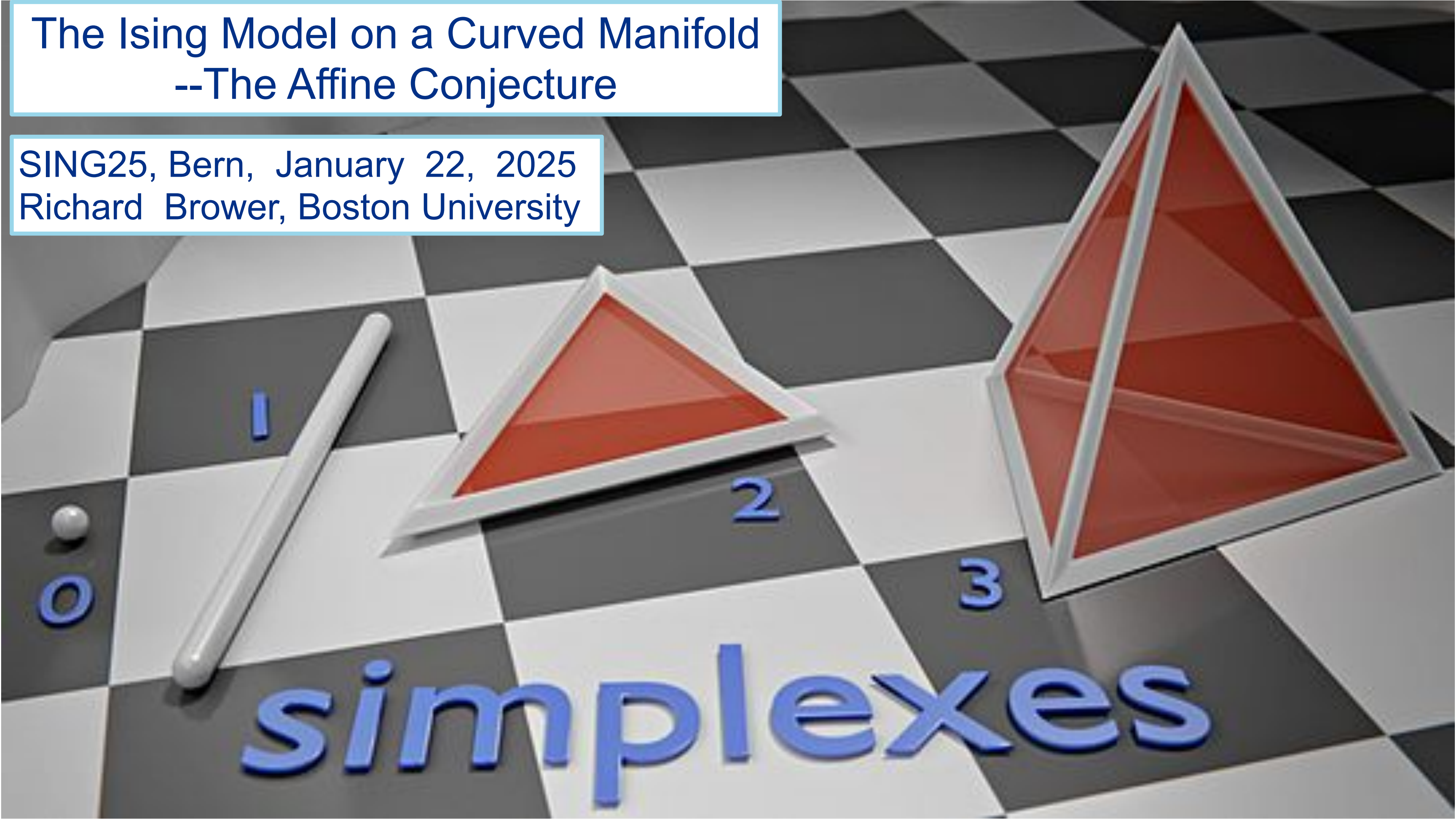
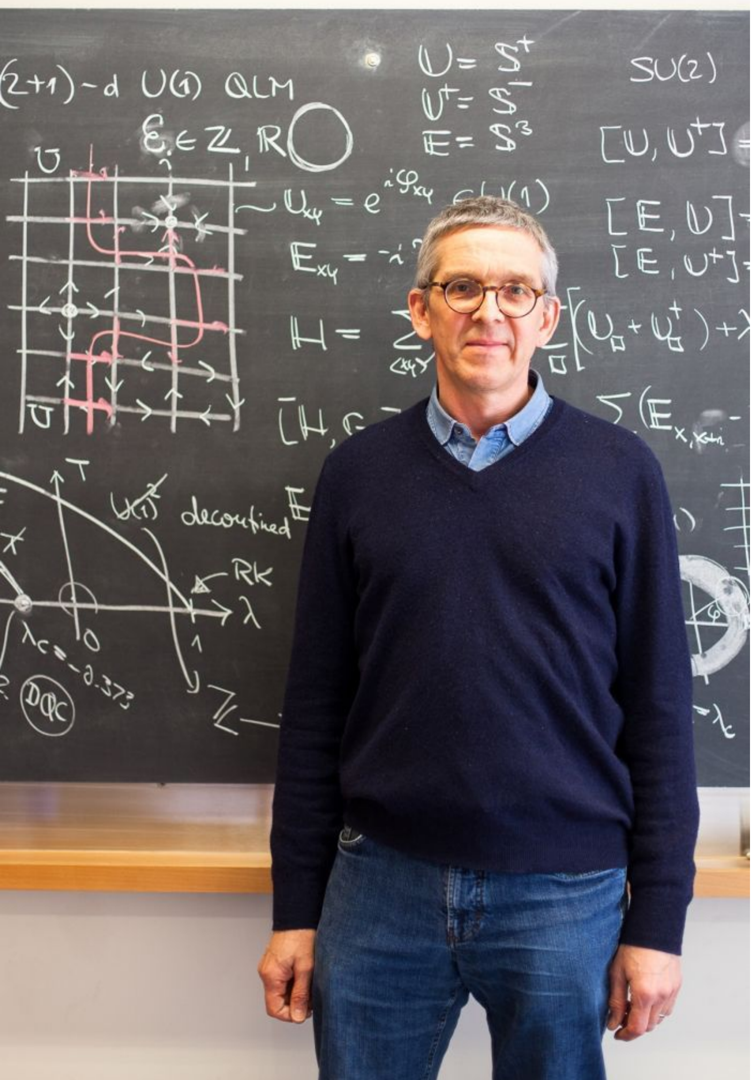
The Ising Model on a Curved Manifold --The Affine Conjecture

SING25, Bern, January 22, 2025

Richard Brower, Boston University





QCD as a Quantum Link Model *

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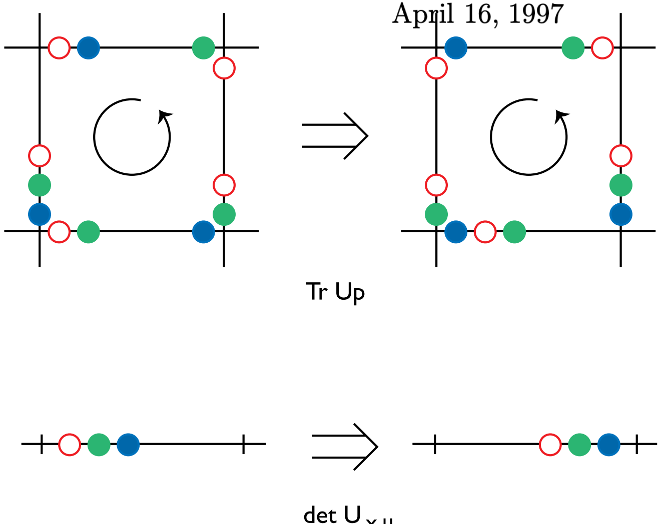
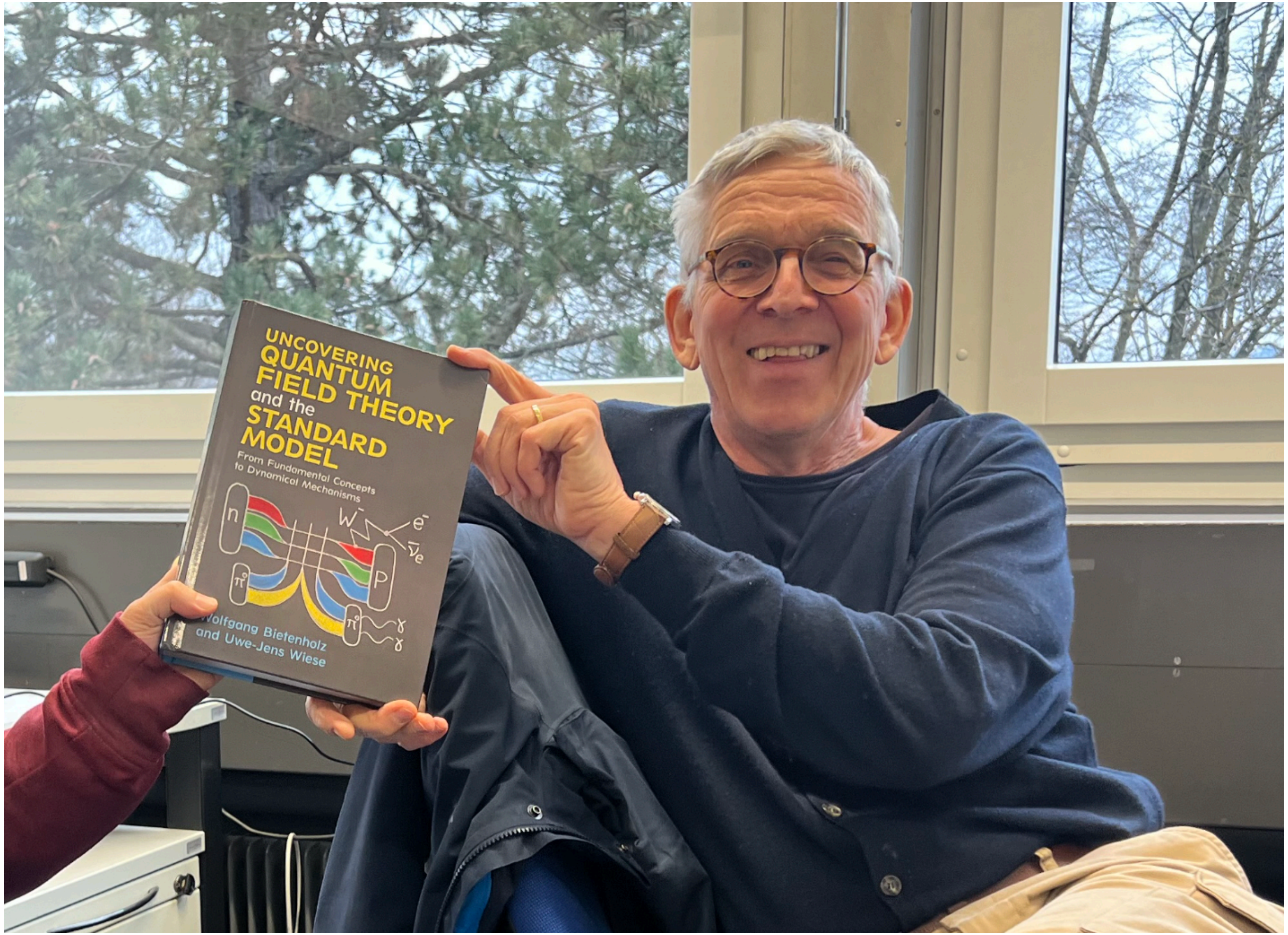


My Reading Assignment



FUNDAMENTAL PROBLEM

- CONSTRUCT LATTICE FIELD THEORIES ON CURVED MANIFOLDS THAT GIVE EXACT* RESULTS IN THE CONTINUUM

- IS THIS POSSIBLE?

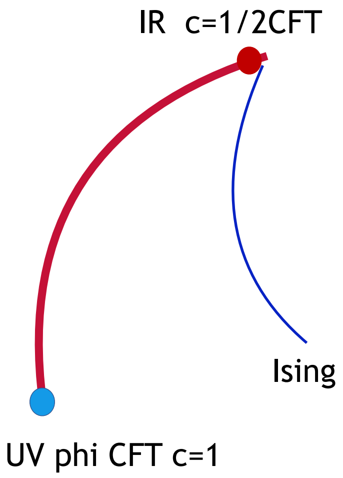
- IS THERE A GENERAL THEORY?

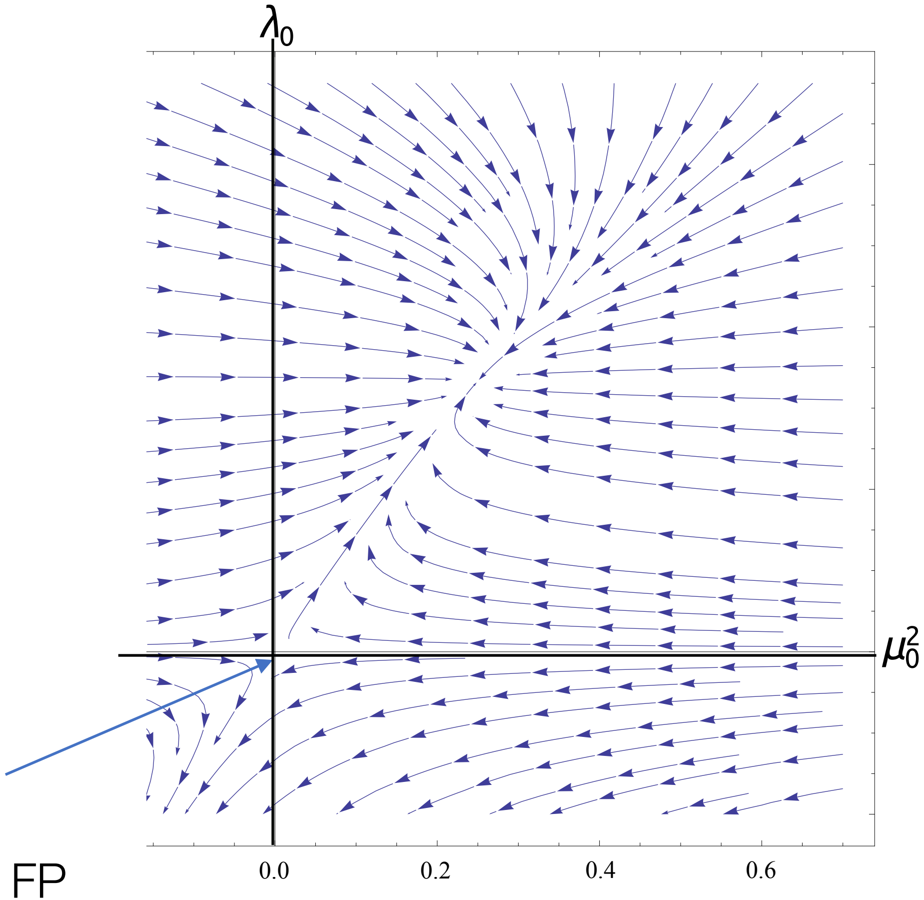
* "Exact" means polynomial complexity in "a = 1/UVcutoff" (aka like Monte Carlo Euclidean lattice QCD)

Outline

- GOAL:
  - Radial Quantize Lattice Conformal and near Conformal Field Theory.
- TEST EXAMPLE:
  - 2d Ising CFT on Sphere and 3d Ising on a Cylinder .
- IS IT GENERAL?
  - Affine Conjecture: There exist a map from flat Affine space to tangent plane

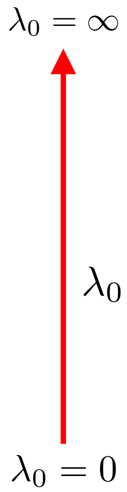
Test Case: Scalar Phi4/Ising Model Universality





UV: Gaussian FP

$$H_{Ising} = \frac{K}{2}\sum_{\langle i,j\rangle}(s_i - s_j)^2 = -K\sum_{\langle i,j\rangle} s_i s_j$$



$$S = \frac{K}{2}\sum_{\langle i,j\rangle}(\phi_i - \phi_j)^2 + \lambda_0(\phi_i^2 - 1)^2$$

First step: Construct the Classical Simplicial Action

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d x \sqrt{g(x)}\,[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)]$$

$g_{\mu\nu}(x)$

Regge Calc Geometry

Quantum field $\phi(x)$

Finite Element Method

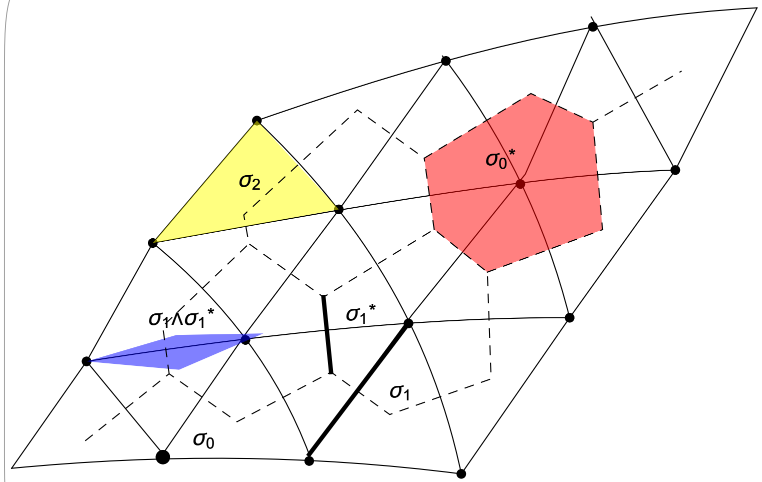
Classical Simplicial Action

$$S_{FEM} = \frac{1}{2}[\sum_{y\in\langle\, x,y\rangle} K_{xy}(\phi_x - \phi_y)^2 + \sqrt{g_x}[\xi\; Ric\; \phi_x^2 + m_0^2\phi_x^2 + \lambda_0\phi_{t,x}^4]]$$

Both Regge's Manifold and Classical Field on the Same Simplicial Complex

REGGE: Piecewise linear metric

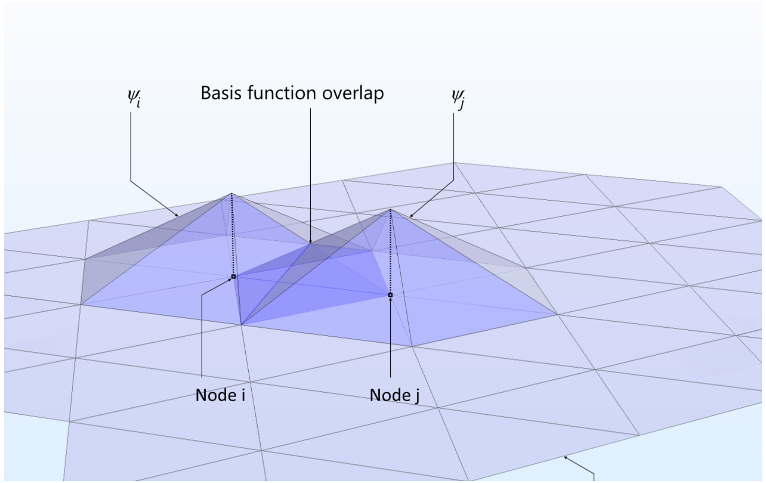
$$(\mathcal{M}, g_{\mu\nu}(x)) \leftrightarrow (\mathcal{M}_\sigma, g_\sigma = \{l_{ij}\})$$



$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

FEM: Piecewise linear fields

$$\phi(x) \leftrightarrow \phi = \sum_i \phi_i W_i(\xi)$$



RADIAL QUANTIZATION D > 2

CARDY (1985 "Universal Amplitude in Finite Size Scaling")
lattice radial quantum is nice BUT very difficult for d>2

$$\mathbb{R}^d \implies \mathbb{R} \times \mathbb{S}^{d-1}$$

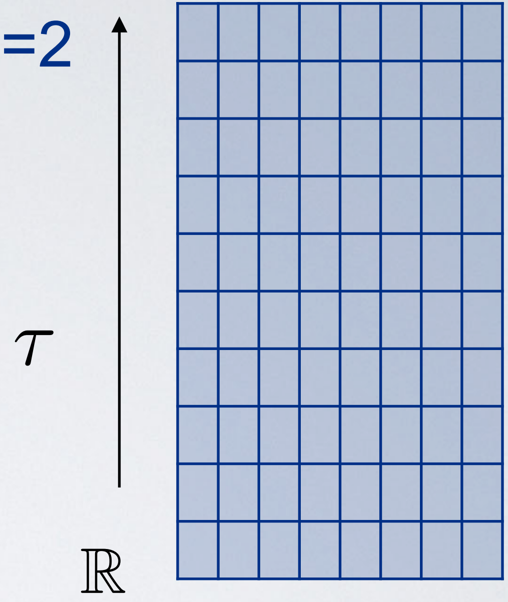
Infinite Cylinder d-1 Sphere

$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

$$1 < t < aL \implies 1 < \tau = log(r) < L$$

Exponential "time" in lattice units:

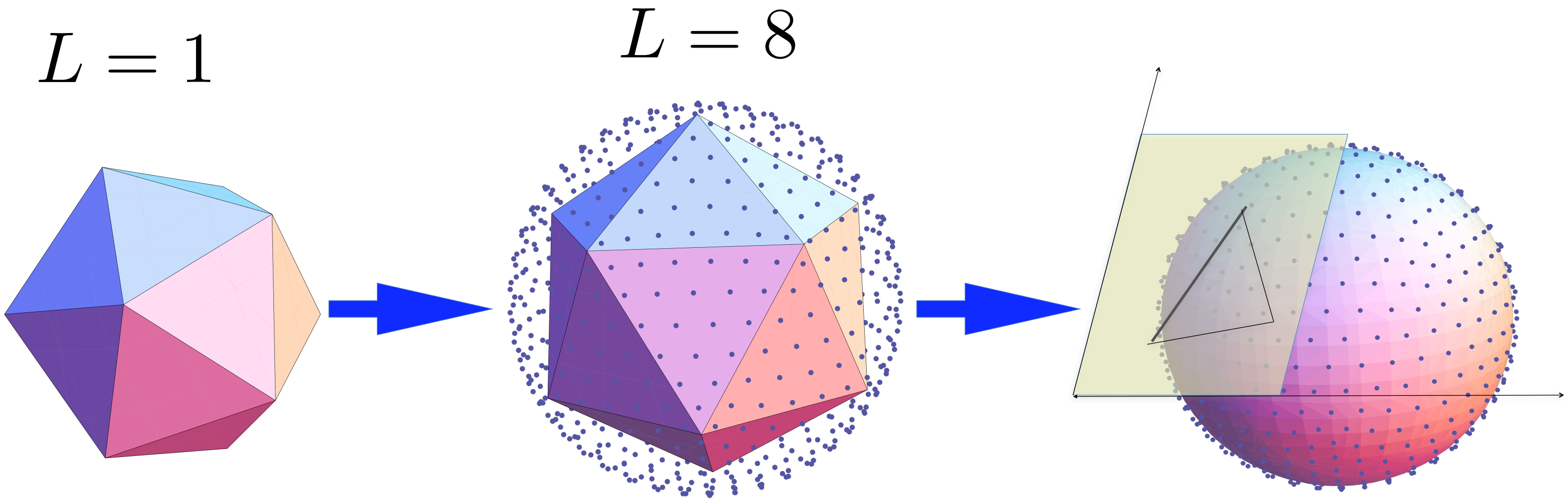
d =2



$\mathbb{S}^1$ with $\theta \in [0, 2\pi]$

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$$
$$= r^2[dlog(r)^2 + d\theta^2]$$
$$\to ds^2_{cylinder} = d\tau^2 + d\theta^2$$

The radial Project Icosahedral Lattice Refinement



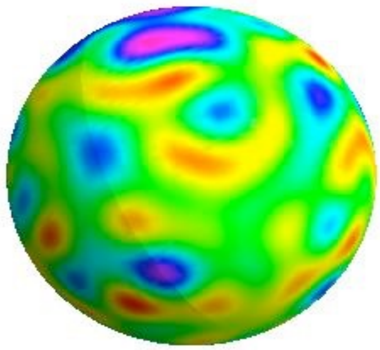
First Attempt (with good results): Classical FEM with UV counter term

Start with maximum regular Tessellation: preserve Icosahedral group upon refinement

I = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

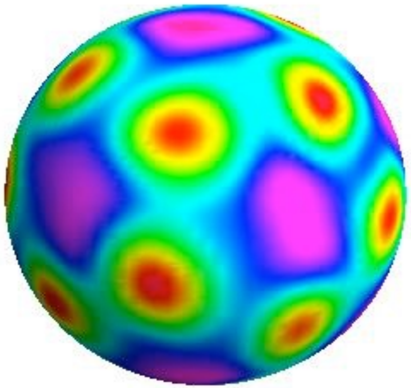
Partial Success Quantum FEM for phi 4th Theory

$\phi^2(x)$



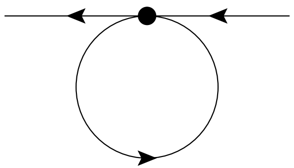
one configuration

$\langle \phi^2(x) \rangle$



Average of config.

Now add $\lambda\phi^4$ term:



$$\delta m^2 = \lambda \langle \phi(x)\phi(x) \rangle \to \frac{1}{K_{xx}}$$

With lambda_0 = 1 NUMERICAL TEST against Exact c=1/2 Ising CFT

μ^2	s	$r_{\min} \leq r \leq r_{\max}$	norm	Δ_ϵ	λ_ϵ^2	c
1.82241	9	$0.25 \leq r \leq 0.75$	0.2900	1.075	0.2536	0.4668
1.82241	9	$0.30 \leq r \leq 0.70$	0.2901	1.075	0.2533	0.4704
1.82241	9	$0.35 \leq r \leq 0.65$	0.2902	1.077	0.2533	0.4738
1.82241	9	$0.40 \leq r \leq 0.60$	0.2902	1.016	0.2427	0.4747
1.82241	18	$0.25 \leq r \leq 0.75$	0.2051	1.068	0.2563	0.4866
1.82241	18	$0.30 \leq r \leq 0.70$	0.2051	1.056	0.2544	0.4878
1.82241	18	$0.35 \leq r \leq 0.65$	0.2051	1.050	0.2535	0.4904
1.82241	18	$0.40 \leq r \leq 0.60$	0.2051	1.046	0.2526	0.4884
1.82241	36	$0.25 \leq r \leq 0.75$	0.1457	1.031	0.2528	0.4926
1.82241	36	$0.30 \leq r \leq 0.70$	0.1458	1.026	0.2519	0.4932
1.82241	36	$0.35 \leq r \leq 0.65$	0.1458	1.018	0.2508	0.4931
1.82241	36	$0.40 \leq r \leq 0.60$	0.1458	1.007	0.2486	0.4933

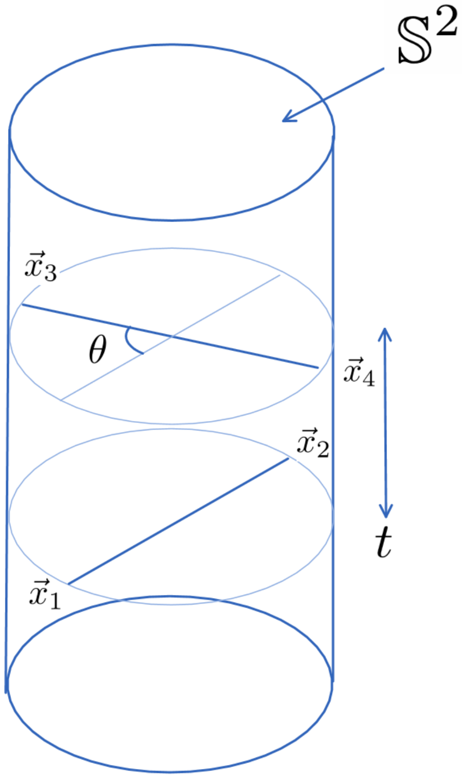
Lattice Sizes: N = 32+ 10 s^2 sites: Very Efficient Brower/Tamayo Cluster algorithm

Fails to have critical surface at large bare lambda and likely any fixed bare lambda as a --0

Antipodal 4-point function on $\mathbb{R} \times \mathbb{S}^2 \ni (t, \vec{x})$

$$\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle \sigma(x_1)\sigma(x_2)\rangle\langle \sigma(x_3)\sigma(x_4)\rangle} = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} G_{\mathcal{O}}(\Delta_{\mathcal{O}}; x_1, x_2, x_3, x_4)$$

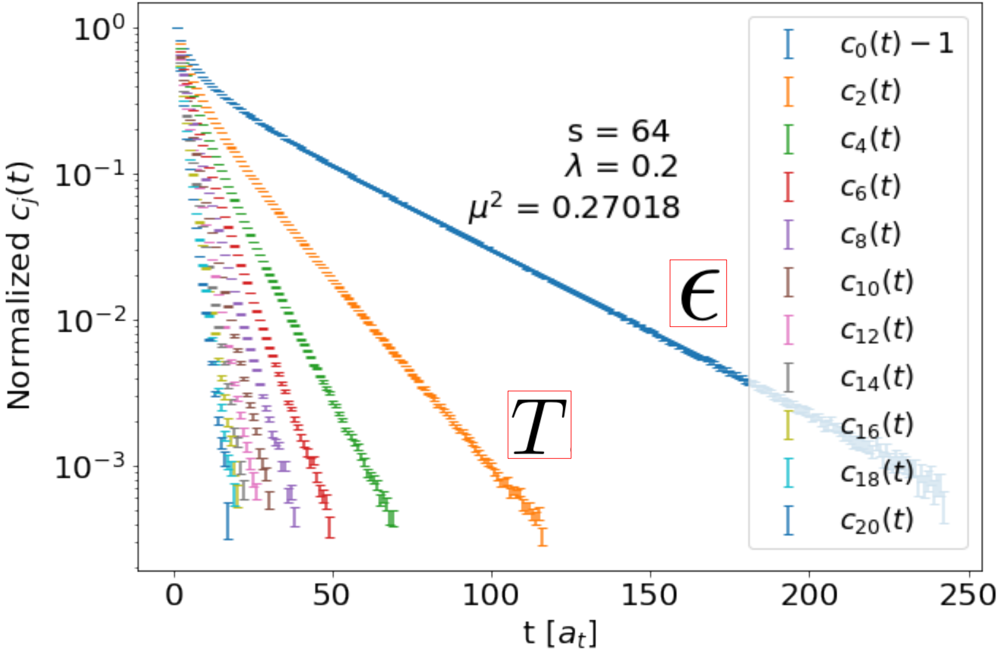
$$G_{\mathcal{O},l} = \sum_{n=0,2,4,\ldots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$



Numerical results

$$j \in \{\max(0, l-n), \ldots, l+n-2, l+n\}$$

$$\frac{\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4)\rangle}{\langle \sigma(x_1)\sigma(x_2)\rangle\langle \sigma(x_3)\sigma(x_4)\rangle} = \sum_{\text{even } j} c_j(\Delta t) P_j(\cos(\theta)) = 1 + \sum_{\mathcal{O}} f^2_{\sigma\sigma\mathcal{O}} \sum_{n=0,2,4,\ldots} \sum_j e^{-(\Delta_{\mathcal{O}}+n)c_R g t} B_{n,j}(\Delta_{\mathcal{O}}) P_j(\cos(\theta))$$

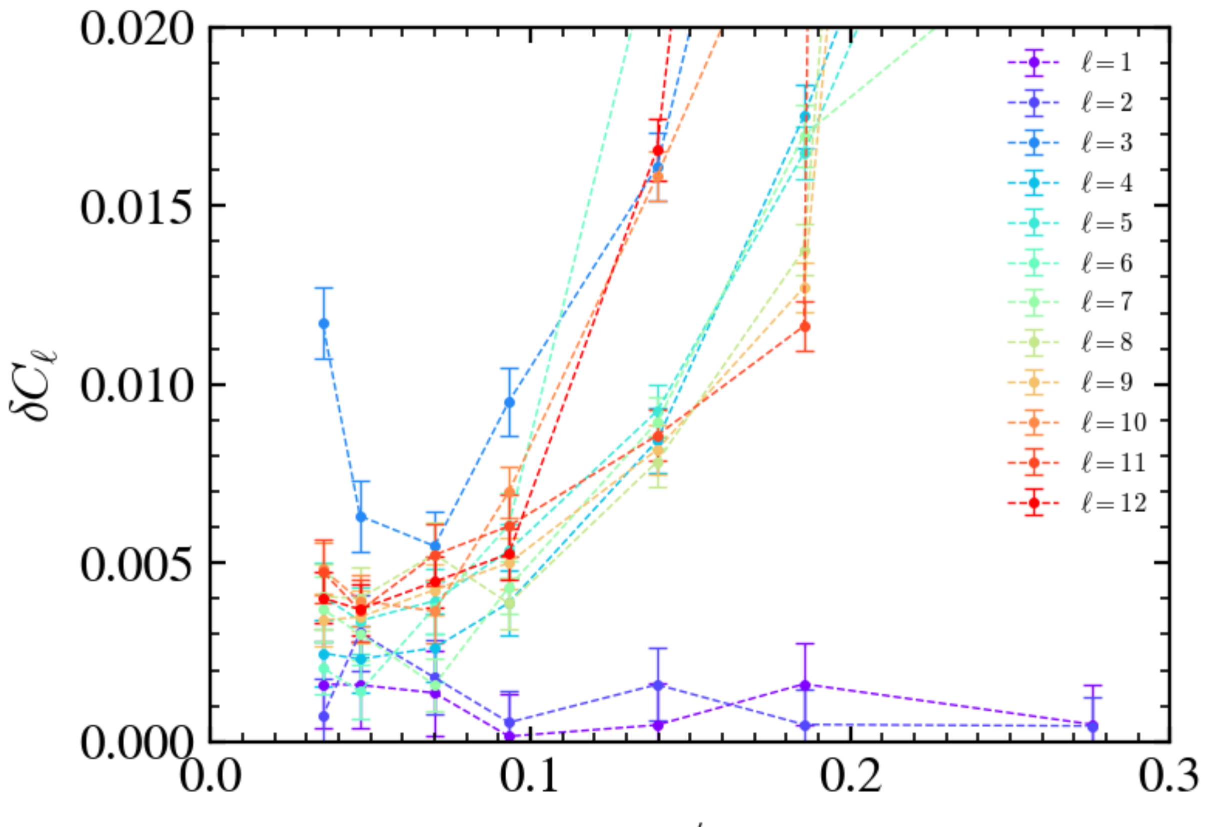


Simultaneous fits of $c_0(t)$ and $c_2(t)$ using primaries $\epsilon,\ T,\ \epsilon',\ T'$ up to n=20

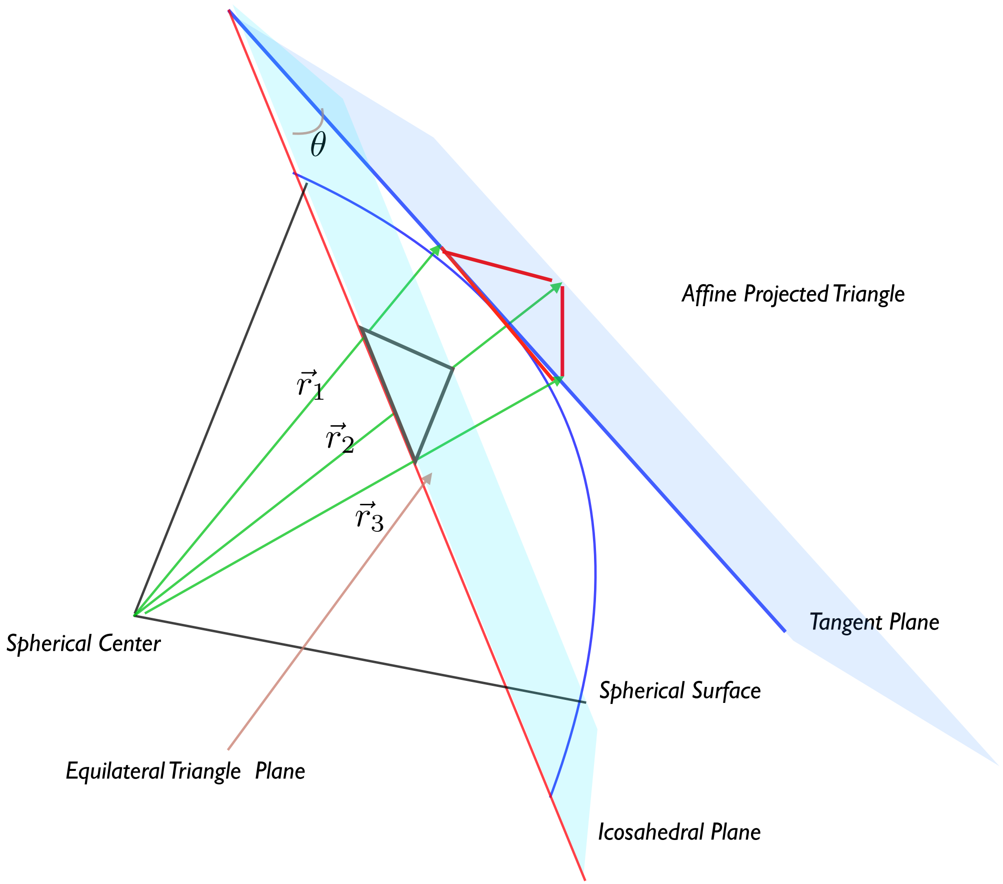
Nice result but central charge appear to violate the bootstrap bound

Also some breaking o f of Rotational Symmetry for

Phi4 with Counter Term



Back to Ising (lambda = infinity)
Project an Affine lattice on each tangent plane to O(a^2)



Affine Parameters:
2d Affine transformation takes circle to ellipse:

$$\langle \phi(x,y)\phi(0)\rangle = \frac{1}{(x^2+y^2)^{\Delta_\phi}} \leftrightarrow \frac{1}{(ax^2+bxy+cy^2)^{\Delta_\phi}}$$

- d = 2 Poincare 1 rotation 2 translation
- New Affine plus 1 major/minor + 1 orientation + 1 scaling
- Poincare d(d+1)/2 plus d(d+1)/2 the number of edge in d-simplex

Back to Simplicial Geometry

$$S = S_{EH} + S_M = \int dx \sqrt{g} [\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_M[\phi]]$$

$$\frac{\delta}{\delta g^{\mu\nu}(x)} \implies \quad R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = \kappa T_{\mu\nu}(x)$$

$$\frac{\delta \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle}{\delta g^{\mu\nu}(x)} = \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n) T_{\mu\nu}(x)\rangle$$

How to put Quantum Fields on a Lattice?

REGGE: "General Relativity without Coordinates" 1960

$$S_{EH} = \int d^d x \sqrt{g(x)} R(x)$$

$$S_{Regge}[\ell_{ij}] = 2 \sum_{h \in G} A_h \epsilon_h$$

$$\{\mathcal{M}, g_{\mu\nu}\}$$

$$\{G, \ell_{ij}\}$$

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

Graph with edge length

The **Whitney embedding theorem** states that any smooth real m-dimensional manifold can be smoothly embedded in the real $2m$-space,

The Simplicial Approximation Theory (L.E.J. Brouwer 1927?)

Equation of Motion:

hinge dim = D -2

$$S_{Regge} = 2\sum_h V_h \epsilon_h$$

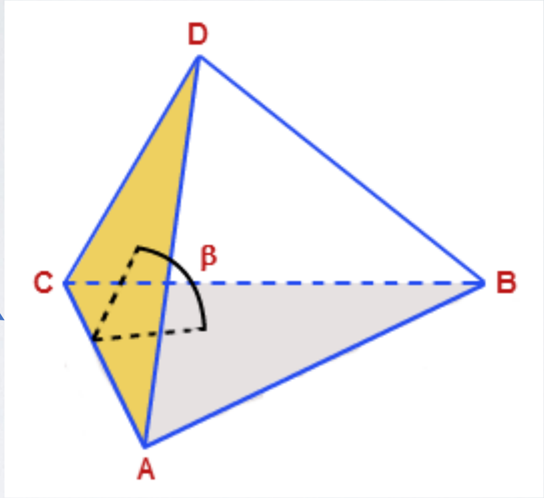
Dihedral Angle in simplex

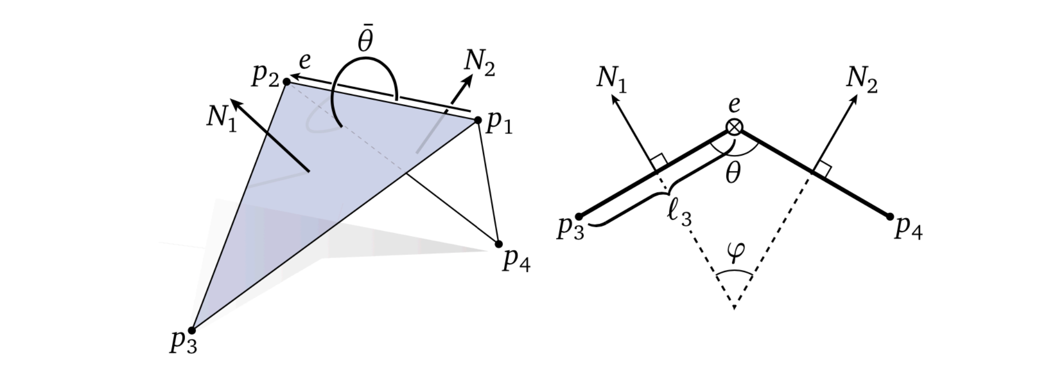
$$\epsilon_h = 2\pi - \sum_{h\in\sigma} \theta_{\sigma,h}$$

$$\frac{\partial S}{\partial \ell_{ij}} = \frac{\partial V_h}{\partial \ell_{ij}} \epsilon_h + \sum_h V_h \frac{\partial \epsilon_h}{\partial \ell_{ij}}$$

0

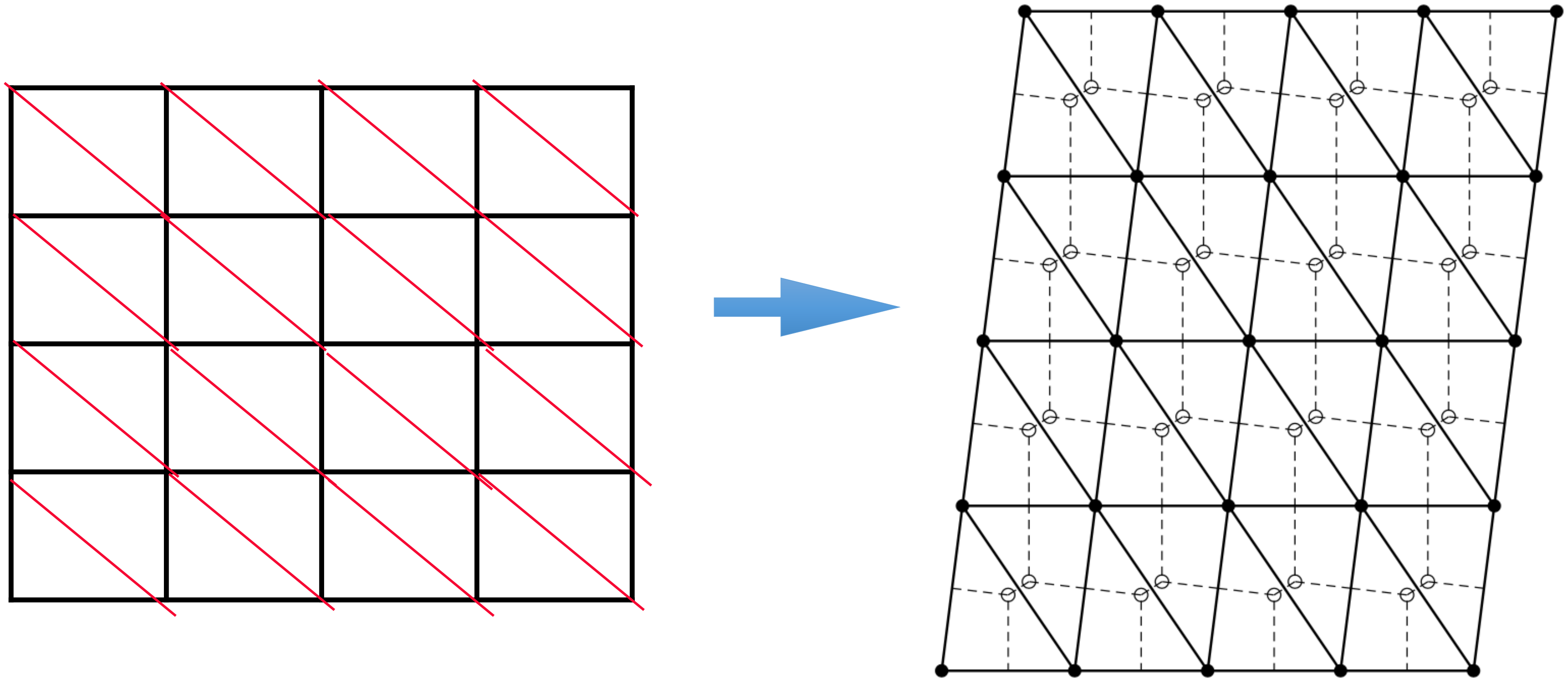
Schlefli Identity



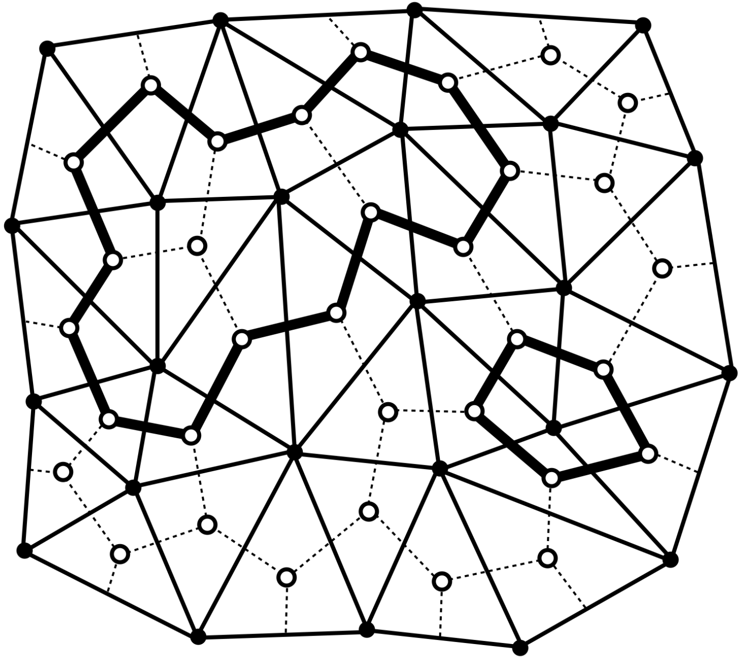


In search of Ludwig Schlefli ?

Analytical solution to 2d affine Ising model



3 Equivalent Loop Expansion for Partition Functions!



$$S_\triangle = -\sum_{\langle ij \rangle} K_{ij} s_i s_j$$

$$S_{\rm dual} = -\sum_{\langle ij \rangle} L_{ij} s_i s_j$$

$$S_\psi = \frac{1}{2}\sum_i \bar\psi_i \psi_i - \sum_{\langle ij \rangle} \kappa_{ij} \bar\psi_i P_{ij} \psi_j \; ,$$

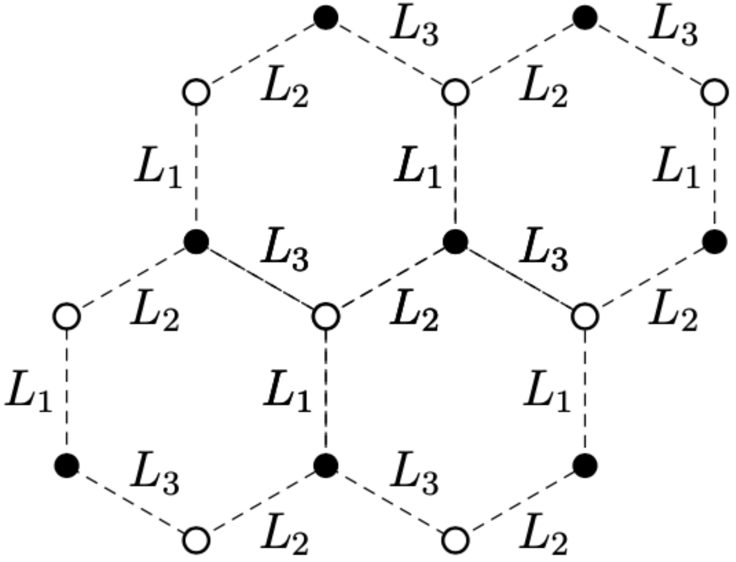
U. Wolff

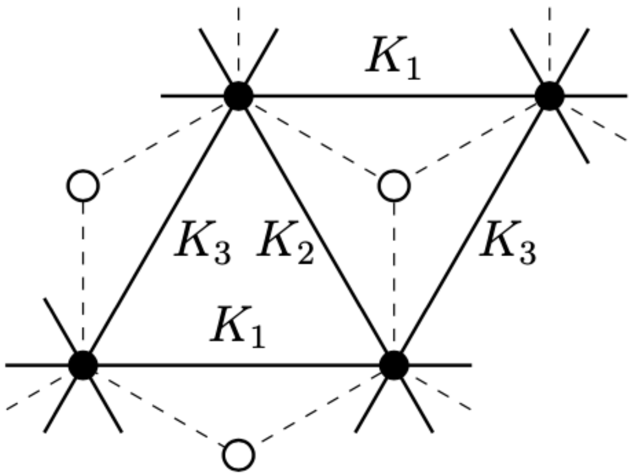
Kramers Wannier High T/Low T Loop expansion + Wilson-Majorana Lattice Fermions

$$\sinh 2K_{ij} \sinh 2L_{ij} = 1$$

$$P_{ij} = \frac{1}{2}(1 + \hat e_{ij} \cdot \vec\sigma)$$

Step I : Star Triangle ID: Hex to Triangle Map





$$h \sinh(2K_1)\sinh(2L_1) = h\sinh(2K_2)\sinh(2L_2) = h\sinh(2K_3)\sinh(2L_3) = 1$$

$$h(K_1, K_2, K_3) = \frac{(1-v_1^2)(1-v_2^2)(1-v_3^2)}{4\sqrt{(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_3v_1)(v_3+v_1v_2)}} \qquad \text{with } v_i = \tanh(K_i)$$

Proved Emergent Geometry Required

$$\sinh(2K_1) = \ell_1^*/\ell_1 \quad , \quad \sinh(2K_2) = \ell_2^*/\ell_2 \quad , \quad \sinh(2K_3) = \ell_3^*/\ell_3$$

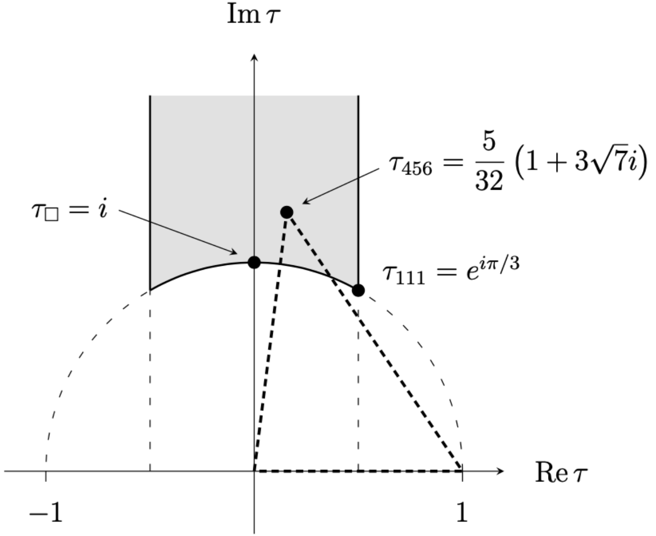
- Implies Critical at $p_1 p_2 + p_2 p_3 + p_3 p_1 = 1 \quad \text{with} \quad p_i = \exp(-2K_i)$

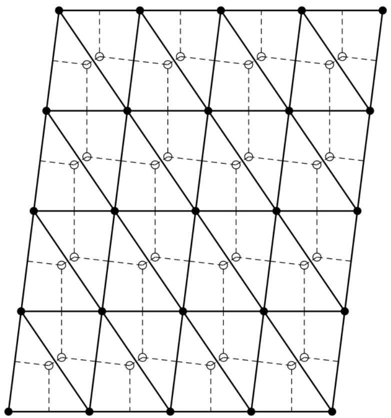
- Not the same as Free (FEM) scalar CFT.

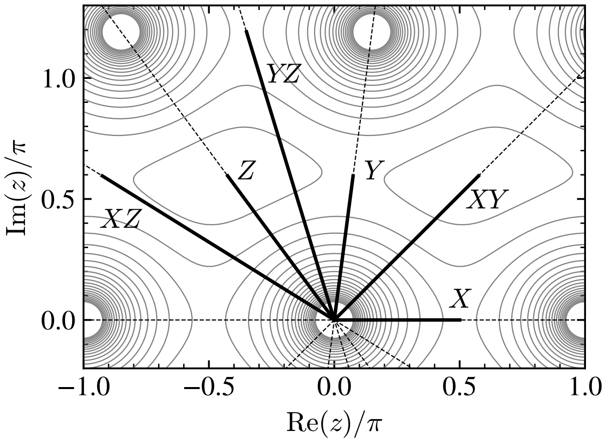
$$S_{\text{free}} = \frac{1}{2} \sum_n [K_1(\phi_n - \phi_{n+\hat{1}})^2 + K_2(\phi_n - \phi_{n+\hat{2}})^2 + K_3(\phi_n - \phi_{n+\hat{3}})^2]$$

$$2K_1 = \ell_1^*/\ell_1 \quad , \quad 2K_2 = \ell_2^*/\ell_2 \quad , \quad 2K_3 = \ell_3^*/\ell_3 \ .$$

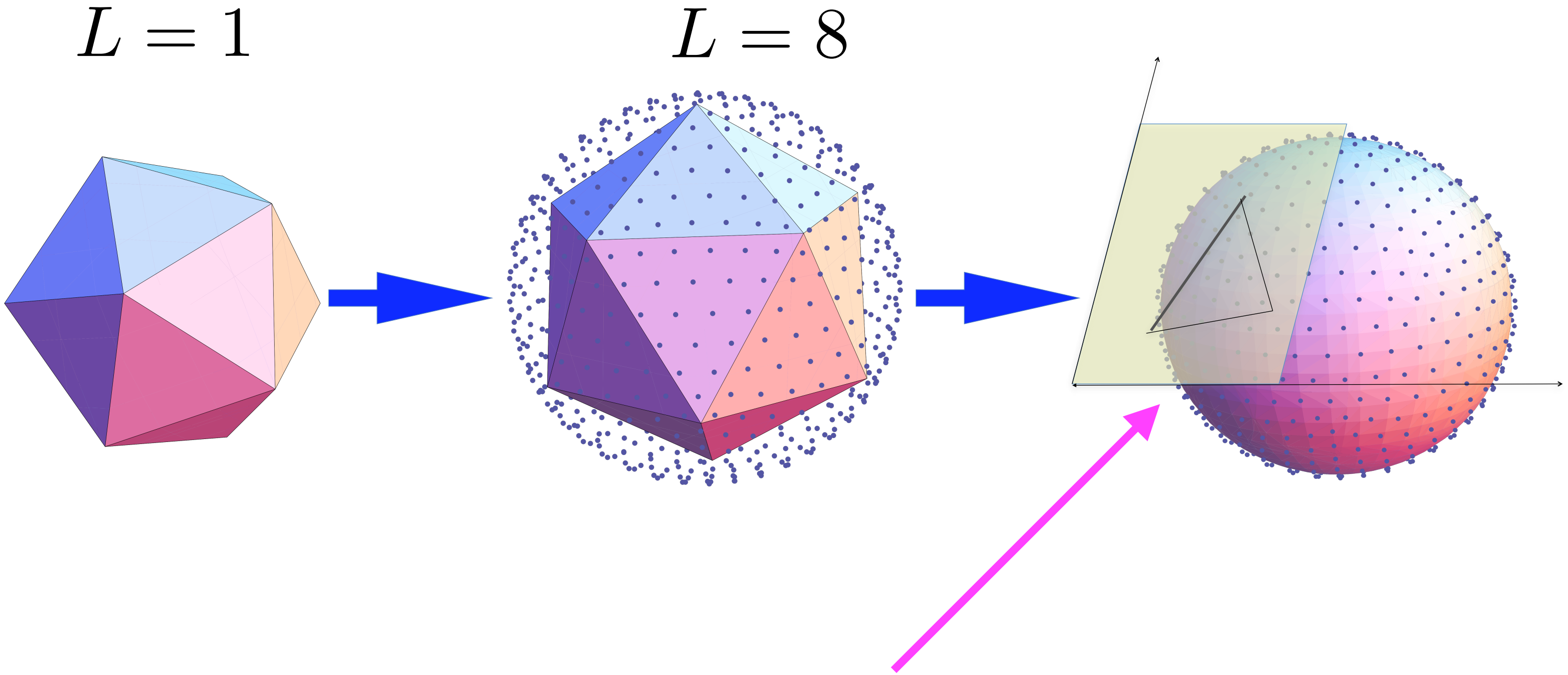
Can check Modular dependent on the torus







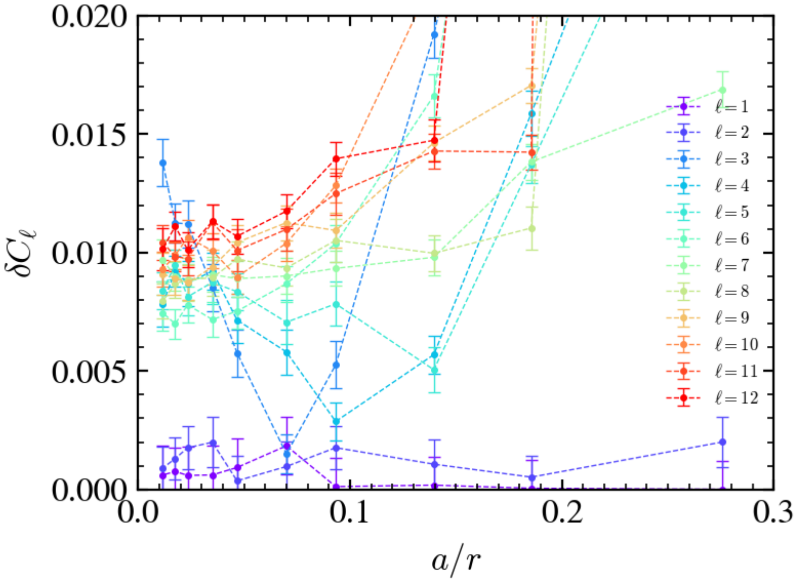
$$\langle \sigma(0)\sigma(z)\rangle = \left|\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z|\tau)}\right|^{1/4} \frac{\sum_{\nu=1}^{4} |\vartheta_\nu(z/2|\tau)|}{\sum_{\nu=2}^{4} |\vartheta_\nu(0|\tau)|}$$

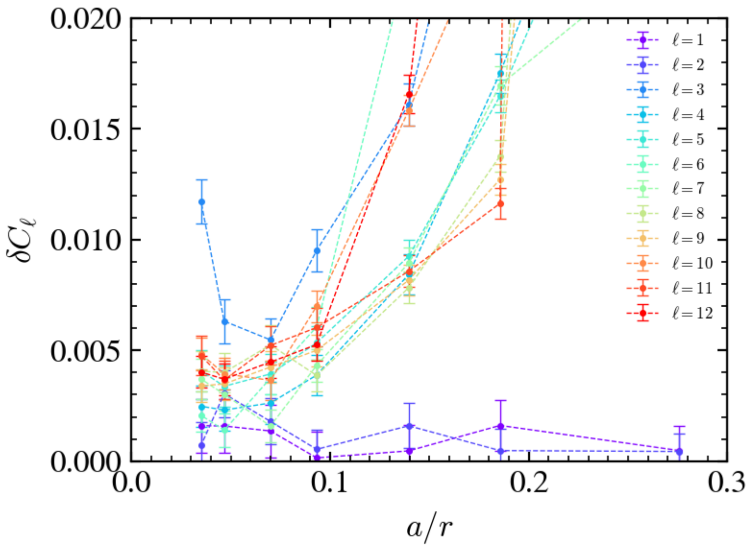


Now use these identities locally on each tangent plane so it uniformly critical in UV

Breaking of Rotational Symmetry on Projected Icosahedron

Ising on S2 Not better than Phi4 with Counter Term





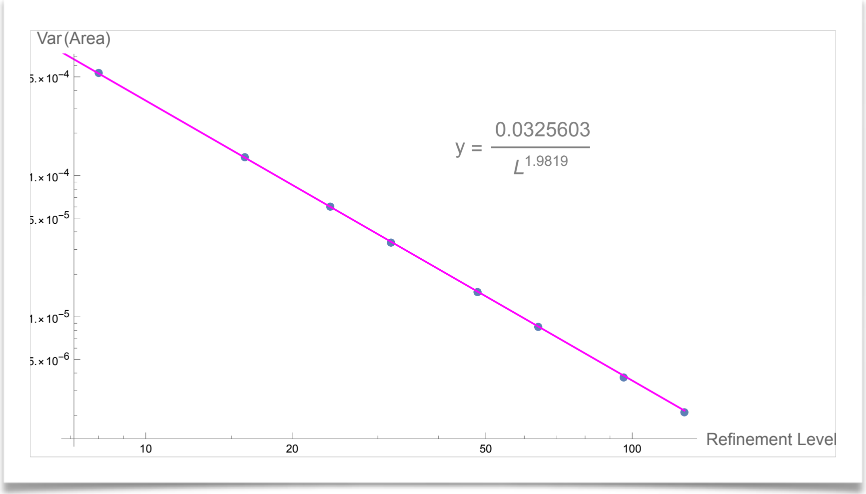
$$C_{l_1 m_1; l_2 m_2} = \sum_{i,j} \sqrt{g_i} Y^*_{l_1 m_1}(\hat{r}_i) \underbrace{\frac{1}{|g|} \sum_{g \in Ih} \langle s(g\hat{r}_i) s(g\hat{r}_j) \rangle}_{\text{e.g. } \frac{1}{(2-2\cos\theta_{ij})^{\Delta_\sigma}}} \sqrt{g_j} Y_{l_2, m_2}(\hat{r}_j) \quad \to c_l P_{l_1}(\hat{r}_i \cdot \hat{r}_j) \delta_{l_1, l_2} \delta_{m_1, m_2} + O(a^2) \qquad ?$$

Gauge Fix Co-ordinate on the Manifold by Area Optimization to smooth scalar curvature

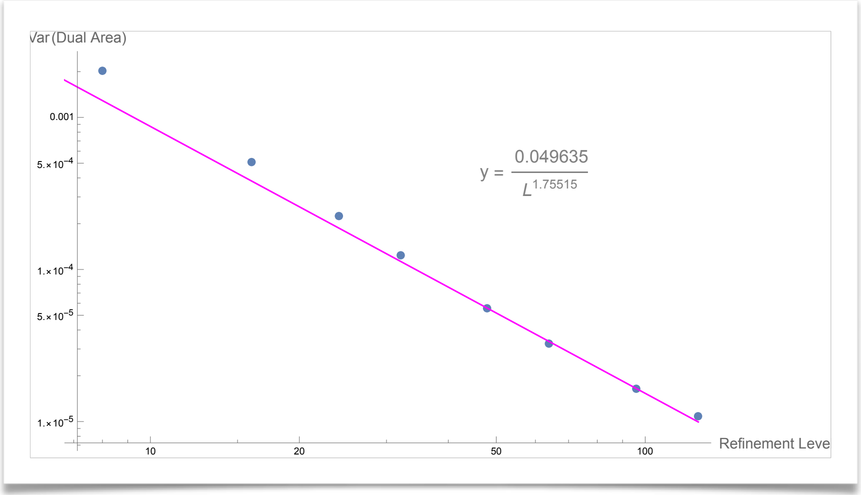
$$S(\ell_{ij}) = N^{-1} \sum_{\triangle} A^2_{\triangle}(l_{ij})$$

dof: $2N = 4 + 20L^3$

Area Variance



Dual Area Variance

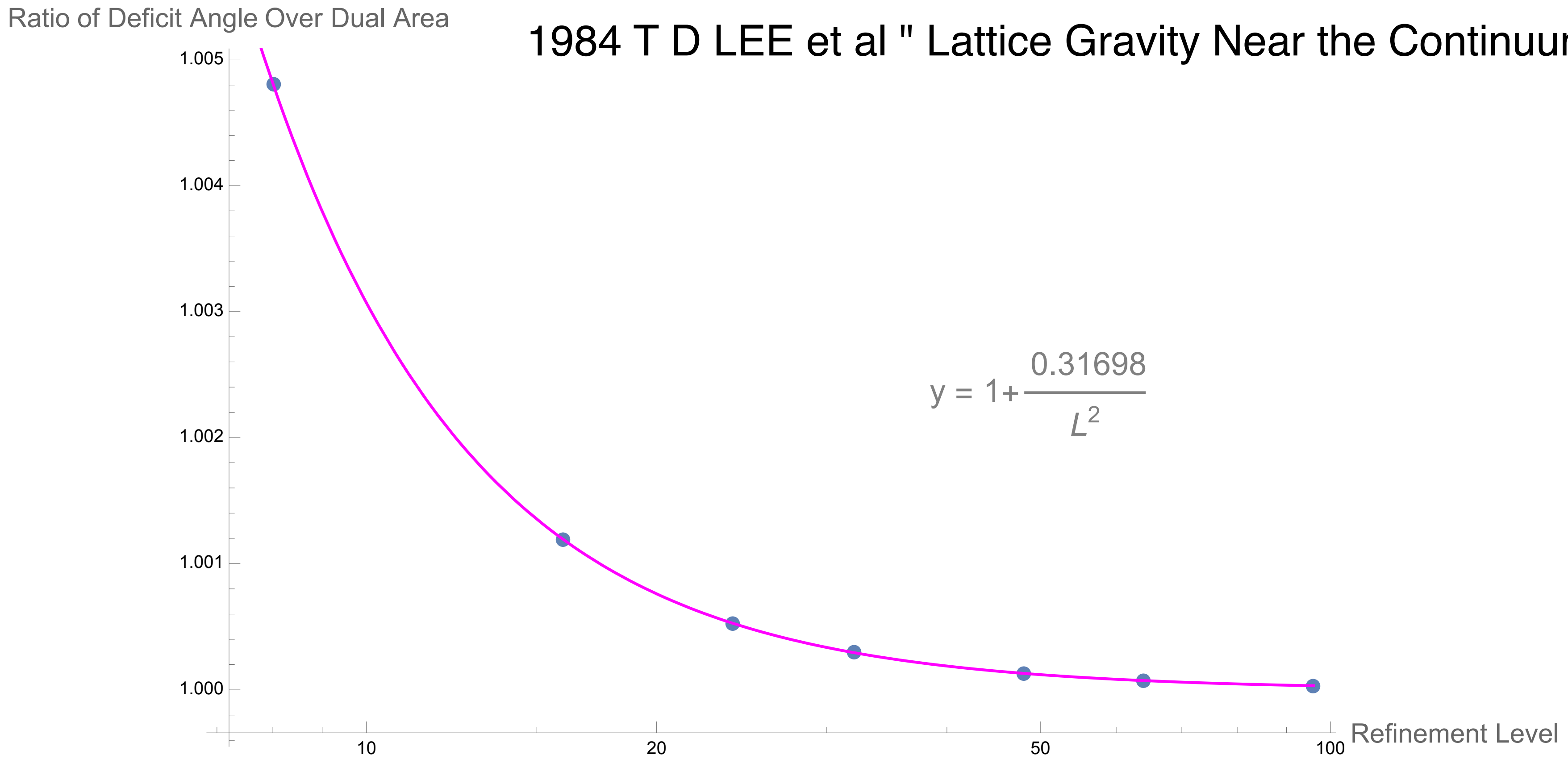


$$4A(a,b,c)^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$
$$= a^2b^2c^2/R^2_{\triangle}$$

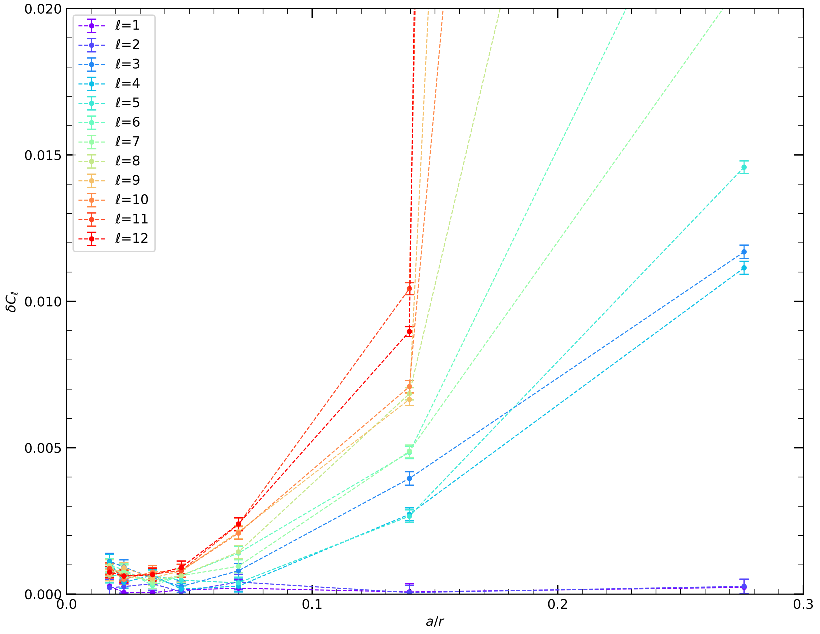
$$a^2 = \ell^2_{12} = |\vec{r}_1 - \vec{r}_2|^2 = 2 - 2\vec{r}_1 \cdot \vec{r}_2$$

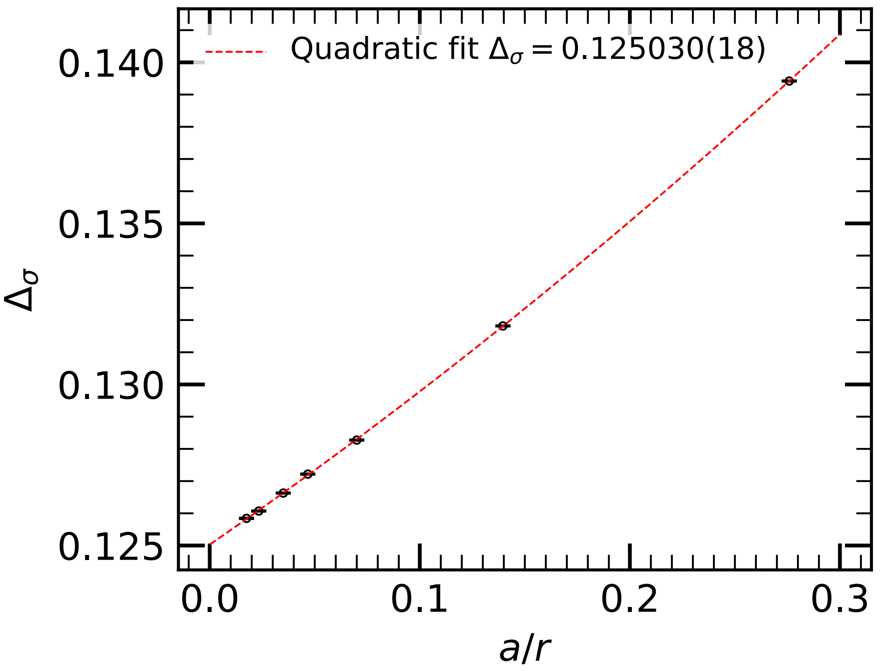
Smooth Scalar Curvature Theorem

1984 T D LEE et al " Lattice Gravity Near the Continuum"



Apparently it "wants" to work

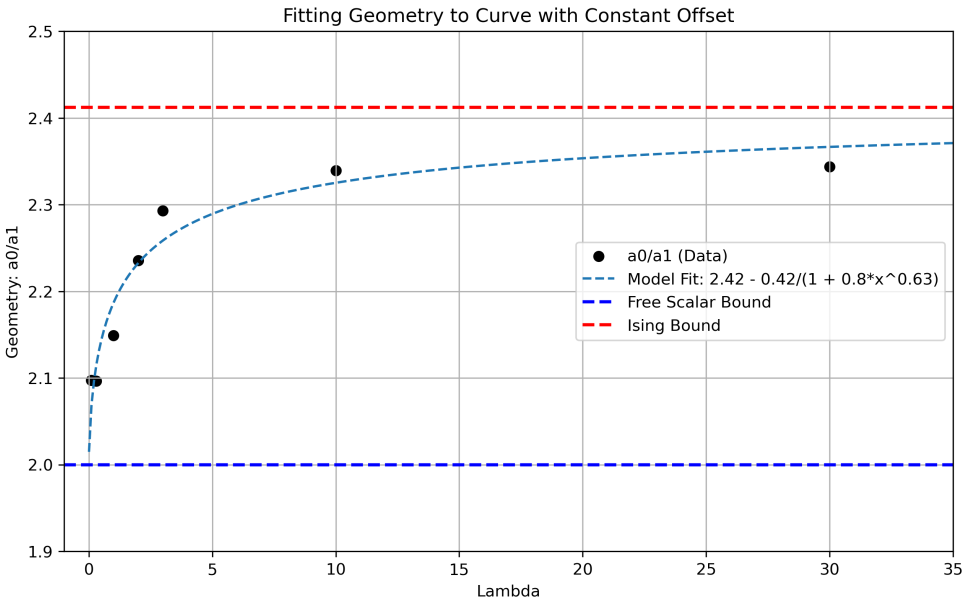




WHAT'S NEXT?

- See if Affine Map is a general non-perturbative (exact?) solution to spherical lattice field theory -- precision/theory.

- Test for 2d phi^4 theory beyond analytical:



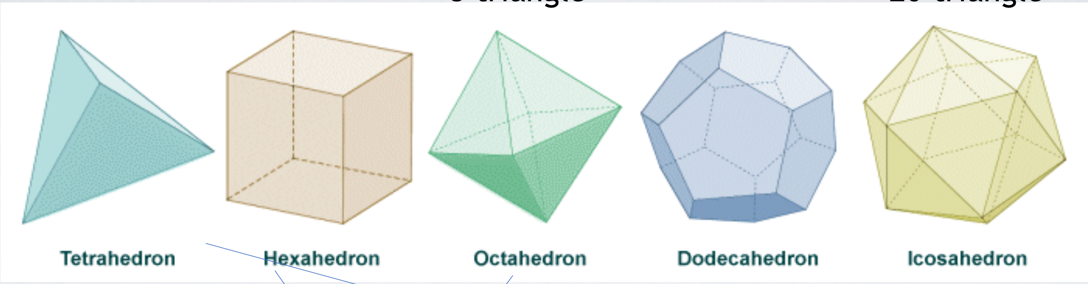
$$f(K_i, \lambda_0) = \ell_i^* / \ell_i$$

2D & 3D SIMPLCIAL PLATONIC SOLIDS

4 triangle

8 triangle

20 triangle



dual

self dual

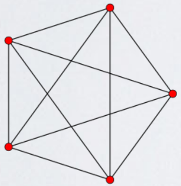
5 tetra

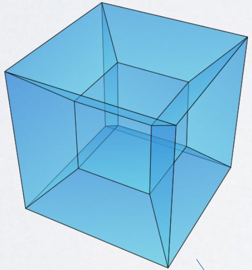
8 cubes

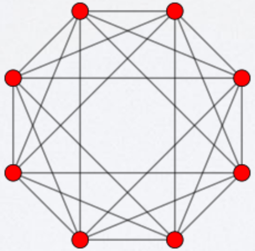
16 tetra

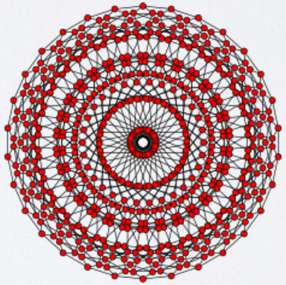
120 dedaca

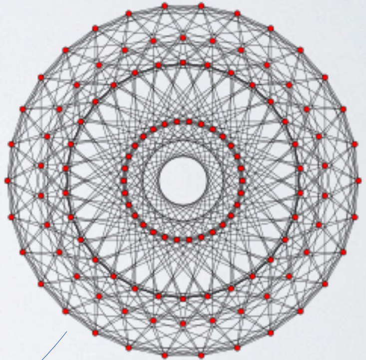
600 tetra











self dual

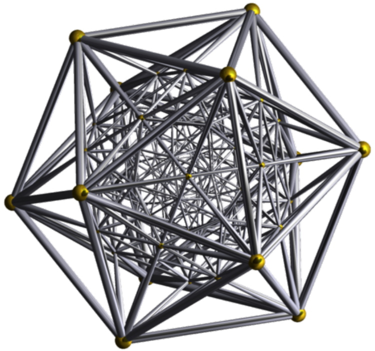
6th self dual with 24 octahedrons

Euler N - E + F - V = 0

https://en.wikipedia.org/wiki/Regular_4-polytope#

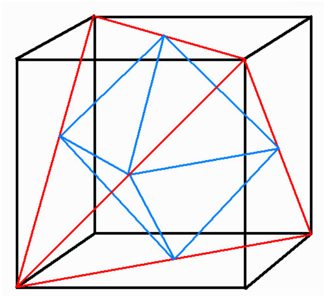
3 Spheres and 4D Radial Simplicial Lattices

$$\mathbb{S}^3 \quad \Longrightarrow \quad \mathbb{R} \times \mathbb{S}^3$$



Aristotle' s 2% Error!

$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$



Fast Code Domains of
Regular 3D Grids on Refinement

600 cell: “Square of the icosahedron” —Symmetries 1440= 120 * 120 the 120 copies of icosahedron

$$O(4) \sim SU(2) \times SU(2)$$

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group.

Of course Schlefli knew this!

https://homeweb.unifr.ch/kellerha/pub/Schlaefli-article2010.pdf

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Das Schläfli-Symbol eines regulären Polytops $P_{\text{reg}} \subset \mathbb{E}^n$ ist rekursiv definiert durch $P_{\text{reg}} =: \{p_1, \ldots, p_{n-1}\}$, falls

- die Fazetten gegeben sind durch $\{p_1, \ldots, p_{n-2}\}$, und
- die Eckenfiguren gegeben sind durch $\{p_2, \ldots, p_{n-1}\}$.

Abb. 3 Ein Ikosaeder $\{3, 5\}$ beim Bahnhof SBB Basel, Juni 2010. (Photographiert von Elisabeth Kellerhals.)

Mit Hilfe der Eulerschen Polyederformel und mit kombinatorischen Relationen zwischen den Zahlen a_i und p_j für $P_{\text{reg}} \subset \mathbb{E}^n$, wie etwa $p_2 a_0 = 2a_1 = p_1 a_2$ für $n = 3$, erhält Schläfli durch Induktion folgende uns heute wohlbekannte Tabelle:

n	Schläfli-Symbol	Bezeichnung
3	$\{3, 3\}$	reguläres Tetraeder
	$\{4, 3\}$	regulärer Würfel
	$\{3, 4\}$	reguläres Oktaeder
	$\{5, 3\}$	reguläres Dodekaeder
	$\{3, 5\}$	reguläres Ikosaeder
4	$\{3, 4, 3\}$	24–Zell
	$\{3, 3, 5\}$	600–Zell
	$\{5, 3, 3\}$	120–Zell
≥ 5	$\{3, \ldots, 3\}$	reguläre Pyramide
	$\{3, \ldots, 3, 4\}$	reguläres Kreuzpolytop
	$\{4, 3, \ldots, 3\}$	regulärer Hyperwürfel

Weiter findet Schläfli vier der insgesamt zehn regelmässigen Sternpolytope, die schon von Kepler und Poinsot entdeckt und von Hess schliesslich vollständig klassifiziert worden sind (cf. [2]).

SUMMARY OF SIMPLICIAL FIELDS

J = 0

$$S_{scalar} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}^2} (\phi_i - \phi_j)^2 \; ,$$

$$l_{ij}^2 = |\sigma_1(ij)|^2$$

J = 1/2

$$S_{Wilson} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}} (\bar{\psi}_i \hat{e}_a^{j(i)} \gamma^a \Omega_{ij} \psi_j - \bar{\psi}_j \Omega_{ji} \hat{e}_a^{i(j)} \gamma^a \psi_i)$$

J = 1

$$S_{gauge} = \frac{1}{2g^2 N_c} \sum_{\triangle_{ijk}} \frac{V_{ijk}}{A_{ijk}^2} Tr[2 - U_{\triangle_{ijk}} - U^\dagger_{\triangle_{ijk}}]$$

FFdual

$$\epsilon^{ijkl} Tr[U_{\triangle_{0ij}} U_{\triangle_{0kl}}] \simeq V_{ijkl} \epsilon^{\mu\nu\rho\sigma} Tr[F_{\mu\nu}(0) F_{\rho\sigma}(0)]$$

$$U_{\triangle_{ijk}} = U_{ij} U_{jk} U_{ki} \quad A_{ijk} = |\sigma_2(ijk)| \quad V_{ijk} = |\sigma_2(ijk) \wedge \sigma_2^*(ijk)|$$

$$U_{0ij} = U_{0i} U_{ij} U_{j0} \quad , \quad U^\dagger_{0ij} = U_{0j} U_{ji} U_{i0} \quad V_{ij} = |\sigma_1(ij) \wedge \sigma_1^*(ij)|$$

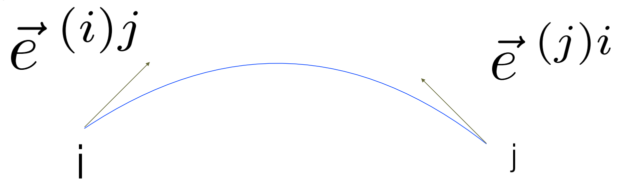
But Dirac needs Spin Connection (Kahler Dirac doesn't)

$$S = \frac{1}{2}\int d^D x \sqrt{g}\,\bar{\psi}[\mathbf{e}^{\mu}(\partial_\mu - \frac{i}{4}\boldsymbol{\omega}_\mu(x)) + m]\psi(x)$$

$$\mathbf{e}^{\mu}(x) \equiv e^{\mu}_a(x)\gamma^a$$ Verbein & Spin connection*

$$\boldsymbol{\omega}_\mu(x) \equiv \omega^{ab}_\mu(x)\sigma_{ab} \quad , \quad \sigma_{ab} = i[\gamma_a, \gamma_a]/2$$

$$S_{naive} = \frac{1}{2}\sum_{\langle i,j\rangle}\frac{V_{ij}}{l_{ij}}[\bar{\psi}_i \vec{e}^{\,(i)j}\cdot\vec{\gamma}\,\Omega_{ij}\psi_j - \bar{\psi}_j\Omega_{ji}\vec{e}^{\,(i)j}\cdot\vec{\gamma}\psi_i] + \frac{1}{2}mV_i\bar{\psi}_i\psi_i$$



Simplicial Tetrad Hypothesis

$$e^{(i)j}_a\gamma^a\Omega_{ij} + \Omega_{ij}e^{(j)i}_a\gamma^a = 0$$

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- Rohan Misra, Boston University
- Jin-Yun Lin, Carnegie Mellon University
- Chung-I Tan, Brown University

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2024 Ising Solution* and "The Affine Conjecture"?

*Ising model on the affine plane Richard C. Brower and Evan K. Owen Phys. Rev. D **108**, 014511 – Published 20 July 2023

*The Ising Model on S2, Richard Brower and Evan K. Owen, arXiv:2407.00459 (2024).