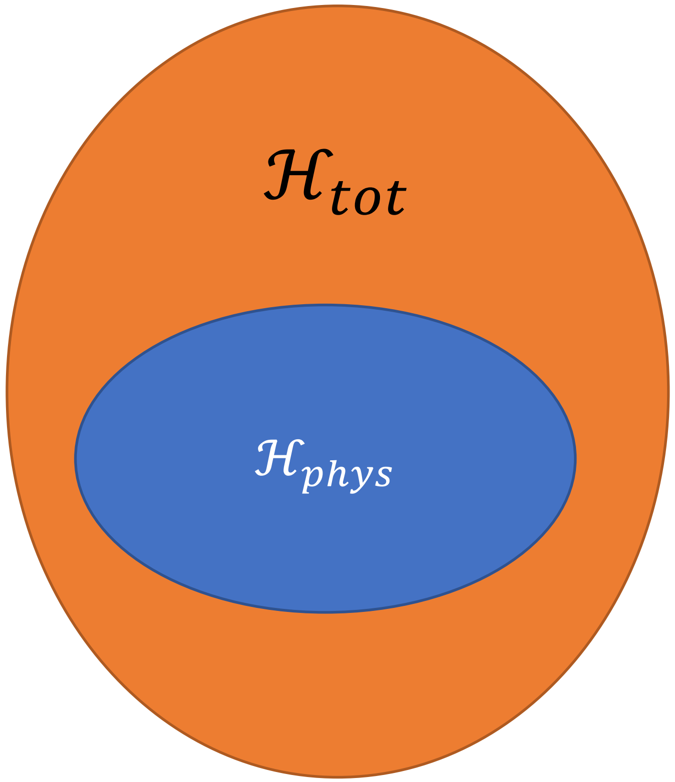
Interest in Hamiltonian methods:
tensor networks, quantum simulation, etc

Gauge-invariant states

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$

Some theories of interest are gauge theories.

Only states which satisfy the **Gauss law** are **physical**.

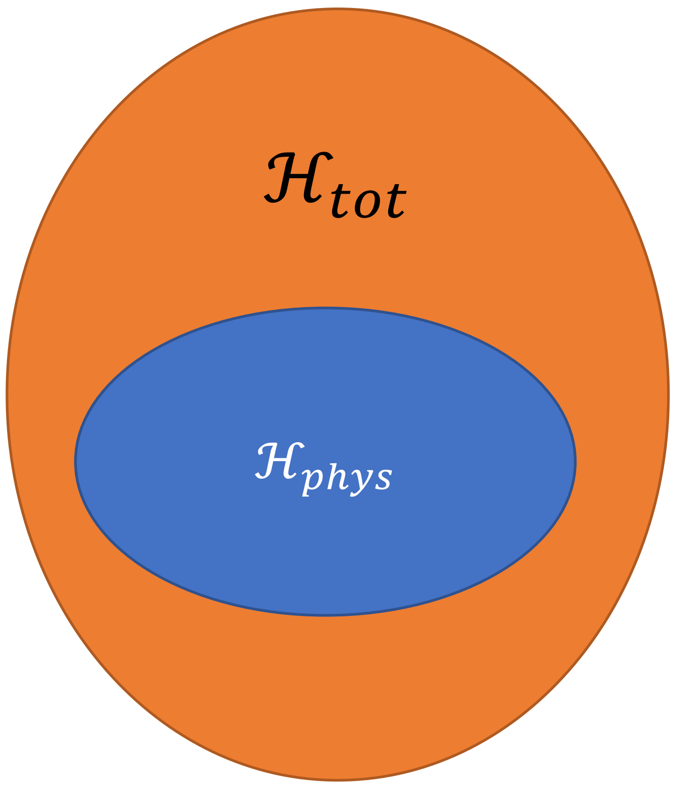


Gauge-invariant states

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$

Some theories of interest are gauge theories.

Only states which satisfy the **Gauss law** are **physical**.



Usually, (e.g.) start with exact diagonalization:
It is useful to know **how many states are gauge-invariant:**
1) Resource estimation
2) Crosscheck

Many schemes for truncating the Hilbert space

Bosonic QFTs have infinite-dimensional Hilbert space

Many ways have been designed to make the Hilbert space finite-dimensional:

Many schemes for truncating the Hilbert space

Bosonic QFTs have infinite-dimensional Hilbert space

Many ways have been designed to make the Hilbert space finite-dimensional:

Quantum Link Models

Truncation in electric field basis

Finite subgroups

Orbifold

q-deformation

Mixed basis

Fuzzy

Finite subsets

Many more…

Many schemes for truncating the Hilbert space

Bosonic QFTs have infinite-dimensional
Hilbert space

Many ways have been designed to make
the Hilbert space finite-dimensional:

Quantum Link Models

Truncation in electric field basis

Finite subgroups

Orbifold

q-deformation

Mixed basis

Fuzzy

Finite subsets

Simple formula for #
gauge-invariant states

Many more…

Many schemes for truncating the Hilbert space

Bosonic QFTs have infinite-dimensional
Hilbert space

Many ways have been designed to make
the Hilbert space finite-dimensional:

Quantum Link Models

Truncation in electric field basis

Finite subgroups

Orbifold

q-deformation

Mixed basis

Fuzzy

Finite subsets

Simple formula for #
gauge-invariant states

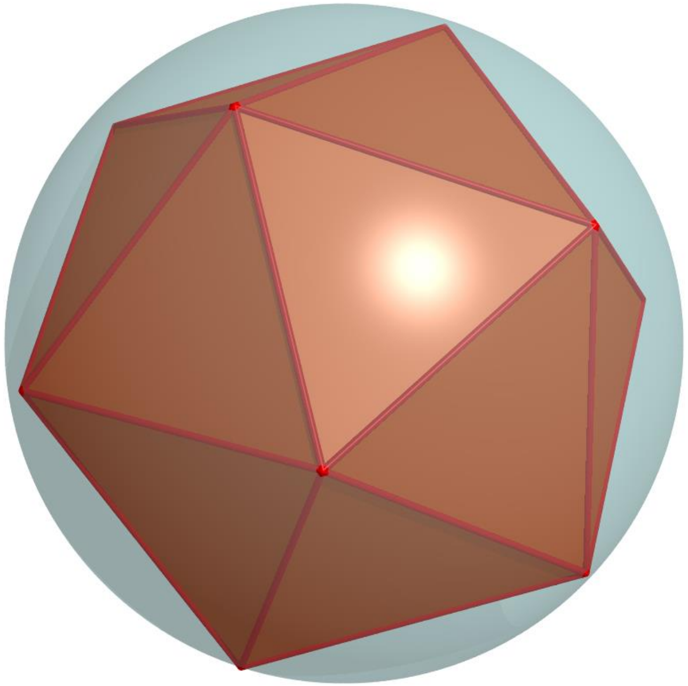
Many more…

Harder

Finite gauge groups

Idea: replace the gauge group (a Lie group) with a **finite subgroup** G.

e.g. $\mathbb{Z}_N \leq U(1)$,
$Q_8 \leq SU(2)$,
$S(1080) \leq SU(3)$



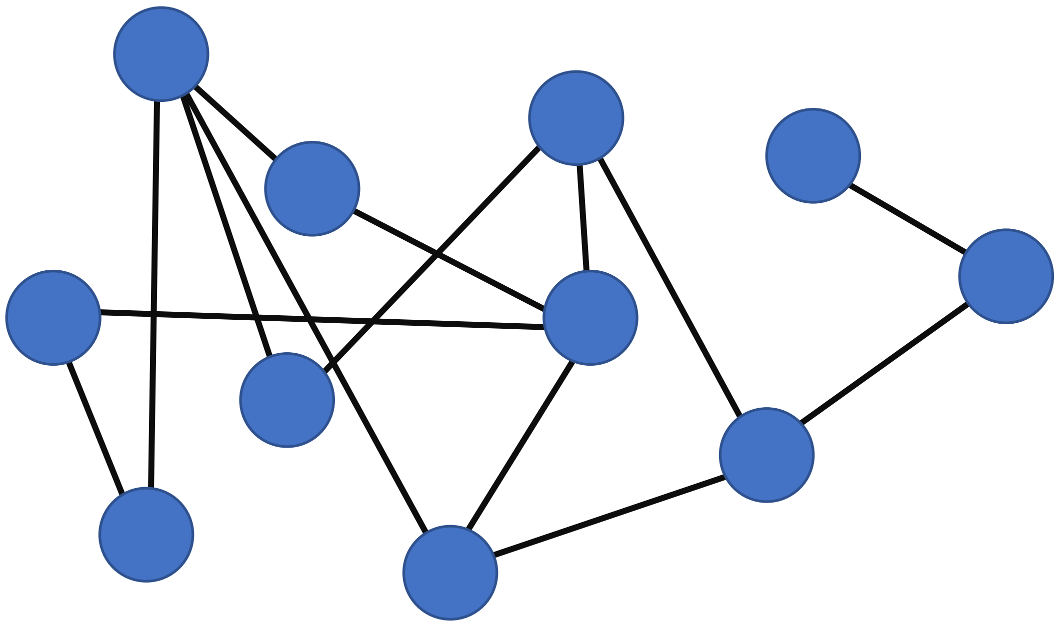
[Hasenfratz & Niedermayer '01]

The link variable $U \in G$ can take only finitely-many values.
⟶ **Hilbert space is finite-dimensional**.

- Continuum limit via improved actions [Alexandru et al '19]
- Can construct Hamiltonian [Orland '91, Harlow & Ooguri '18, Mariani, Pradhan, Ercolessi '23]

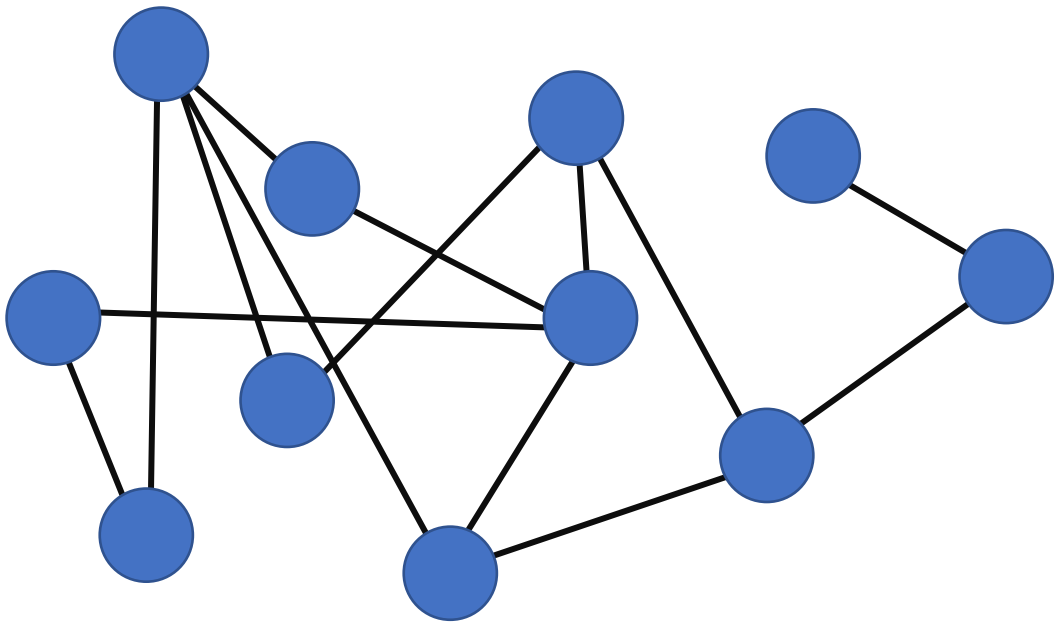
Gauge invariant states

Setting: discretize space as a **graph**:



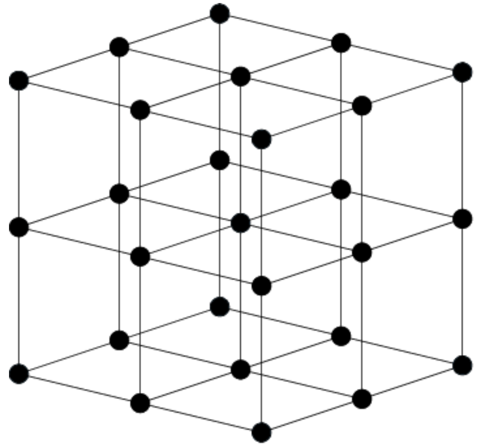
Gauge invariant states

Setting: discretize space as a **graph**:

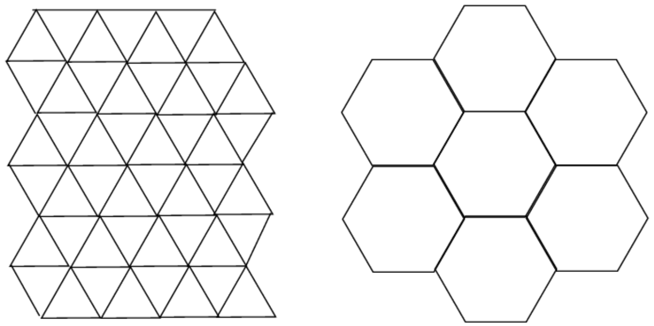


Many interesting special cases:

hypercubic lattices in d dimensions, with open, periodic or mixed boundary conditions

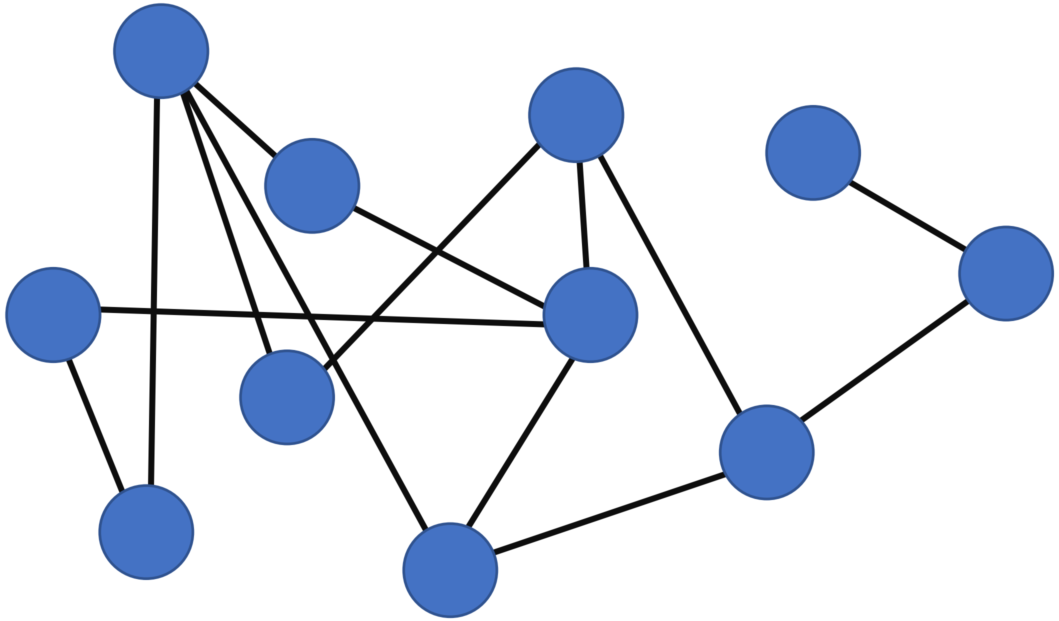


triangular, honeycomb lattices, etc.



Gauge invariant states

Setting: discretize space as a **graph**:



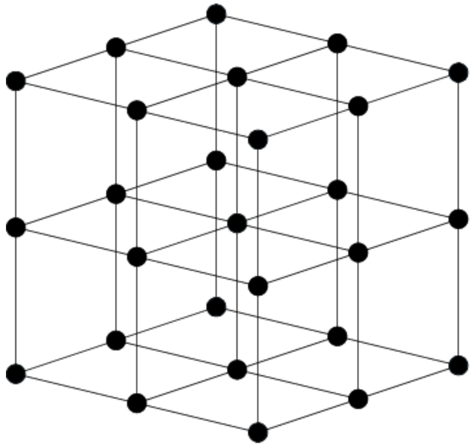
Put one group element $g_l \in G$ per link l

Orthonormal basis of Hilbert space $|g_l\rangle$

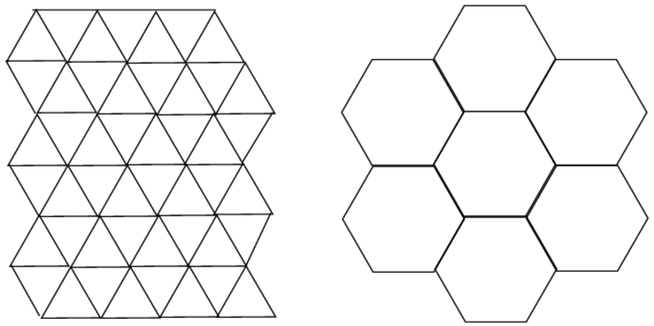
$$\mathcal{H}_{tot} = \bigotimes_{links} \mathbb{C}[G]$$

Many interesting special cases:

hypercubic lattices in d dimensions, with open, periodic or mixed boundary conditions

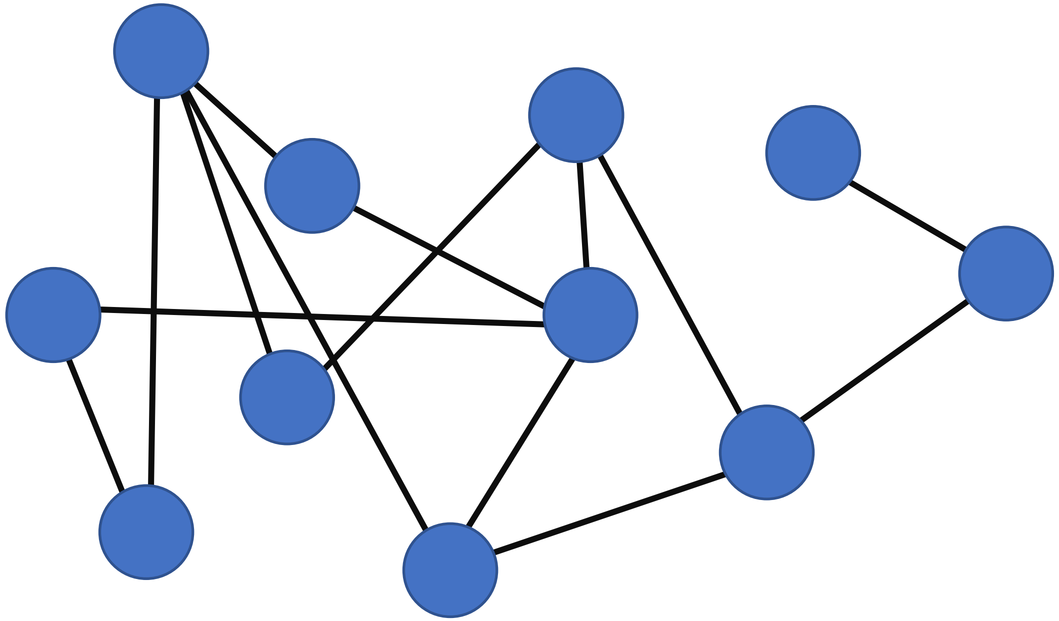


triangular, honeycomb lattices, etc.



Gauge invariant states

Setting: discretize space as a **graph**:



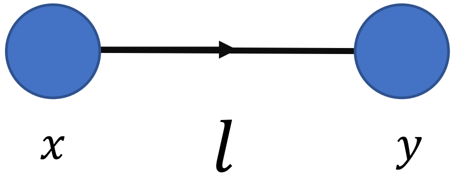
Put one group element $g_l \in G$ per link l

Orthonormal basis of Hilbert space $|g_l\rangle$

$$\mathcal{H}_{tot} = \bigotimes_{links} \mathbb{C}[G]$$

Gauge transformations act as:

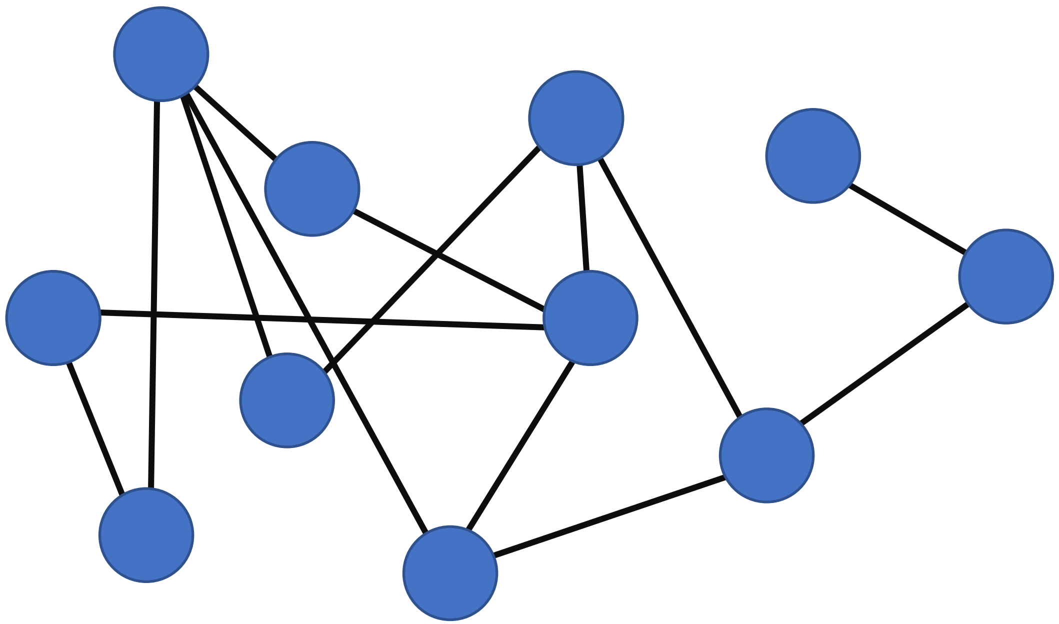
$$\mathcal{G}|g_l\rangle = |g_x g_l g_y^{-1}\rangle$$



Gauge invariant states

Same action
On every link

Setting: discretize space as a **graph**:



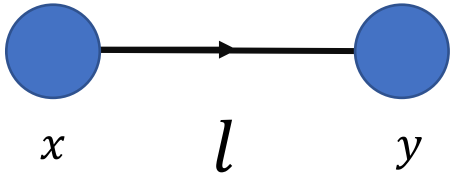
Put one group element $g_l \in G$ per link l

Orthonormal basis of Hilbert space $|g_l\rangle$

$$\mathcal{H}_{tot} = \bigotimes_{links} \mathbb{C}[G]$$

Gauge transformations act as:

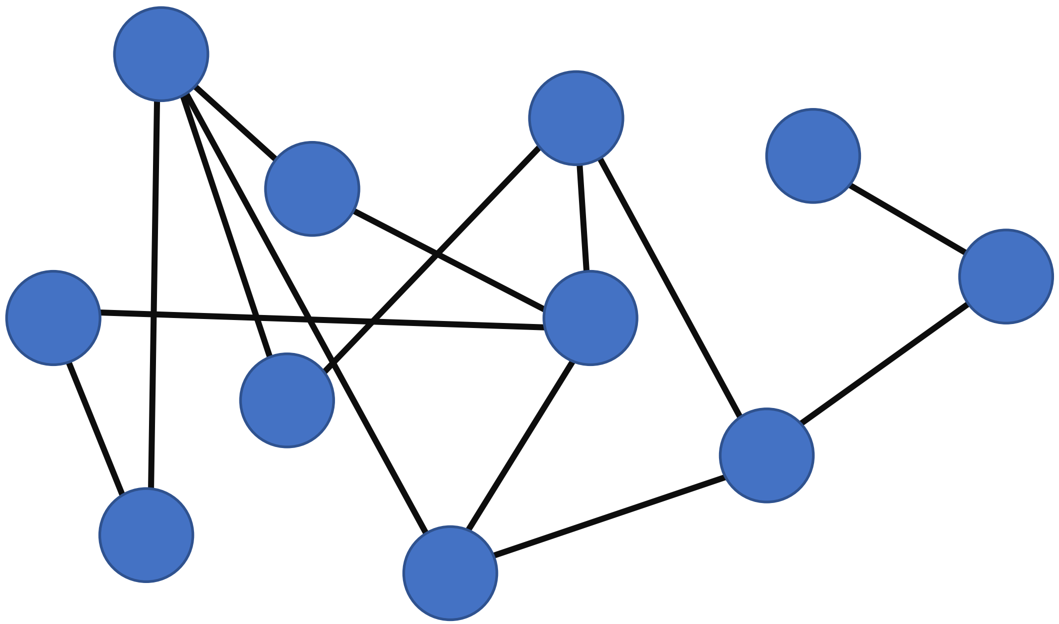
$$\mathcal{G}|g_l\rangle = |g_x g_l g_y^{-1}\rangle$$



Gauge invariant states

Same action
On every link

Setting: discretize space as a **graph**:



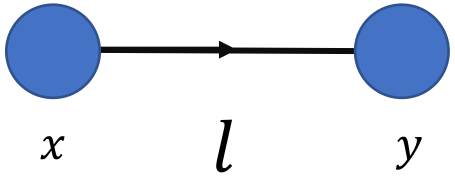
Put one group element $g_l \in G$ per link l

Orthonormal basis of Hilbert space $|g_l\rangle$

$$\mathcal{H}_{tot} = \bigotimes_{links} \mathbb{C}[G]$$

Gauge transformations act as:

$$\mathcal{G}|g_l\rangle = |g_x g_l g_y^{-1}\rangle$$



Gauge-invariant states satisfy:

$$\mathcal{G}|\psi\rangle = |\psi\rangle$$

They form the physical Hilbert space

$$\mathcal{H}_{phys}$$

Counting gauge-invariant states

Write down explicit projector $P\colon \mathcal{H}_{tot} \to \mathcal{H}_{phys}$

$$P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \mathcal{G} \qquad\qquad P^2 = P$$

[Mariani, Pradhan, Ercolessi 2023]

[Mariani 2024, Mariani (in prep.)]

Counting gauge-invariant states

Write down explicit projector $P: \mathcal{H}_{tot} \to \mathcal{H}_{phys}$

$$P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \mathcal{G}$$

$$\dim \mathcal{H}_{phys} = \operatorname{tr} P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \operatorname{tr} \mathcal{G}$$

$P^2 = P$

eigenvalue 1 $\longrightarrow$ $\in \mathcal{H}_{phys}$

eigenvalue 0 $\longrightarrow$ not in $\mathcal{H}_{phys}$

[Mariani, Pradhan, Ercolessi 2023]

[Mariani 2024, Mariani (in prep.)]

The dimension of the physical subspace

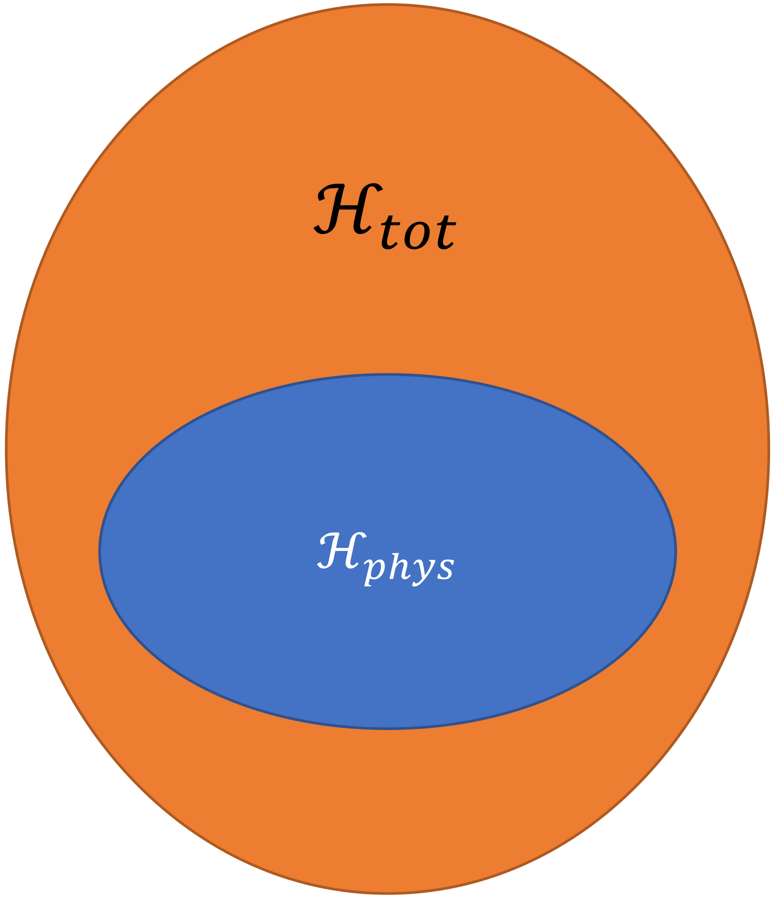
For a **pure gauge theory** with arbitrary finite group G on an arbitrary lattice with V sites and E links:

$$\dim \mathcal{H}_{tot} = |G|^E$$

$$\dim \mathcal{H}_{phys} = \sum_C \left(\frac{|G|}{|C|}\right)^{E-V}$$

C are the **conjugacy classes** of G, i.e. g_1 and g_2 are in the same C iff $g_2 = g\, g_1\, g^{-1}$.

Remember assumption: Gauss law the same everywhere.



[Mariani, Pradhan, Ercolessi 2023]

[Mariani 2024]

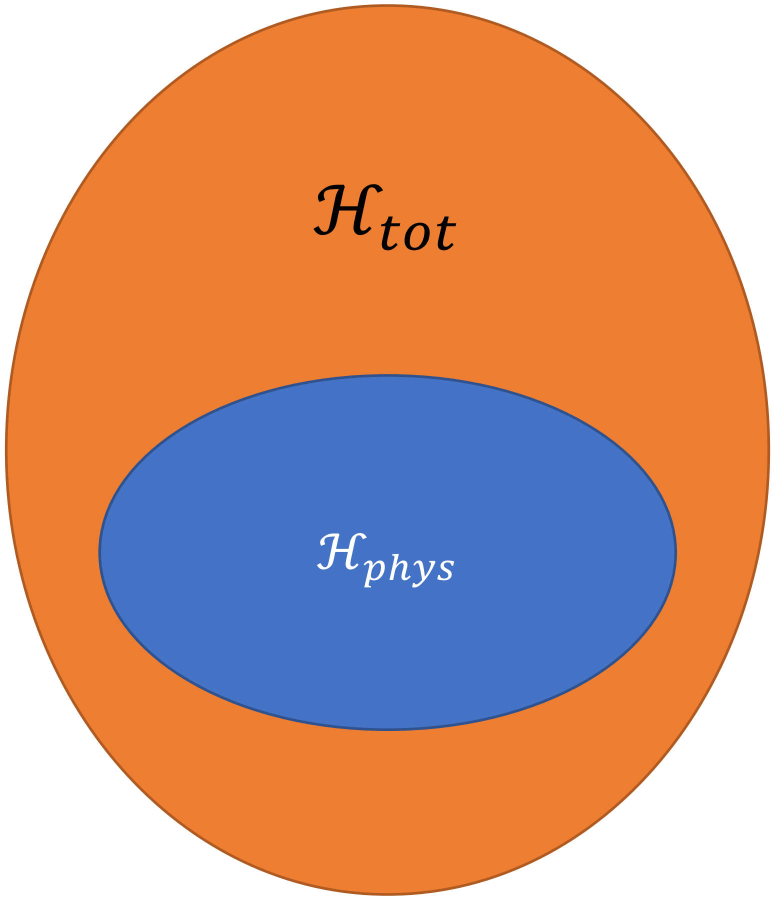
Variants of this formula: scalar fields

Can also derive a formula for gauge+scalar theories:

$$\dim \mathcal{H}_{phys} = \sum_C \left(\frac{|G|}{|C|}\right)^{E-V} \chi(C)^V$$

$\chi = \mathrm{tr}\rho$ is the character of the gauge representation ρ of the scalar field.

Scalar field valued in an arbitrary finite set S, its local Hilbert space is $\mathbb{C}[S]$.



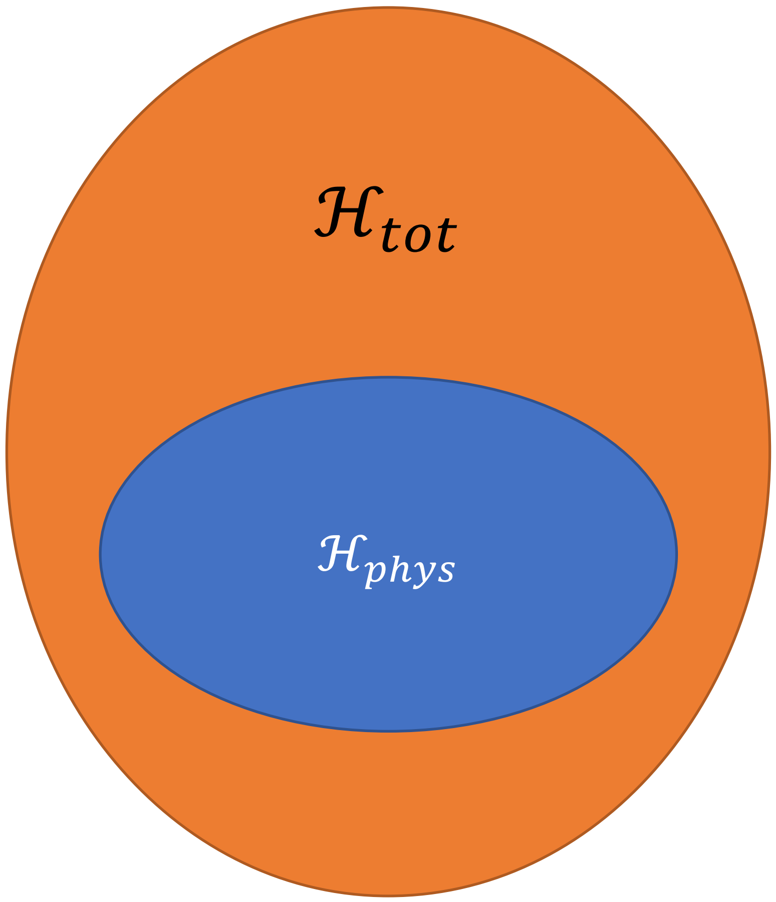
[Mariani (in preparation)]

Variants of this formula: arbitrary charges

If we put arbitrary charges on the lattice sites, get instead:

$$\dim \mathcal{H}_{phys} = \sum_{C} \left(\frac{|G|}{|C|}\right)^{E-V} \prod_{x} \chi_x(C)$$

$\chi_x = \mathrm{tr}\rho_x$ is the character of the representation ρ_x at site x.



[Mariani (in preparation)]

Variants of this formula: arbitrary charges

If we put arbitrary charges on the lattice sites, get instead:

$$\dim \mathcal{H}_{phys} = \sum_C \left(\frac{|G|}{|C|}\right)^{E-V} \prod_x \chi_x(C)$$

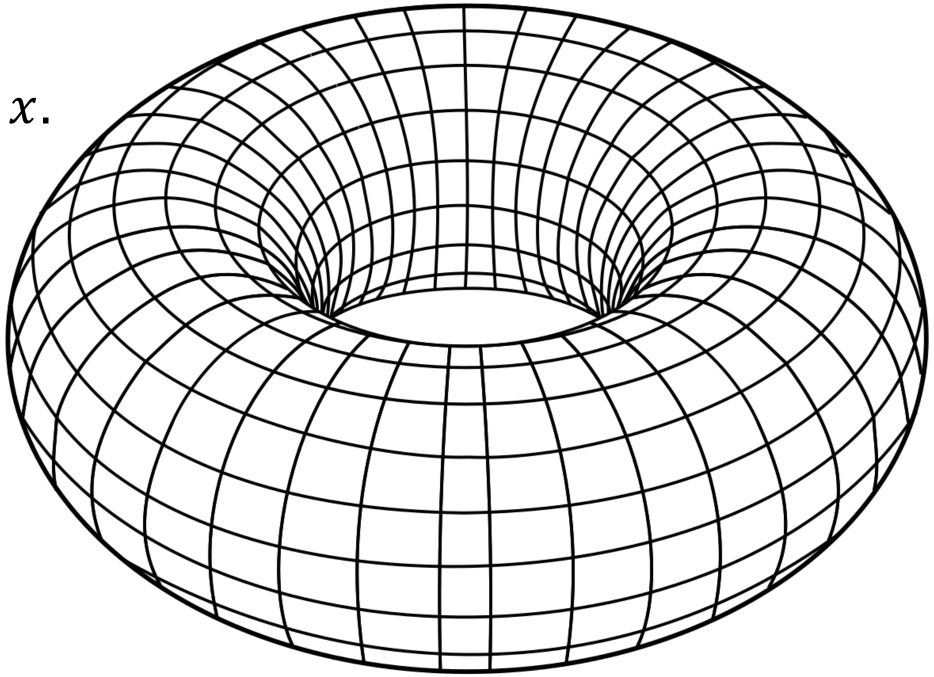
$\chi_x = \mathrm{tr}\rho_x$ is the character of the representation ρ_x at site x.

Example: no charged states on a torus.

$\mathbb{Z}_{\boldsymbol{N}}$ theory. Place $q = 1$ charge on each site.

$$\dim \mathcal{H}_{phys} = N^{E-V} \sum_{k=0}^{N-1} e^{2\pi i k \frac{V}{N}}$$

which is zero unless $Q_{tot} = V \equiv 0 \pmod N$.



[Mariani (in preparation)]

Variants of this formula: arbitrary charges

If we put arbitrary charges on the lattice sites, get instead:

$$\dim \mathcal{H}_{phys} = \sum_C \left(\frac{|G|}{|C|}\right)^{E-V} \prod_x \chi_x(C)$$

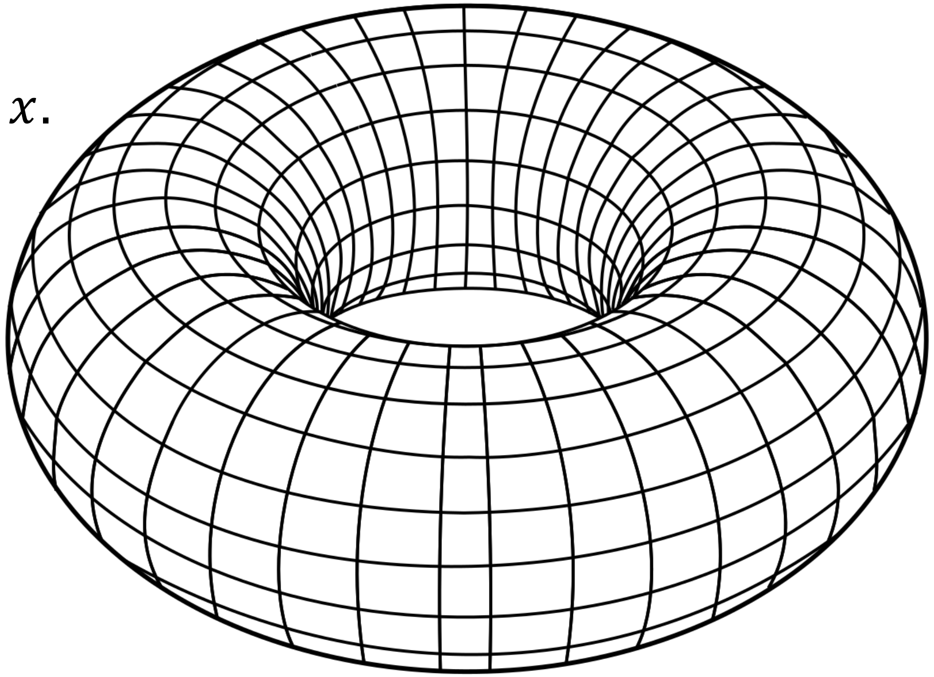
$\chi_x = \mathrm{tr}\rho_x$ is the character of the representation ρ_x at site x.

Example: Dimer model.

$\mathbb{Z}_{\boldsymbol{N}}$ theory. Stagger $q = \pm 1$ charges.

$$\dim \mathcal{H}_{phys} = N^{E-V} \sum_{k=0}^{N-1} |e^{2\pi i k \frac{V}{N}}| = N^{E-V+1}$$

which is not zero!



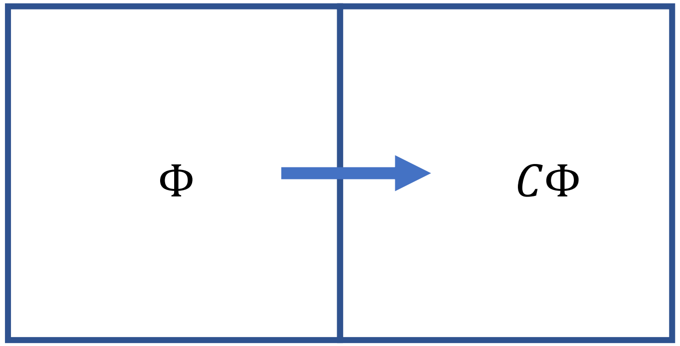
[Mariani (in preparation)]

Variants of this formula: C-periodic boundaries

More general boundary conditions can also be treated with the same method.

[Kronfeld & Wiese (1991), Wiese (1992)]

Example: C-periodic boundary conditions



Total Hilbert space is the same, but extended operators (e. g. Gauss law) are modified.

[Mariani (in preparation)]

Variants of this formula: C-periodic boundaries

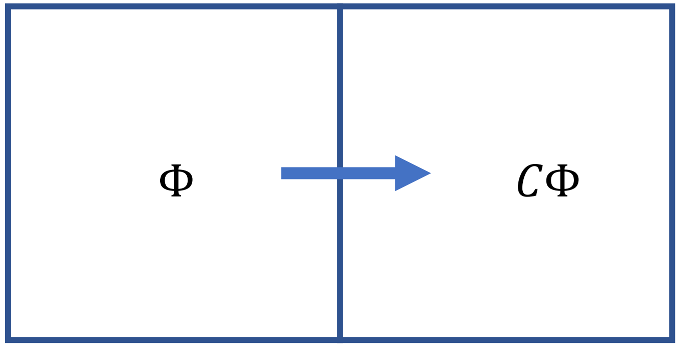
More general boundary conditions can also be treated with the same method.

[Kronfeld & Wiese (1991), Wiese (1992)]

Example: C-periodic boundary conditions

$$\dim \mathcal{H}_{phys} = \sum_{C,\, C=C^{-1}} \left(\frac{|G|}{|C|}\right)^{E-V}$$

i.e. sum over only those conjugacy classes C which are self-inverse (they contain all their inverses).



Total Hilbert space is the same, but extended operators (e. g. Gauss law) are modified.

[Mariani (in preparation)]

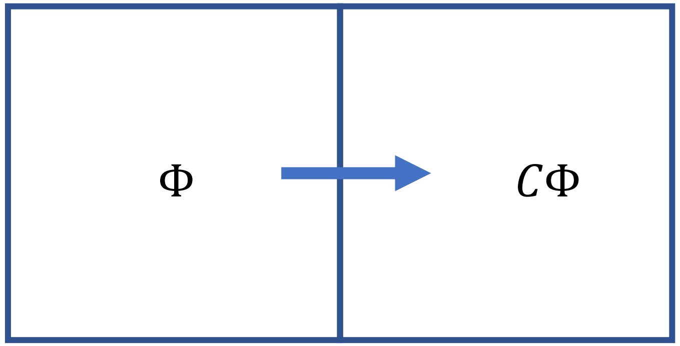
Variants of this formula: C-periodic boundaries

More general boundary conditions can also be treated with the same method.

[Kronfeld & Wiese (1991), Wiese (1992)]

Example: C-periodic boundary conditions

$$\dim \mathcal{H}_{phys} = \sum_{C,\, C=C^{-1}} \left(\frac{|G|}{|C|}\right)^{E-V}$$



Example: charged states on a C-per torus.

$\mathbb{Z}_N$ theory. Place $q = 1$ charge on each site.

$$\dim \mathcal{H}_{phys} = N^{E-V} \sum_{k=0}^{0} e^{2\pi i k \frac{V}{N}} = N^{E-V}$$

which is not zero!

Total Hilbert space is the same, but extended operators (e. g. Gauss law) are modified.

[Mariani (in preparation)]

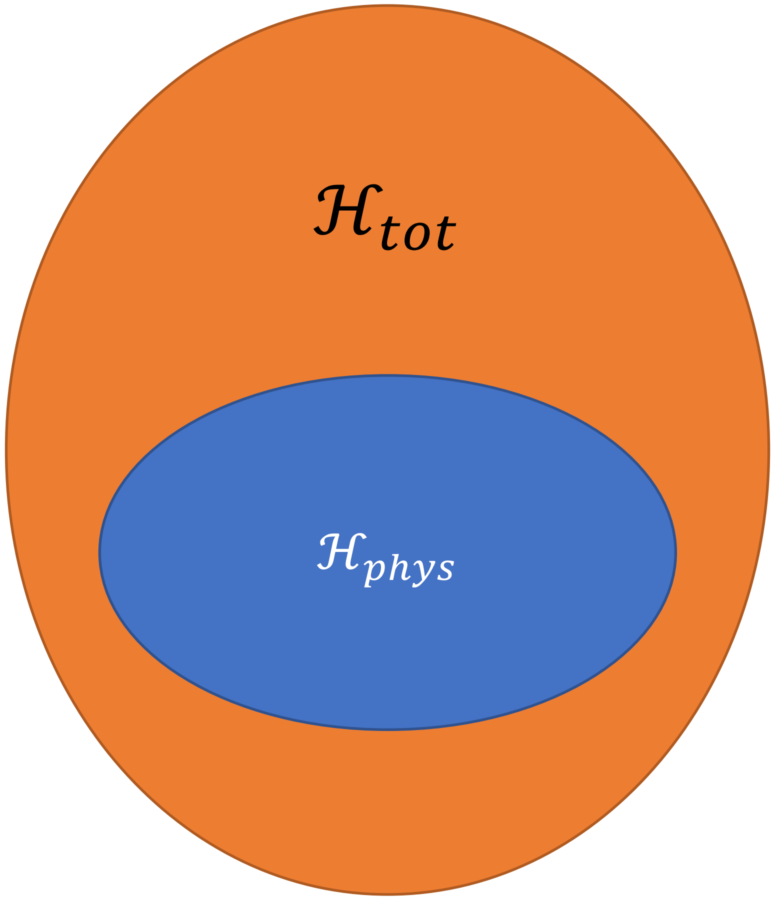
How to describe the physical subspace?

[Durhuus ’80, Sengupta ’94, Lévy ‘04]

$\mathcal{H}_{tot}$ $\xrightarrow{\text{Gauss' Law}}$ $\mathcal{H}_{phys}$

(traced) Wilson loops **do not** necessarily span $\mathcal{H}_{phys}$

For $SU(N)$ Wilson loops in the fundamental span $\mathcal{H}_{phys}$.



See [Mariani ‘24] for a summary.

How to describe the physical subspace?

[Durhuus ’80, Sengupta ’94, Lévy ‘04]

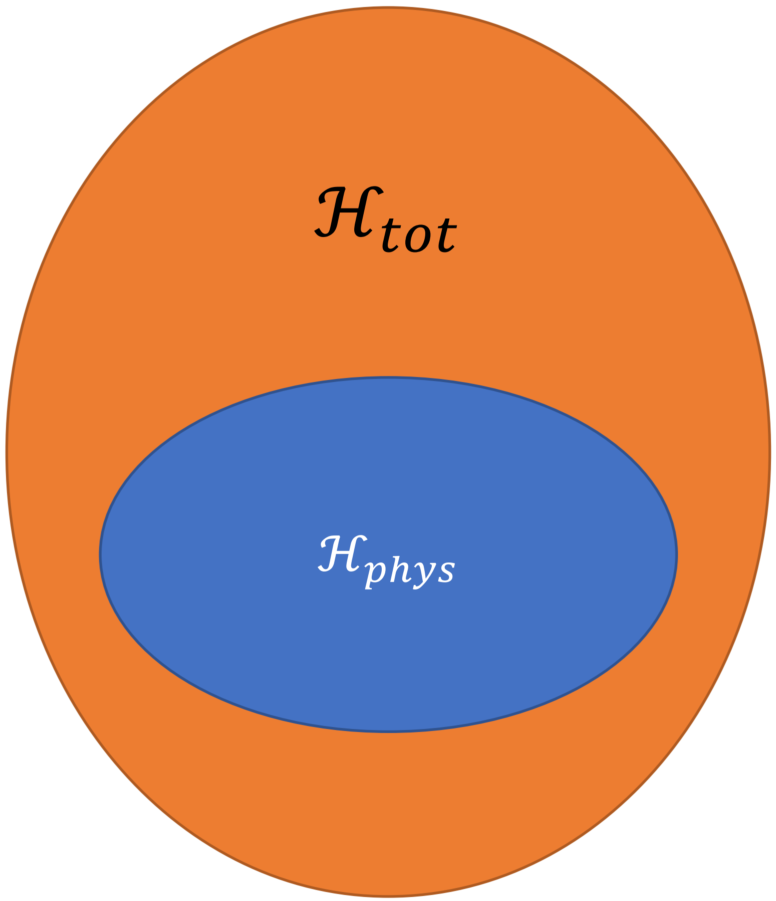
$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$

(traced) Wilson loops **do not** necessarily span $\mathcal{H}_{phys}$

For $SU(N)$ Wilson loops in the fundamental span $\mathcal{H}_{phys}$.

For direct products of $SU(N)$, $U(N)$, $SO(N), O(N)$ and Abelian groups, Wilson loops span $\mathcal{H}_{phys}$, but **all irreps may be needed** (e.g. $SO(2N)$).

For other groups such as G_2 it is **not known**.



See [Mariani ‘24] for a summary.

How to describe the physical subspace?

[Durhuus ’80, Sengupta ’94, Lévy ‘04]

$$\mathcal{H}_{tot} \xrightarrow{\text{Gauss' Law}} \mathcal{H}_{phys}$$

(traced) Wilson loops **do not** necessarily span $\mathcal{H}_{phys}$

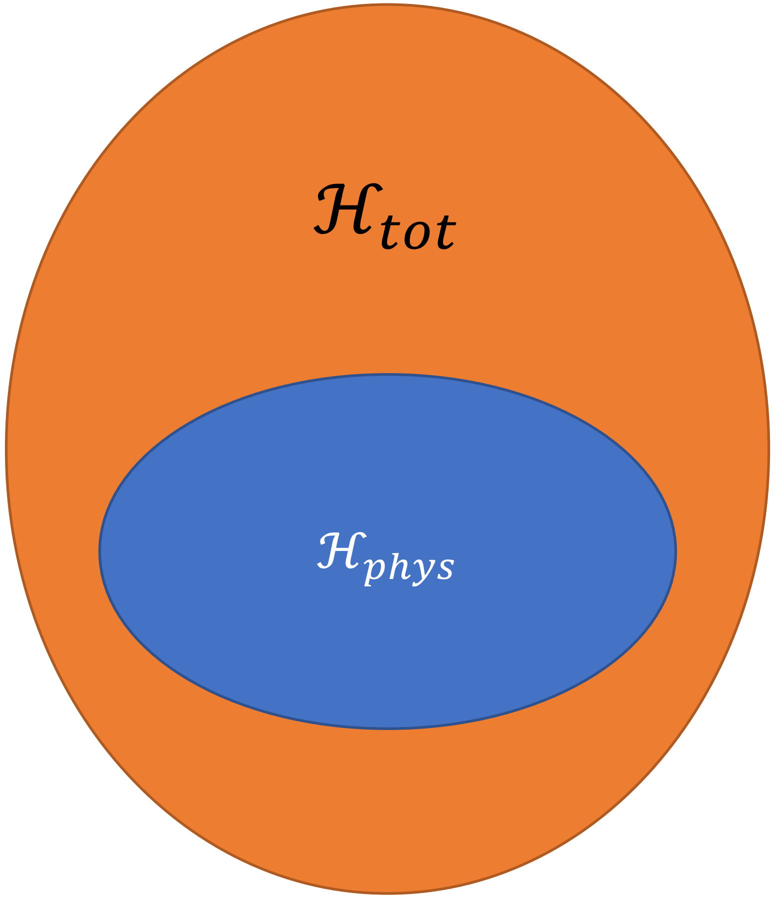
For $SU(N)$ Wilson loops in the fundamental span $\mathcal{H}_{phys}$.

For direct products of $SU(N)$, $U(N)$, $SO(N), O(N)$ and Abelian groups, Wilson loops span $\mathcal{H}_{phys}$, but **all irreps may be needed** (e.g. $SO(2N)$).

For other groups such as G_2 it is **not known**.

↓

Cannot use Wilson loops for general description.
(various other implications: entanglement entropy, etc)



See [Mariani ‘24] for a summary.

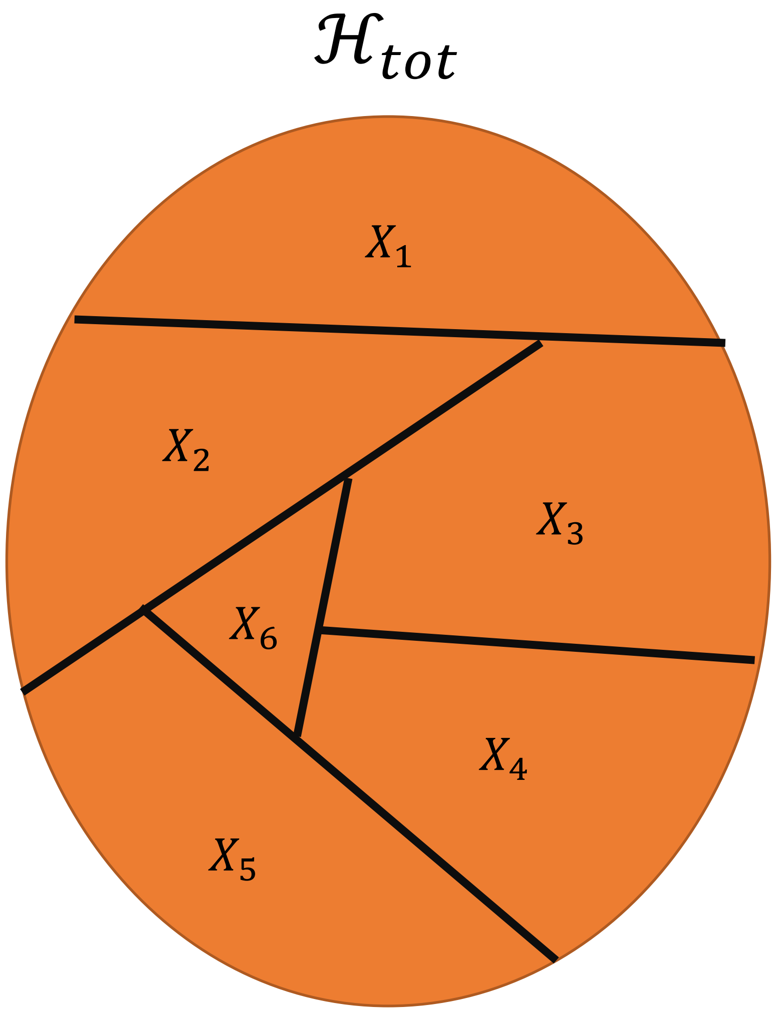
How to describe the physical subspace?

Gauge-equivalence classes?

Split configurations into **gauge-equivalence classes** X_i

$$|X_i\rangle = \frac{1}{\sqrt{|X_i|}} \sum_{\vec{g} \in X_i} |\vec{g}\rangle$$

i.e. superimpose all gauge-equivalent configurations.



Gauge-equivalence classes?

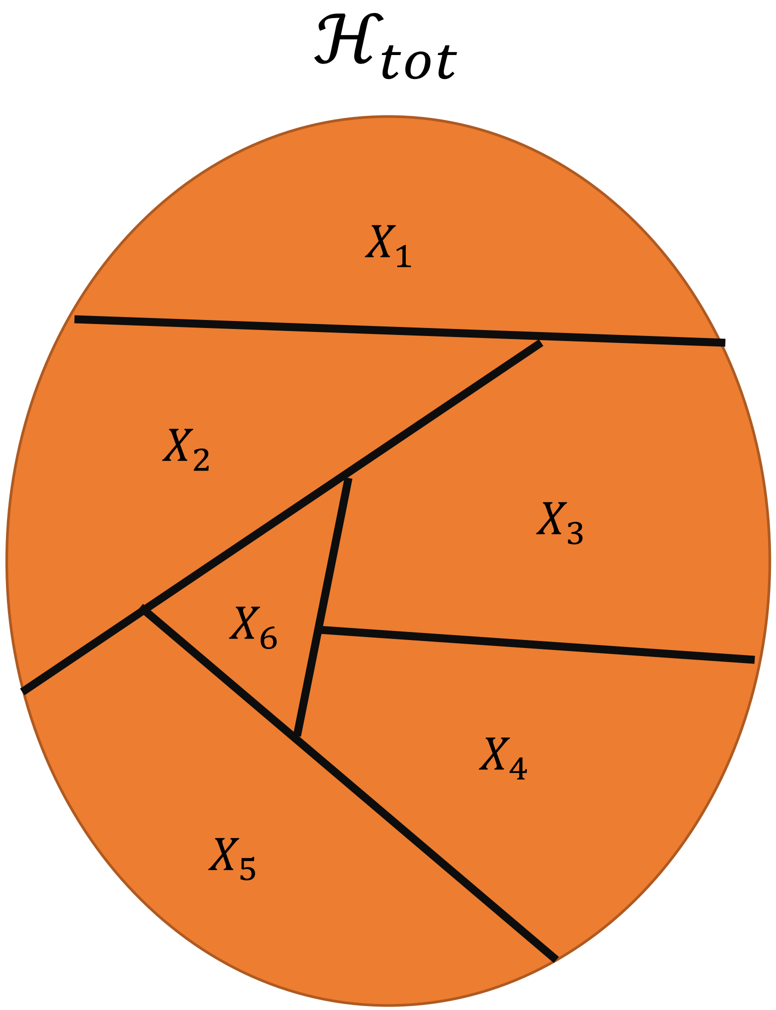
Split configurations into **gauge-equivalence classes** X_i

$$|X_i\rangle = \frac{1}{\sqrt{|X_i|}} \sum_{\vec{g}\in X_i} |\vec{g}\rangle$$

i.e. superimpose all gauge-equivalent configurations.

We don't know the normalization

Gauge-equivalence classes do not all have the same size



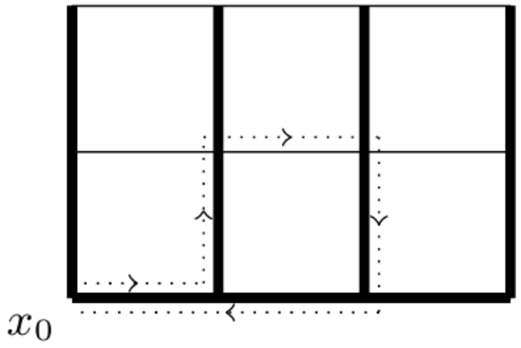
How to describe the physical subspace?

Better way: use **holonomies** (untraced Wilson loops)

[Mariani ‘24]
see also: [Durhuus ‘80]
similar ideas in:
[Grabowska, Kane, Bauer ‘24]
[Burbano, Bauer ‘24]

Holonomy states

Construct basis of holonomies **based at the same point** (need $E - V + 1$):

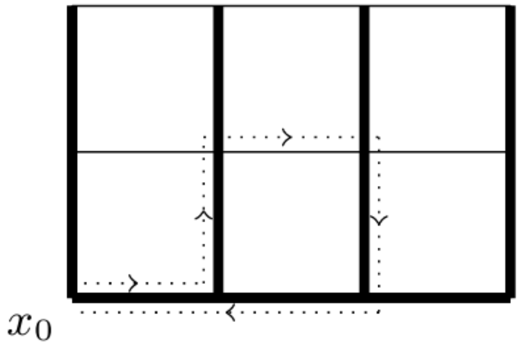


$h = g_1 g_2 g_3 g_4^{-1} g_1^{-1}$

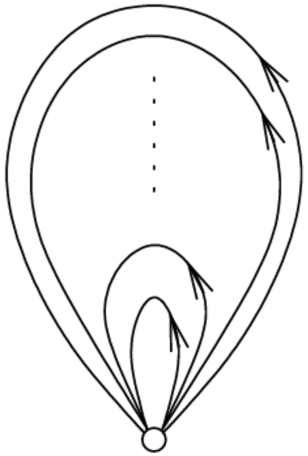
[Durhuus ‘80]

Holonomy states

Construct basis of holonomies **based at the same point** (need $E - V + 1$):



$h = g_1 g_2 g_3 g_4^{-1} g_1^{-1}$



looks like this

$h_1, h_2, \dots, h_{E-V+1}$

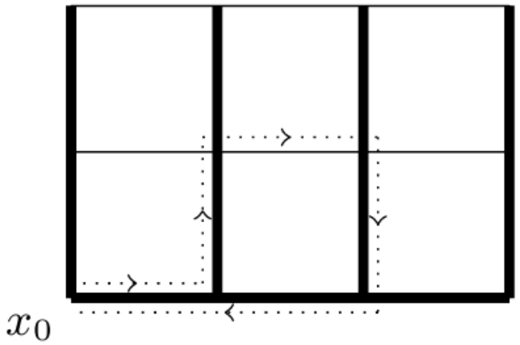
Gauge-invariant except at the base point:

$h_i \to g h_i g^{-1}$ for all i

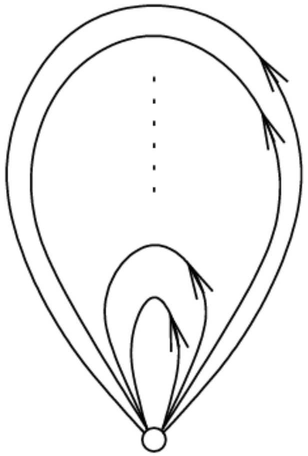
[Durhuus ‘80]

Holonomy states

Construct basis of holonomies **based at the same point** (need $E - V + 1$):



$h = g_1 g_2 g_3 g_4^{-1} g_1^{-1}$



looks like this

$h_1, h_2, \dots, h_{E-V+1}$

Gauge-invariant except at the base point:

$h_i \to g h_i g^{-1} \quad \text{for all } i$

Two configurations $\vec{g}$ and $\vec{g}'$ are gauge-equivalent iff their holonomies are related by conjugation, $h_i \to g h_i g^{-1} \quad \text{for all } i.$

[Durhuus ‘80]

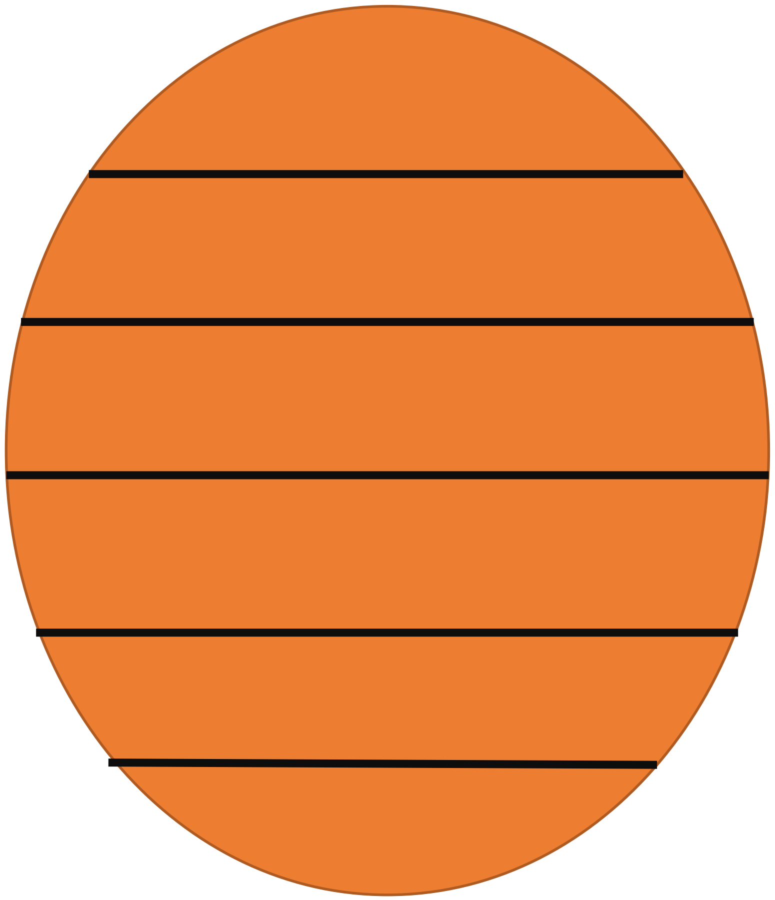
[Mariani ‘24]

How to describe the physical subspace?

Holonomy states: $|h_1, h_2, \ldots, h_{E-V+1}\rangle = \frac{1}{\sqrt{|G|^{V-1}}} \sum_{\vec{g} \in h} |\vec{g}\rangle$

i.e. same number of configurations in each holonomy class.

$\mathcal{H}_{tot}$

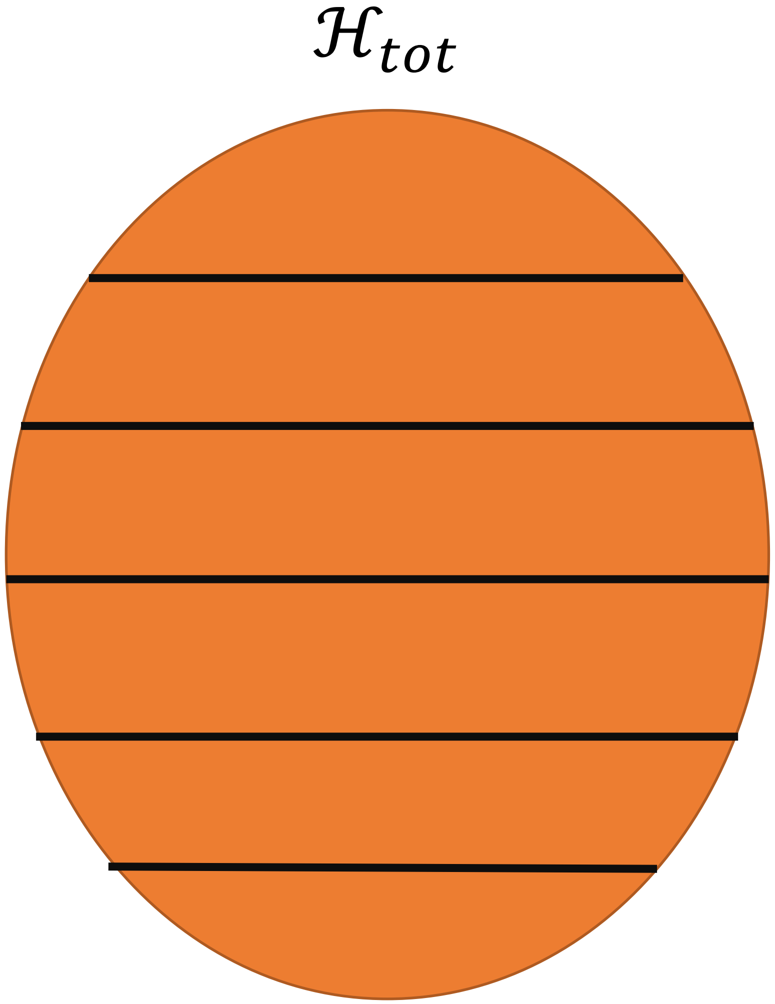


[Mariani '24]

How to describe the physical subspace?

Holonomy states: $|h_1, h_2, \ldots, h_{E-V+1}\rangle = \frac{1}{\sqrt{|G|^{V-1}}} \sum_{\vec{g} \in h} |\vec{g}\rangle$

i.e. same number of configurations in each holonomy class.



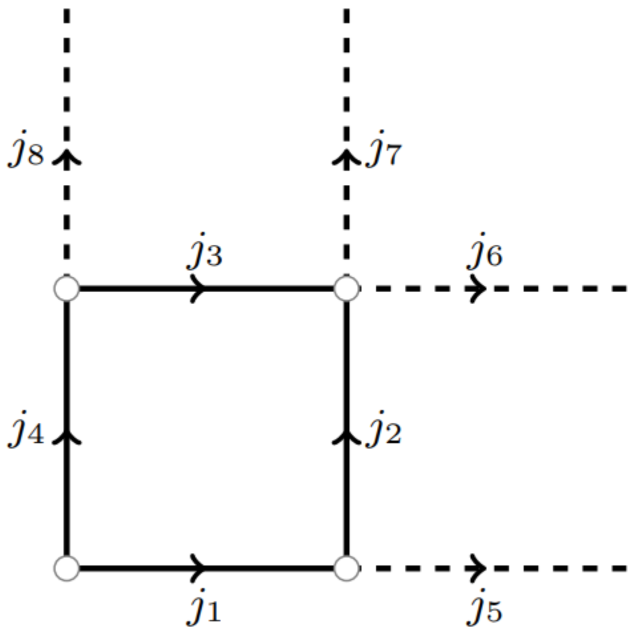
- Holonomy states are an analytical solution to almost all gauge constraints.
- They form a tensor product Hilbert space.
- Local operators are s-sparse.

What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

On each link the electric field takes a value

$$j_l = -s, -s+1, \ldots, s-1, s$$



What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

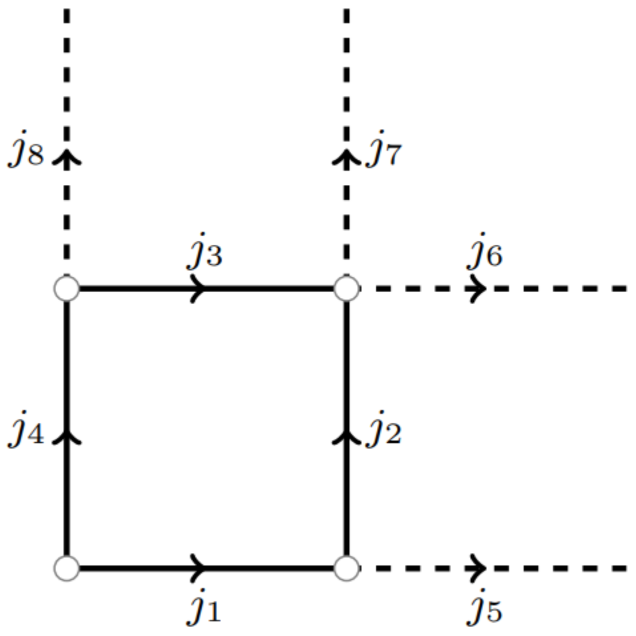
On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

For the four links attached to the site.



What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

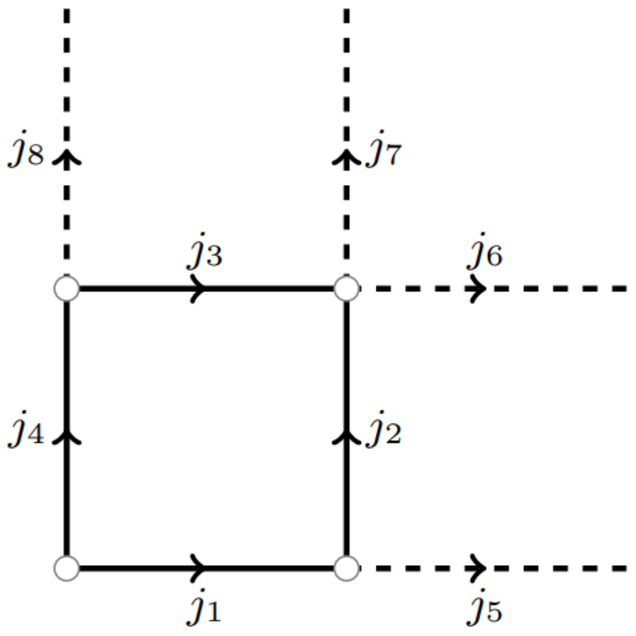
On each link the electric field takes a value

$$j_l = -s, -s+1, \ldots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

For the four links attached to the site.



Already known to mathematicians [Beck, Zlaslavsky '03]:

- s integer: **integer k-flow** $(k = s + 1)$
- s half-integer: **odd-valued integer k-flow** $(k = 2s + 1)$
- $s = 1/2$: **nowhere-zero integer 2-flow**

What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

For the four links attached to the site.

Counting gauge-invariant states
in the U(1) quantum link model
is an **open problem** in graph theory.

Mathematicians have shown that
$\dim \mathcal{H}_{phys}^{QLM}$ = polynomial in s. [Kochol ‘02]

Already known to mathematicians [Beck, Zlaslavsky ‘03]:

- s integer: **integer k-flow** ($k = s + 1$)
- s half-integer: **odd-valued integer k-flow** ($k = 2s + 1$)
- $s = 1/2$: **nowhere-zero integer 2-flow**

What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

For the four links attached to the site.

Counting gauge-invariant states in the U(1) quantum link model is an **open problem** in graph theory.

Mathematicians have shown that $\dim \mathcal{H}_{phys}^{QLM}$ = polynomial in s. [Kochol '02]

$\dim \mathcal{H}_{phys}^{QLM}$ does not depend simply on E and V (more geometric info needed)

Already known to mathematicians [Beck, Zlaslavsky '03]:

- s integer: **integer k-flow** $(k = s + 1)$
- s half-integer: **odd-valued integer k-flow** $(k = 2s + 1)$
- $s = 1/2$: **nowhere-zero integer 2-flow**

What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

For the four links attached to the site.

Already known to mathematicians [Beck, Zlaslavsky '03]:
- s integer: **integer k-flow** ($k = s + 1$)
- s half-integer: **odd-valued integer k-flow** ($k = 2s + 1$)
- $s = 1/2$: **nowhere-zero integer 2-flow**

Counting gauge-invariant states in the U(1) quantum link model is an **open problem** in graph theory.

Mathematicians have shown that $\dim \mathcal{H}_{phys}^{QLM}$ = polynomial in s. [Kochol '02]

$\dim \mathcal{H}_{phys}^{QLM}$ does not depend simply on E and V (more geometric info needed)

On an arbitrary graph, with arbitrary spin on each link, and arbitrary charges the problem is #P-hard.
[e.g. Baldoni-Silva et al '03]

What about Quantum Link Models?

Consider the $U(1)$ Quantum Link Model on an arbitrary graph, with spin $s \in \frac{1}{2}\mathbb{Z}$.

On each link the electric field takes a value

$$j_l = -s, -s+1, \dots, s-1, s$$

Gauge-invariant states (for example on a square lattice) satisfy

$$j_1 + j_2 - j_3 - j_4 = 0$$

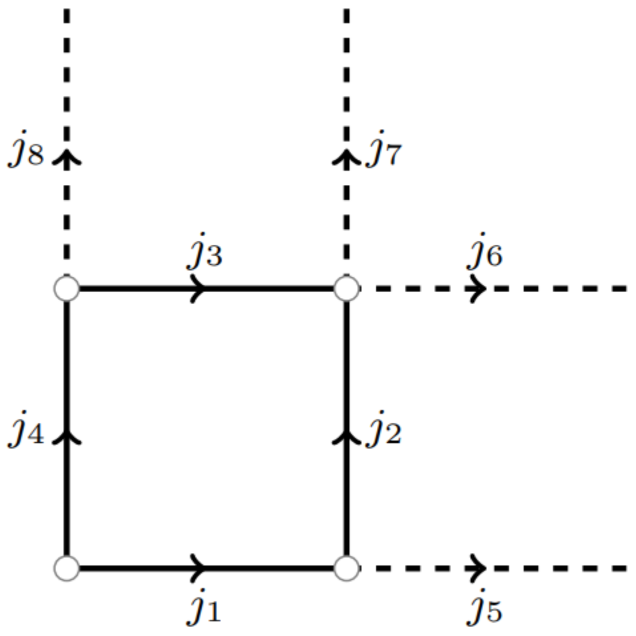
For the four links attached to the site.

Compare with $\mathbb{Z}_N$, $N = 2s+1$ where the condition is

$$j_1 + j_2 - j_3 - j_4 = 0 \ (\text{mod } N)$$

where $j = 0, 1, \dots, N-1$. Then here the answer is

$$N^{E-V+1}$$



Conclusions

Can compute $\dim \mathcal{H}_{phys}$ for finite groups in various settings.

Fermions anticommute at arbitrary distance: non-local Hilbert space.
Difficult (work in progress)

Can we do gauge anomalies? [Witten ’82]
i.e. Gauge anomaly $\leftrightarrow$ $\dim \mathcal{H}_{phys} = 0$?

Backup slides

Gauge anomalies?

Can compute $\dim \mathcal{H}_{phys}$ for finite groups in various settings.

Fermions anticommute at arbitrary distance: non-local Hilbert space. **Difficult (work in progress)**

Idea: can we detect anomalies by computing $\dim \mathcal{H}_{phys}$?

Example. [Witten ’82] Global anomaly for odd no. of $SU(2)$ left-handed fermion doublets.

Consequence: $Z = 0$ $\langle O \rangle = \frac{0}{0}$

Gauge anomalies?

Can compute $\dim \mathcal{H}_{phys}$ for finite groups in various settings.

Fermions anticommute at arbitrary distance: non-local Hilbert space. **Difficult (work in progress)**

Idea: can we detect anomalies by computing $\dim \mathcal{H}_{phys}$?

Example. [Witten '82] Global anomaly for odd no. of $SU(2)$ left-handed fermion doublets.

Consequence: $Z = 0$ $\quad \langle O \rangle = \frac{0}{0}$

Why? $Z = \mathrm{tr}(e^{-H}P)$ $\quad \langle O \rangle = \frac{\mathrm{tr}(e^{-H}OP)}{\mathrm{tr}(e^{-H}P)}$

P is the projector on the gauge-invariant states

i.e. no gauge-invariant states!

Anomalies on the lattice: [Lüscher '98, Bär '02]

What about electric field truncations?

Choose an eigenbasis of the electric field $|jmn\rangle$, where j indexes the irreps of $SU(N)$. Truncate to $j \leq j_{\text{max}}$.

What about electric field truncations?

Choose an eigenbasis of the electric field $|jmn\rangle$, where j indexes the irreps of $SU(N)$. Truncate to $j \leq j_{\max}$.

Can write again:

$$\dim \mathcal{H}_{phys} = \operatorname{tr} P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \operatorname{tr} \mathcal{G}$$

What about electric field truncations?

Choose an eigenbasis of the electric field $|jmn\rangle$, where j indexes the irreps of $SU(N)$. Truncate to $j \leq j_{\text{max}}$.

Can write again:

$$\dim \mathcal{H}_{phys} = \operatorname{tr} P = \frac{1}{|G|^V} \sum_{\mathcal{G} \in G^V} \operatorname{tr} \mathcal{G}$$

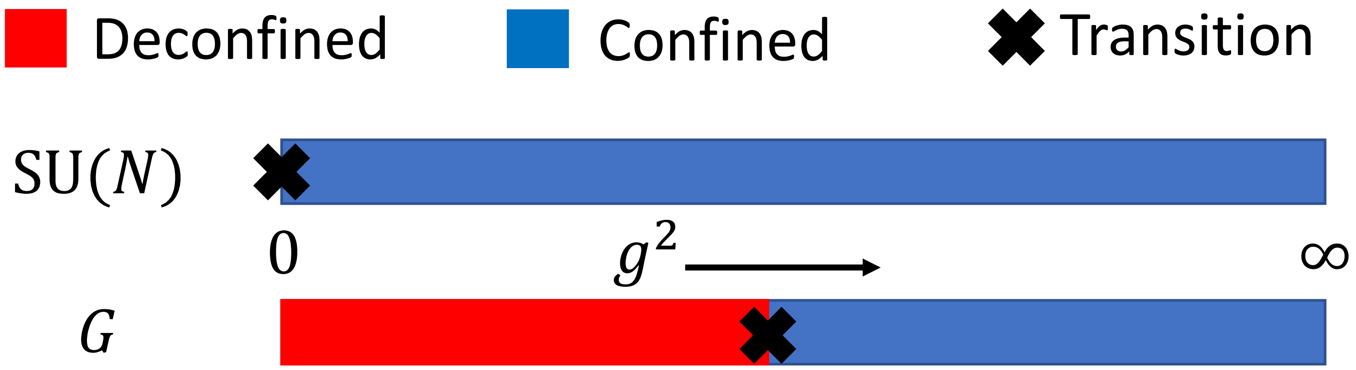
To compute the trace, need character identity:

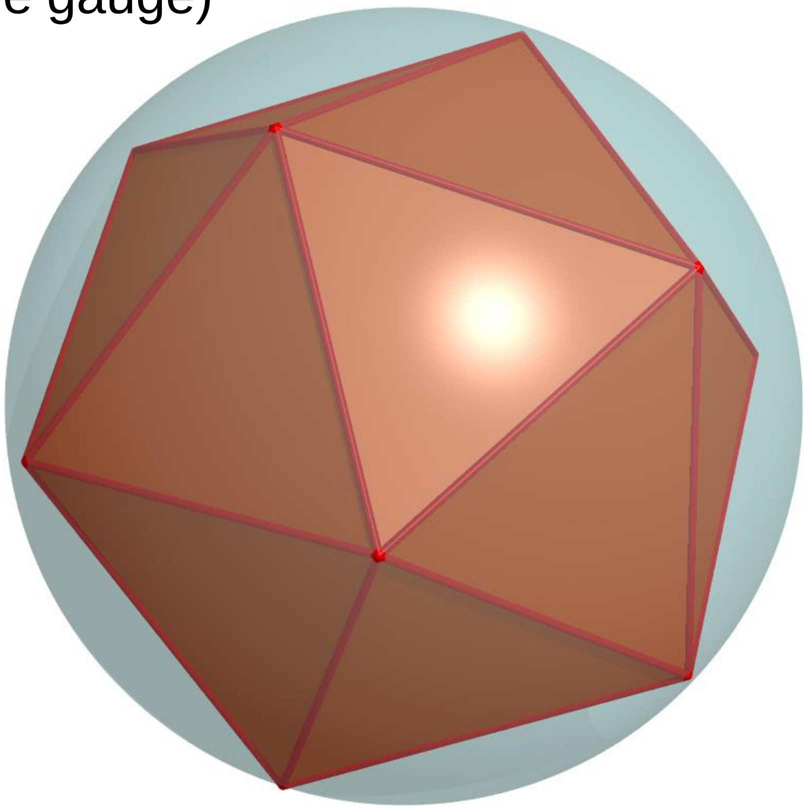
$$\sum_{j \in \text{Irrep}} \chi_j(g)^* \chi_j(h) = \begin{cases} \frac{|G|}{|C|} & \text{if } g, h \in C \text{ (same conjugacy class)} \\ 0 & \text{otherwise} \end{cases}$$

But if we keep only some irreps (i.e. $j \leq j_{\text{max}}$) the formula no longer simplifies.

Finite subgroups

For finite subgroup $G \leq \mathrm{SU}(N)$ in 4D (zero temperature, pure gauge)





For G finite, the transition is only **first order**

→ effective theory

→ requires **improvement**

Q: Is there an action/formulation with a *second-order* transition?

[Hasenfratz & Niedermayer ’01]

Laplacian on a discrete group

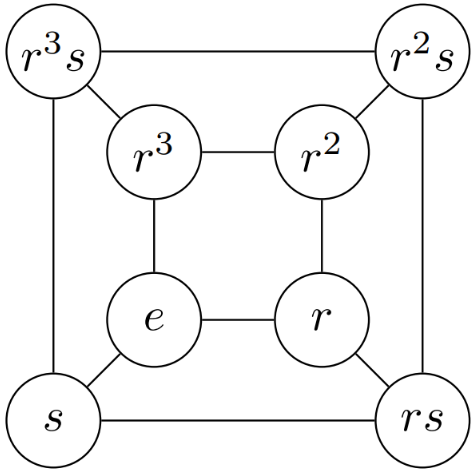
Hamiltonian requires $\mathrm{tr}(E^2)$ but **for a finite group $\boldsymbol{E}$ does not exist!**

↓

But this is a Laplacian in group space! ⟶ Construct Laplacian for finite groups

Geometric structure of a finite group is a **graph:**

Vertices = group elements
Edges based on **multiplication structure**.



Example for the finite group

$$D_4 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$$

[Mariani & Ercolessi '20]

but see also
[Orland '91]

[Caspar et al '16]

[Harlow & Ooguri '18]

Every graph has a graph Laplacian!