Kernels and Integration Cycles in Complex Langevin Simulations

Michael Mandl

with Michael Hansen, Dénes Sexty and Erhard Seiler

based on arXiv:2412.17137

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FWF Austrian Science Fund

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Possible solution: Complex Langevin

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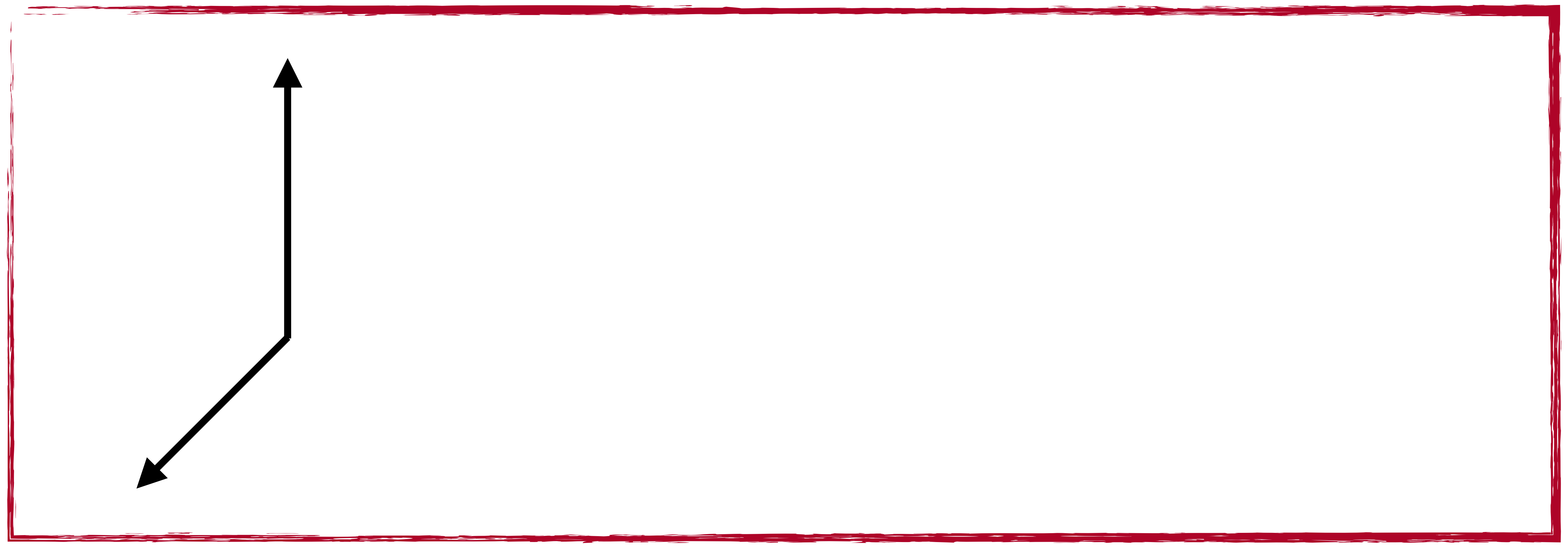

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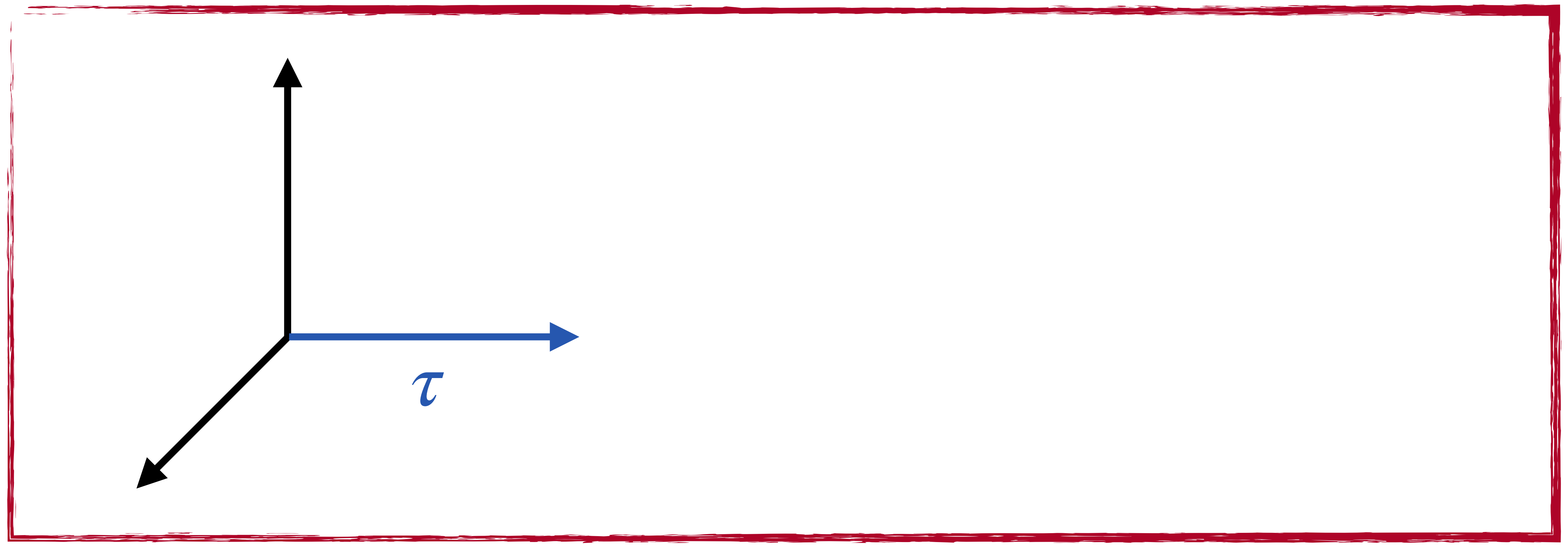

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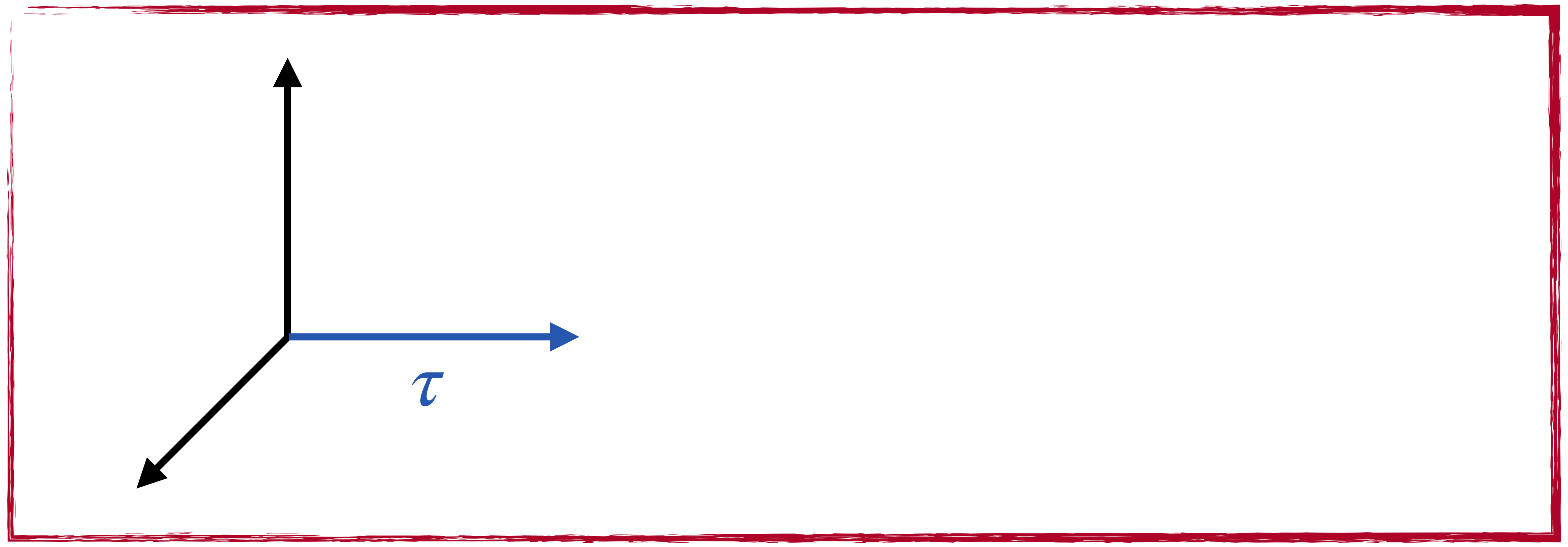

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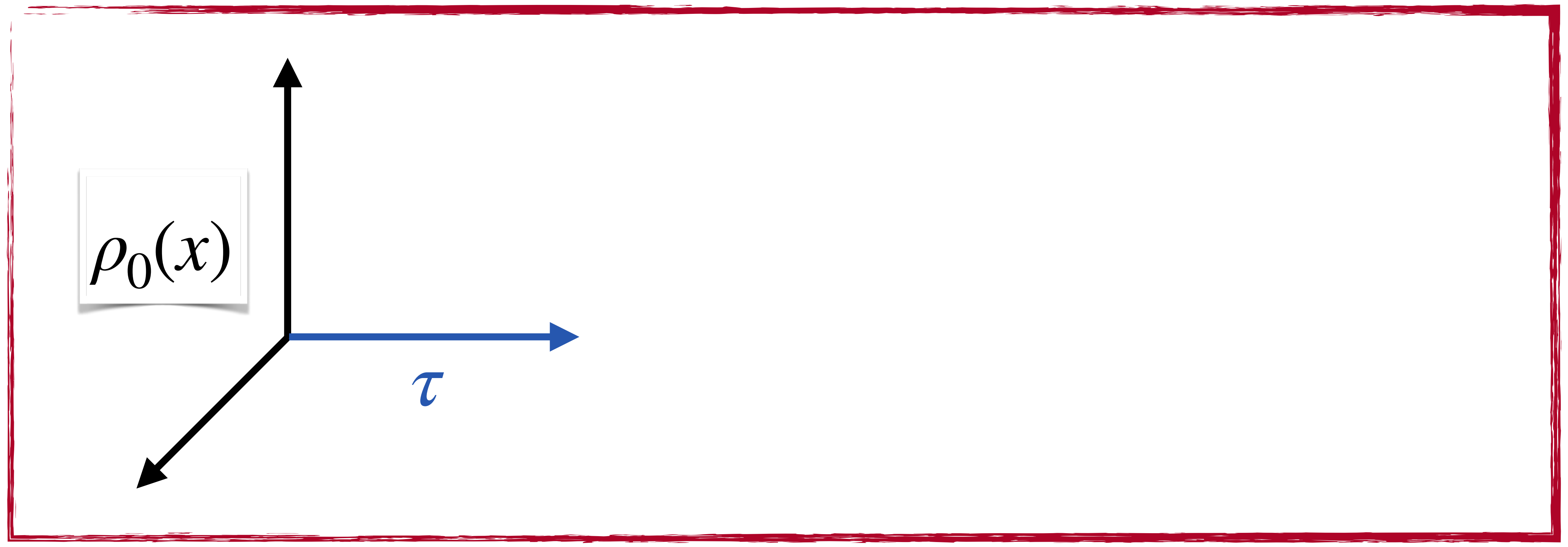

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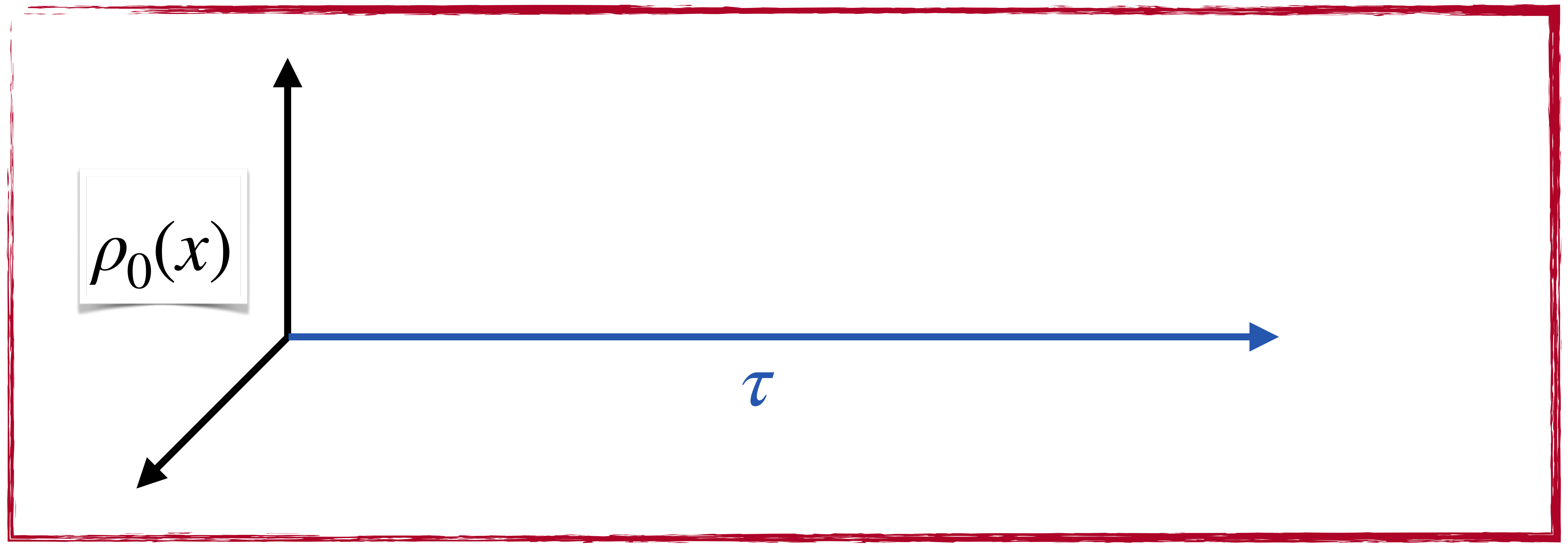

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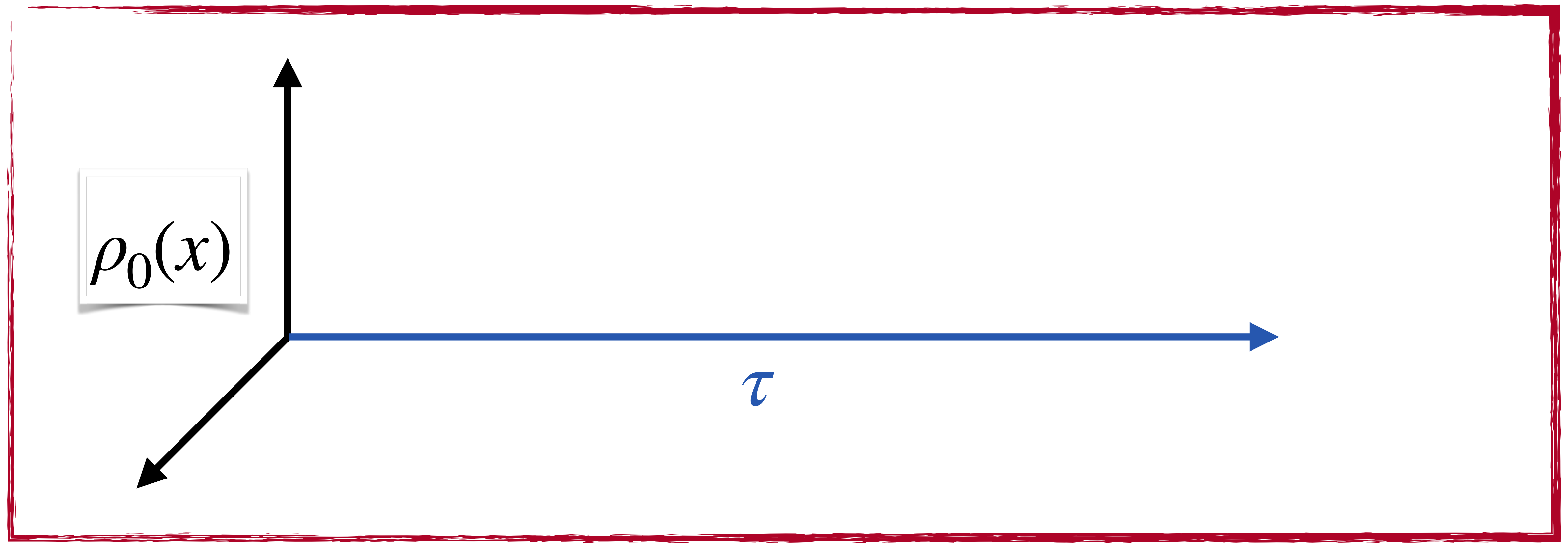

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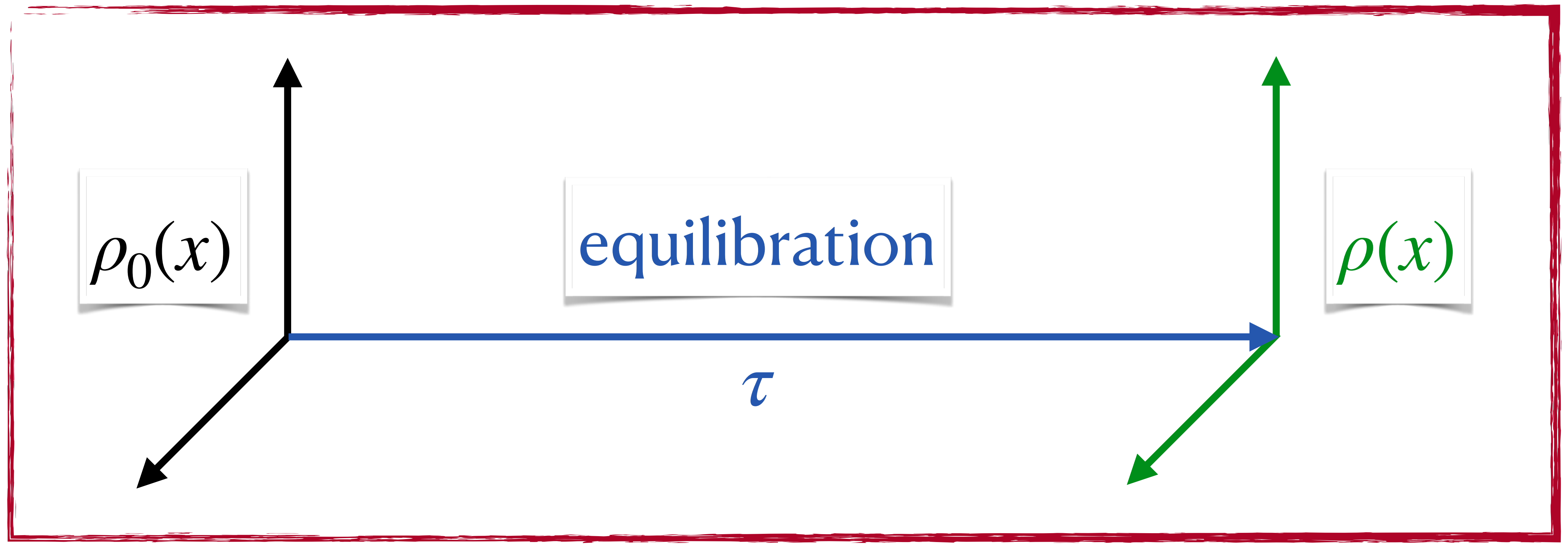

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What if $S(x)$ is complex?

The complex Langevin equation

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- $\Longrightarrow^{*}$ probability density $P(x, y, \tau)$.

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- Complexify $x \to z = x + \mathrm{i}y$.
- $\Longrightarrow^*$ probability density $P(x, y, \tau)$.
- Does it obey $$\lim_{\tau\to\infty} \int dxdy\, \mathcal{O}(x+\mathrm{i}y) P(x,y,\tau) = \int dx \mathcal{O}(x)\rho(x)$$?

Complex Langevin simulation

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- Simulate the process

Discretized evolution equation

$$z_{n+1} = z_n - \varepsilon \frac{\partial S(z)}{\partial z}\bigg|_{z=z_n} + \sqrt{\varepsilon}\,\eta_n$$

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- Generate configurations to produce equilibrium distribution for averaging.

Drawbacks and pitfalls

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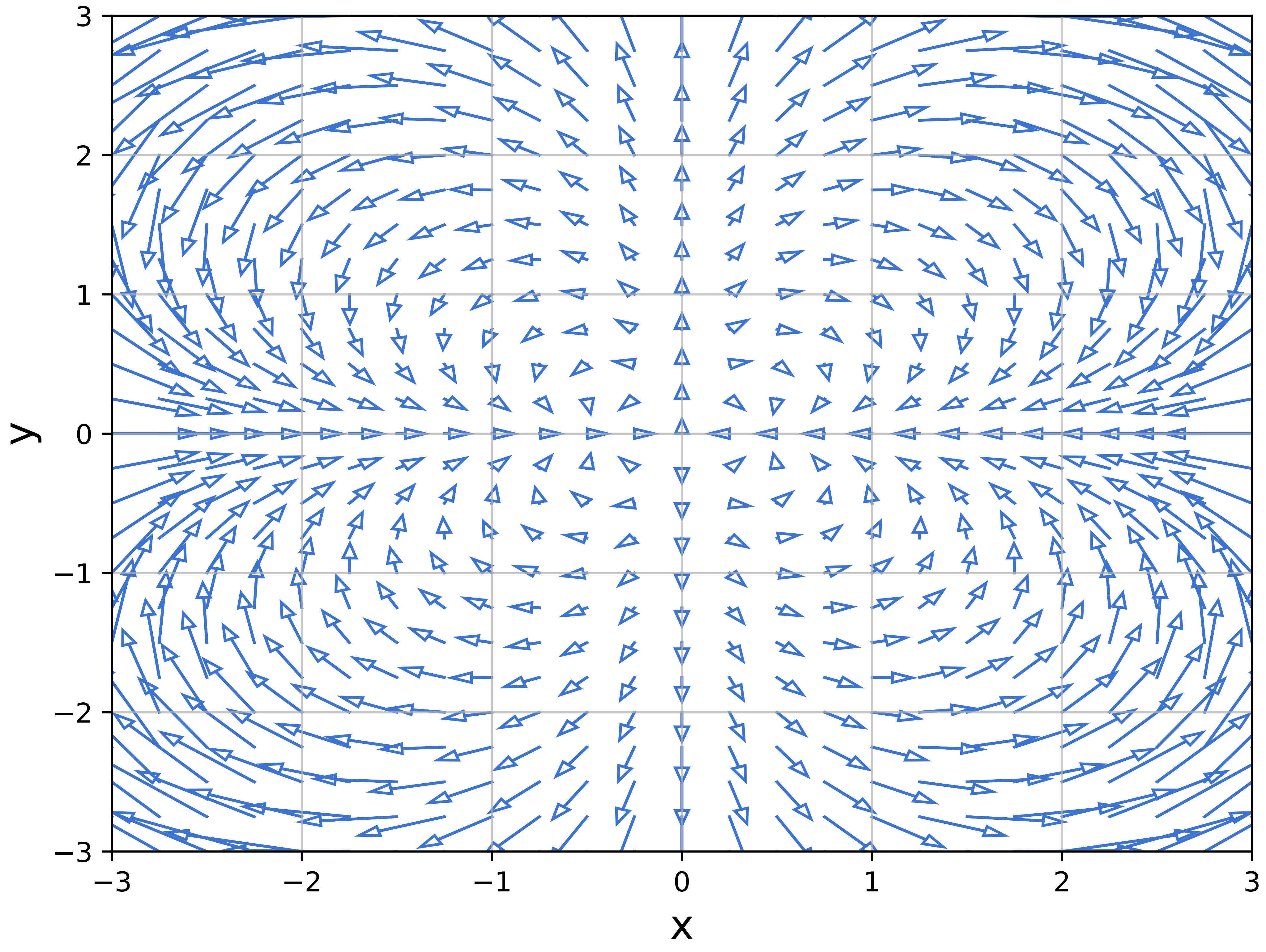

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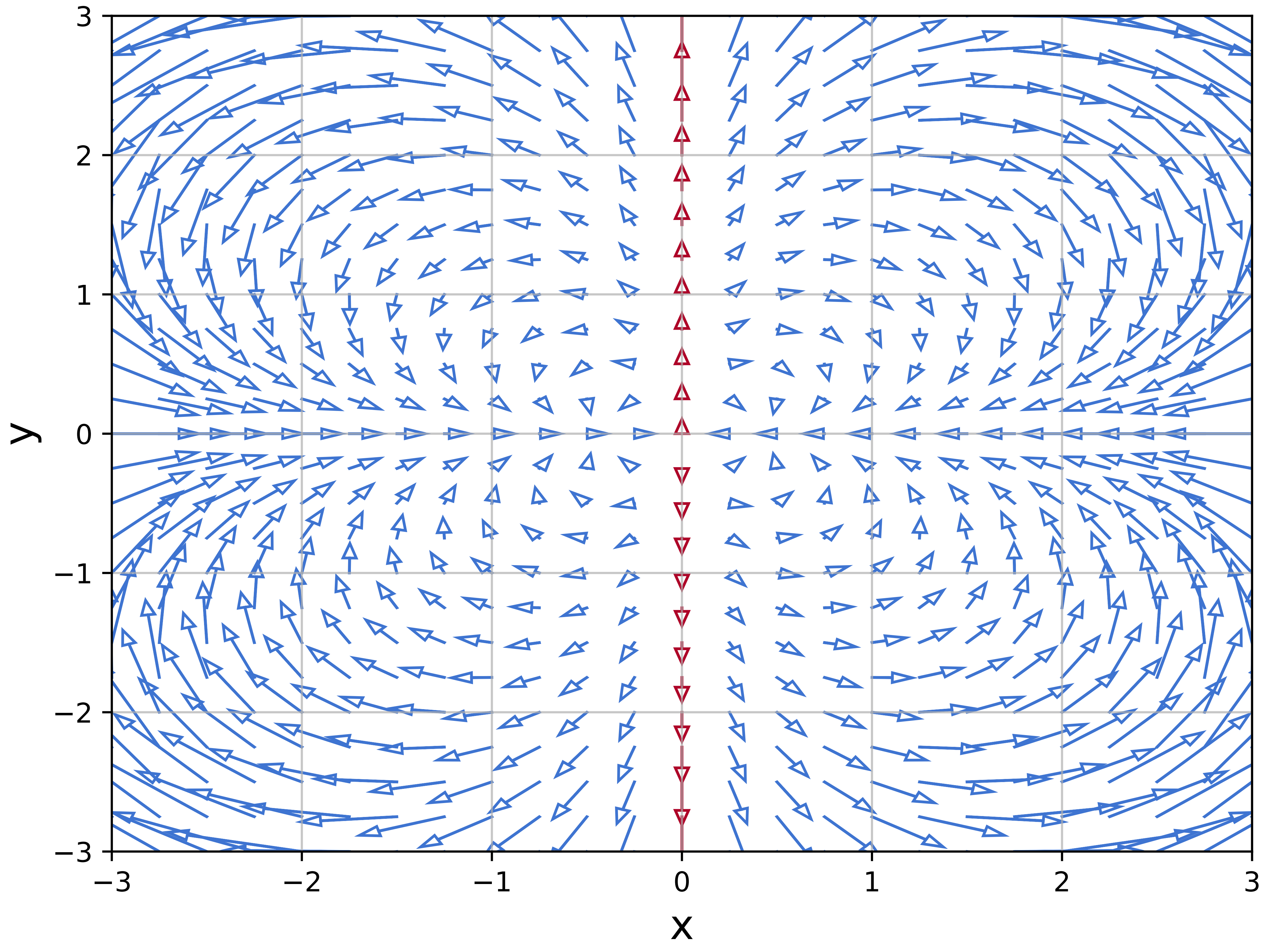

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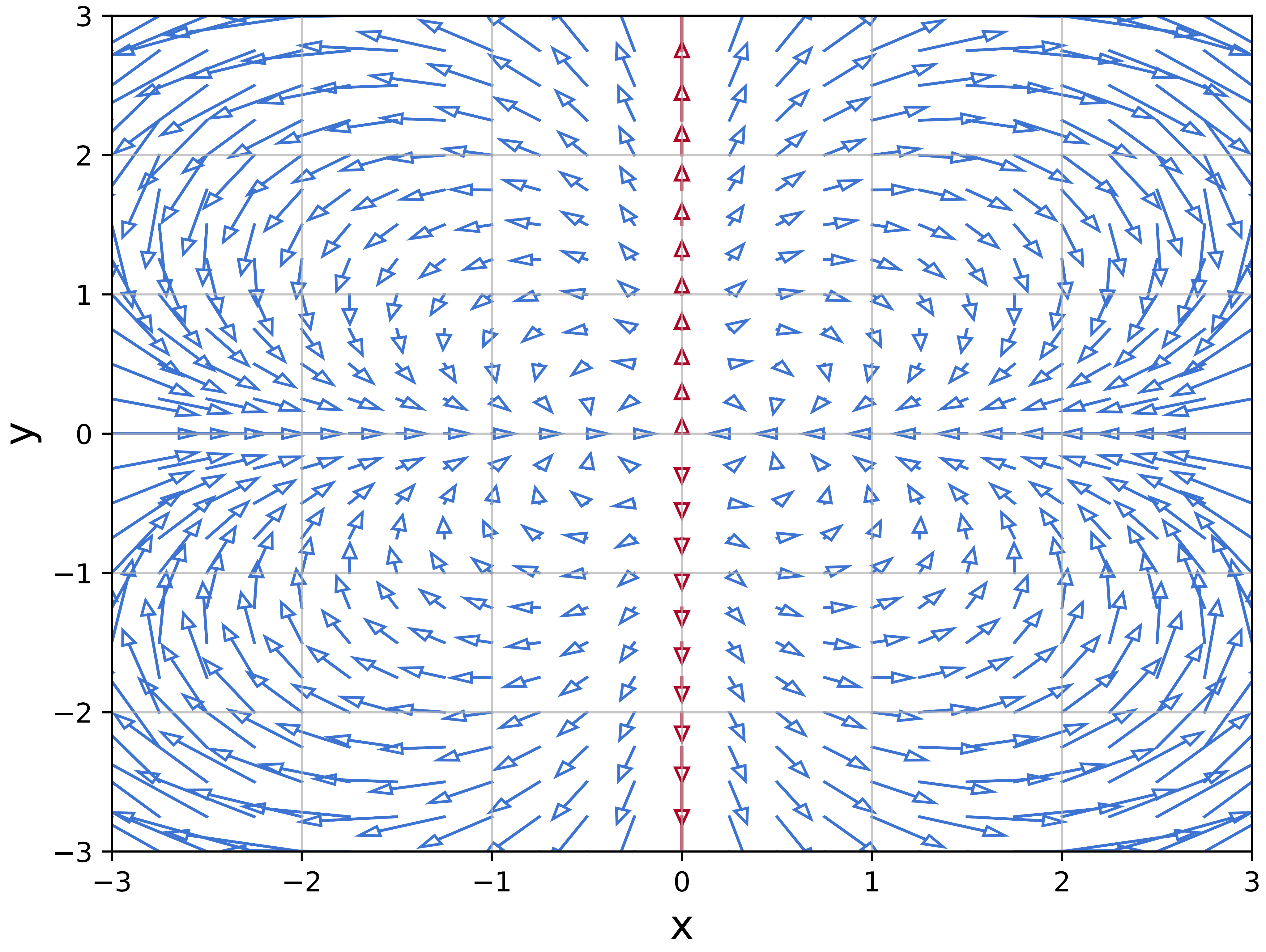

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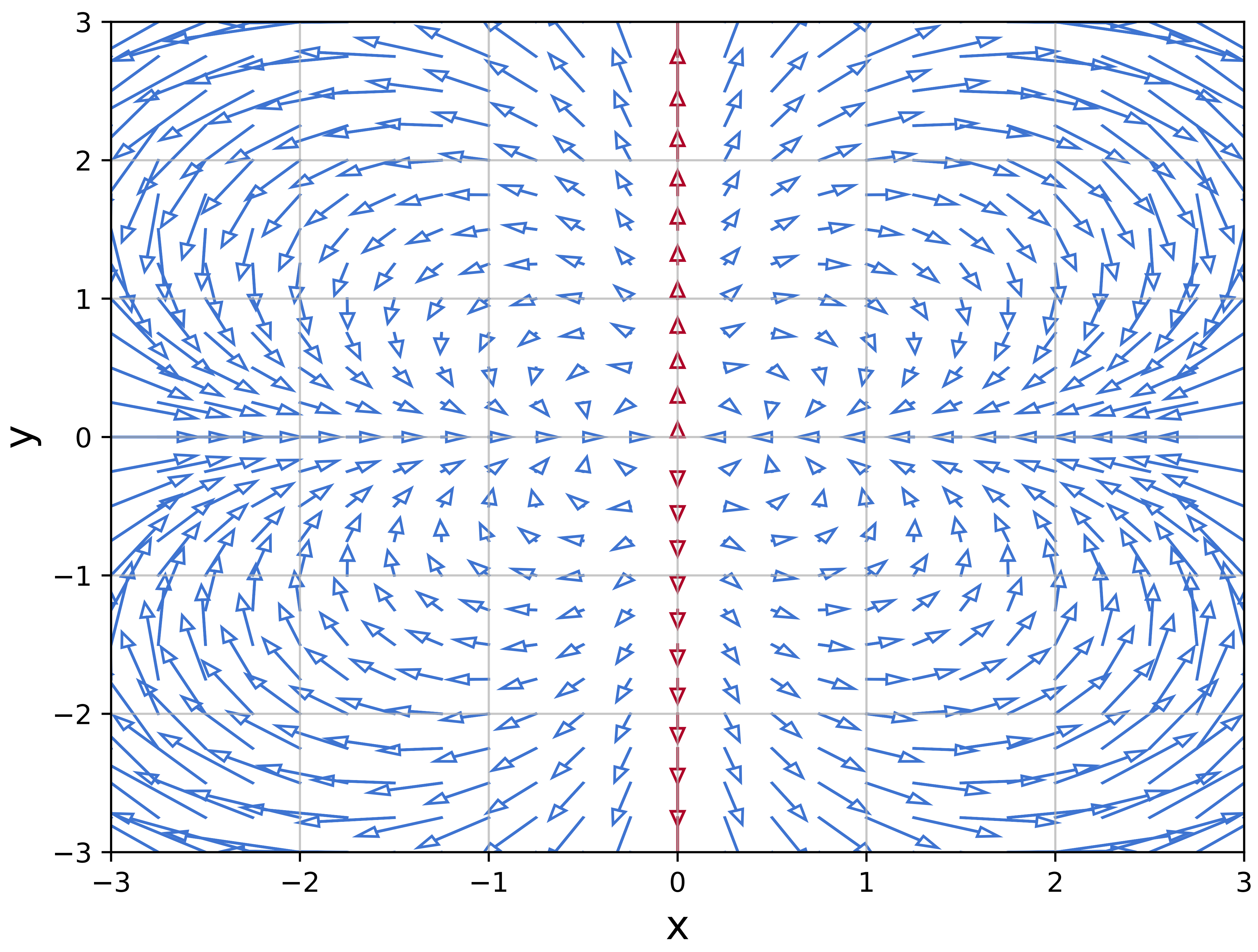
- Overcome via adaptive step-size control.

Aarts et al. '10



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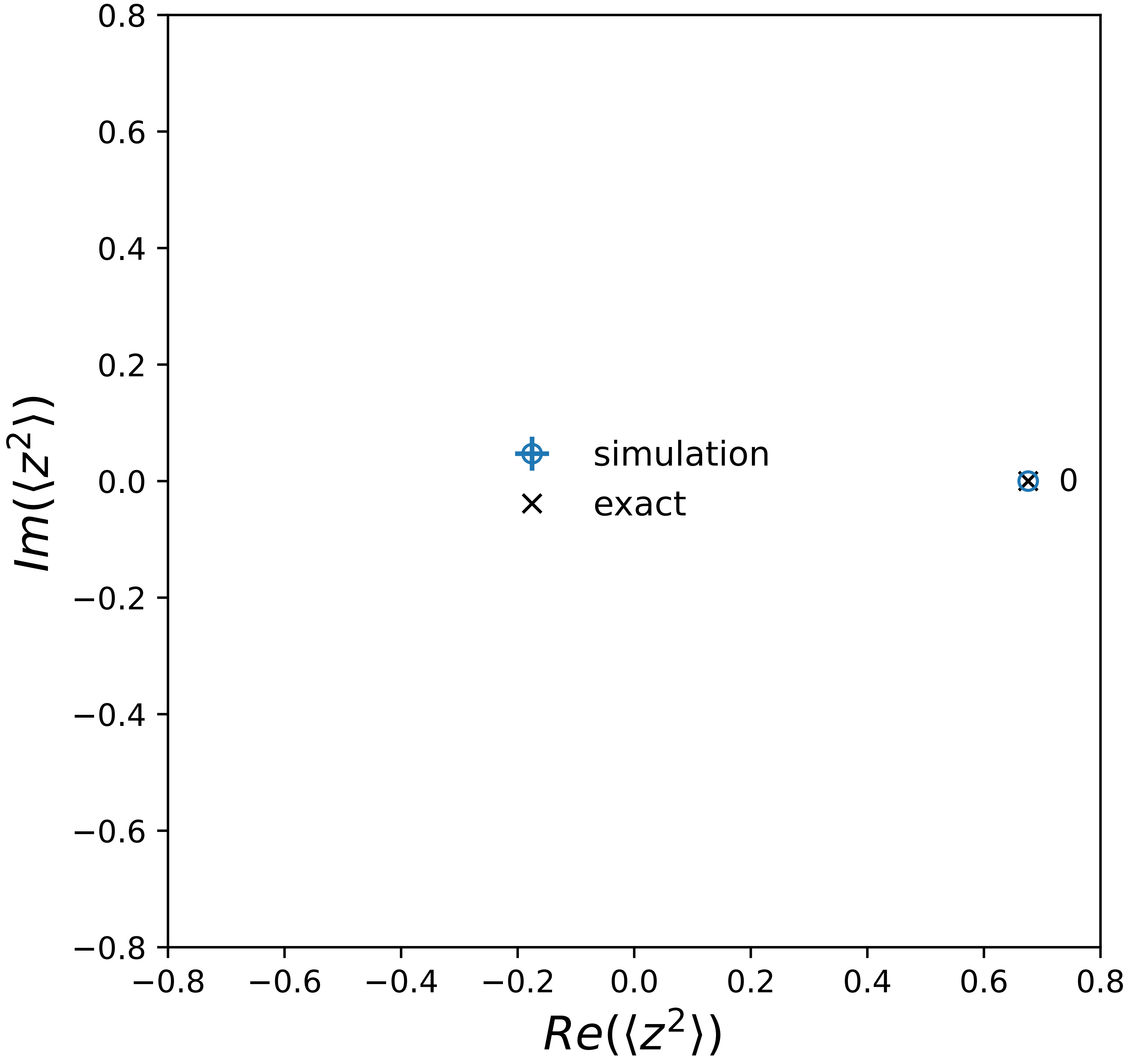

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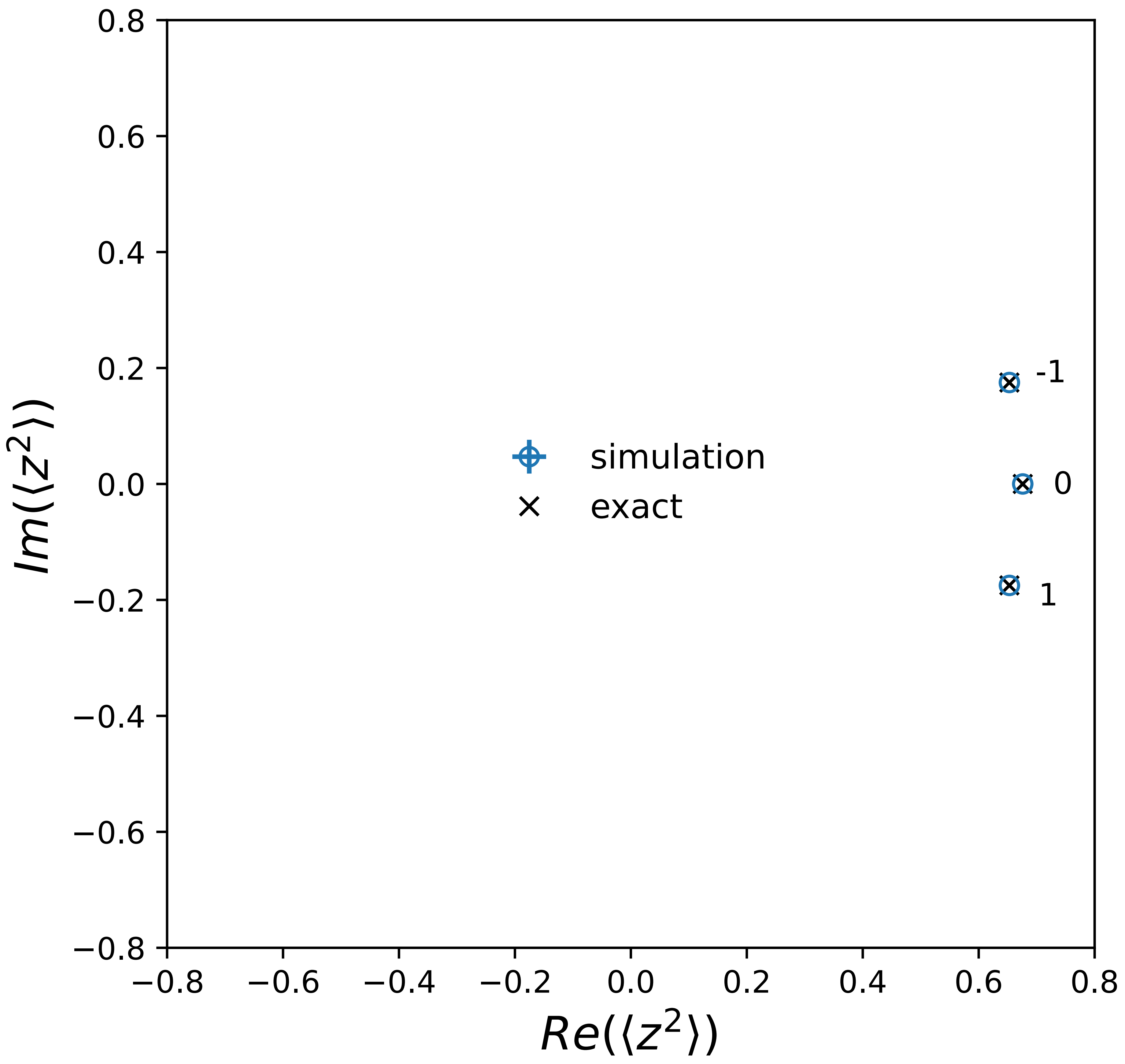

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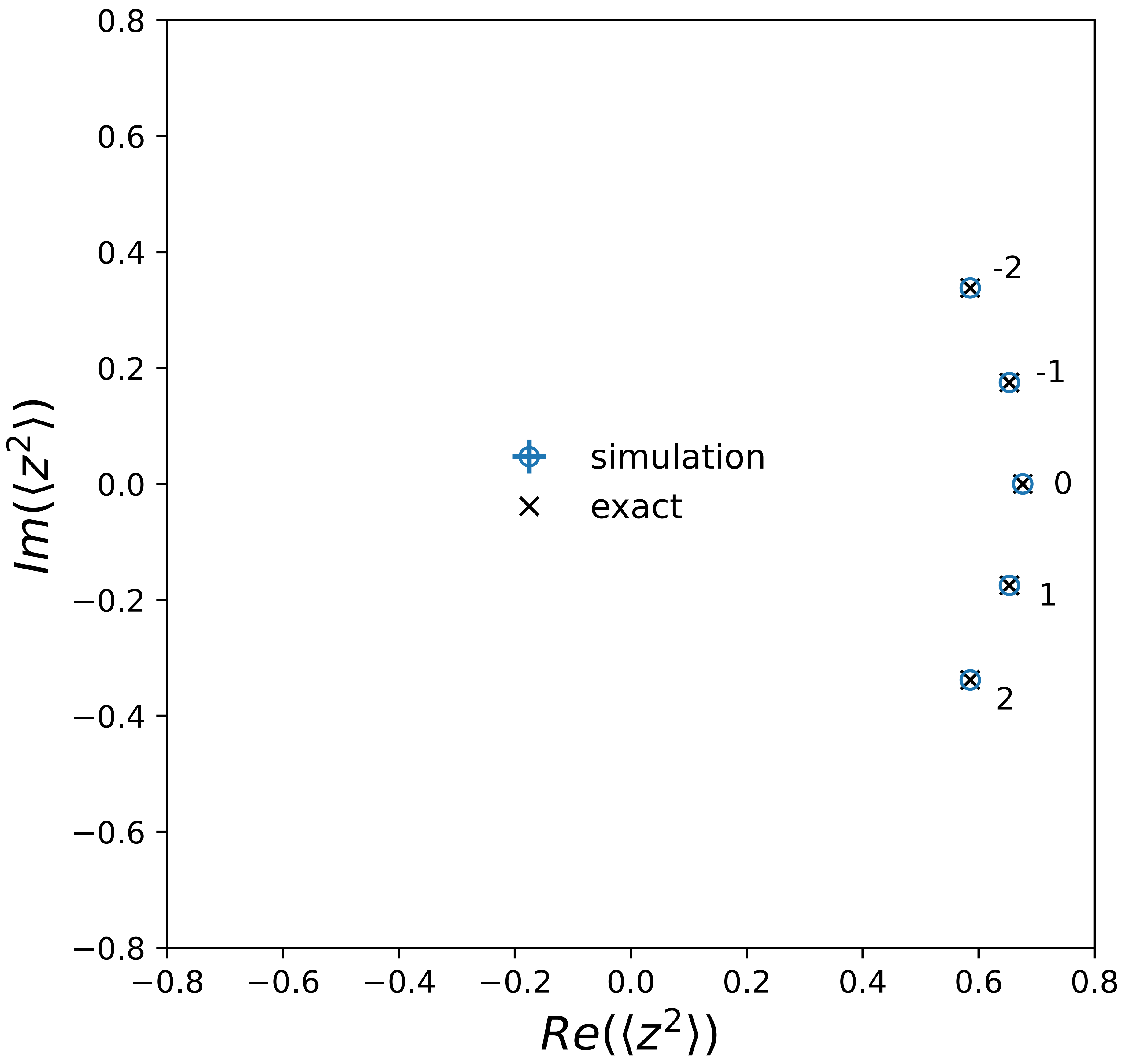

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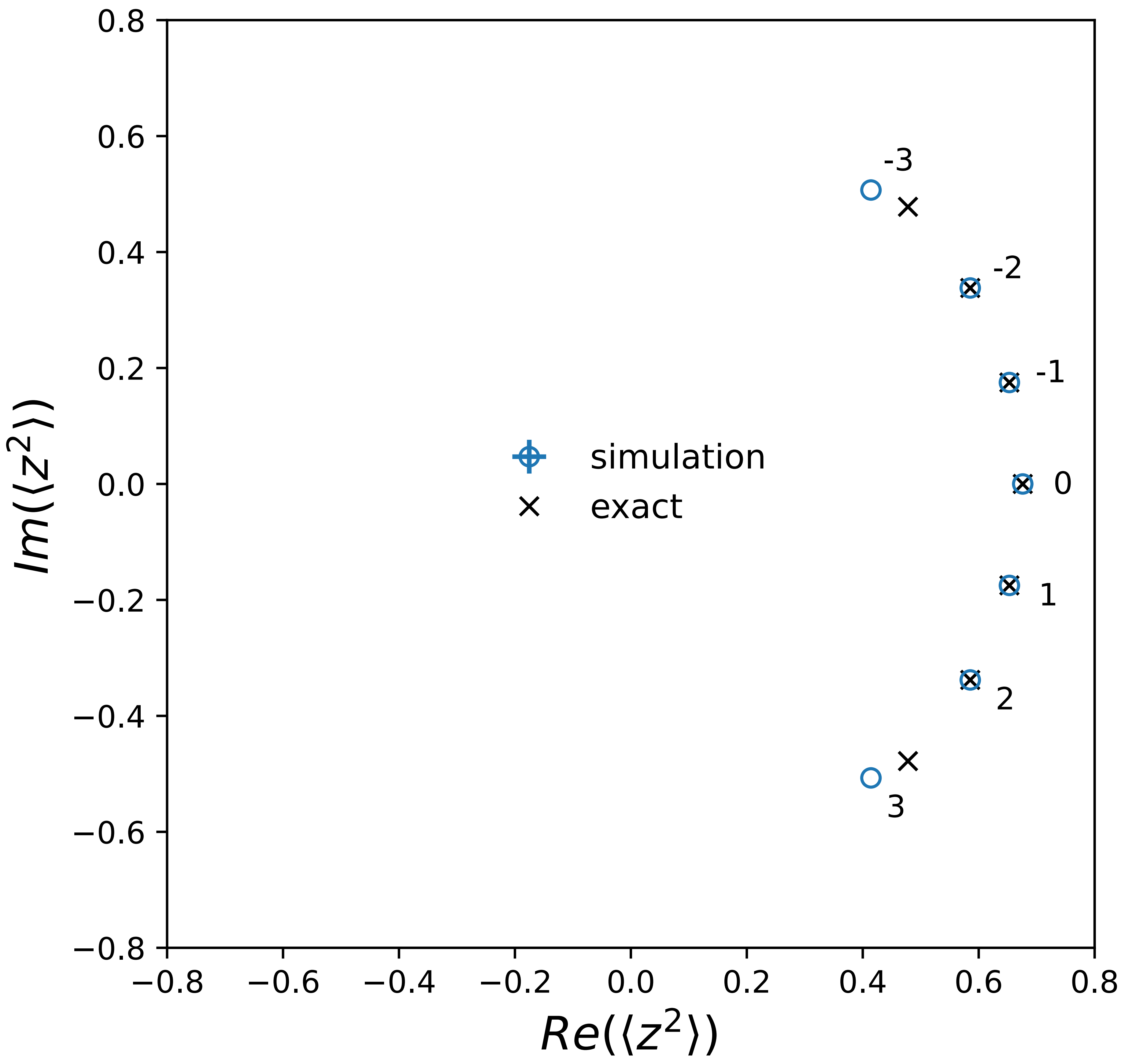

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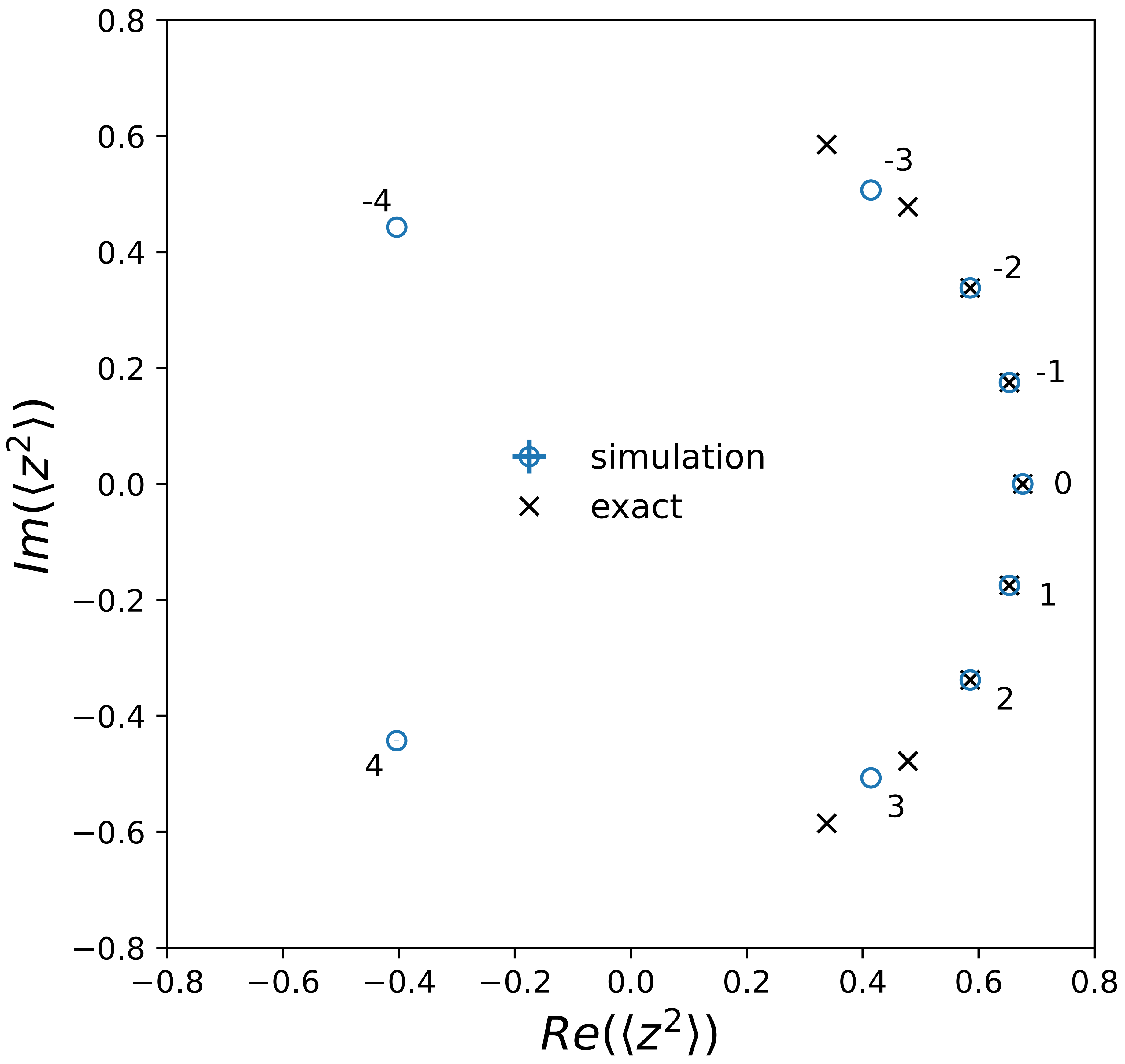

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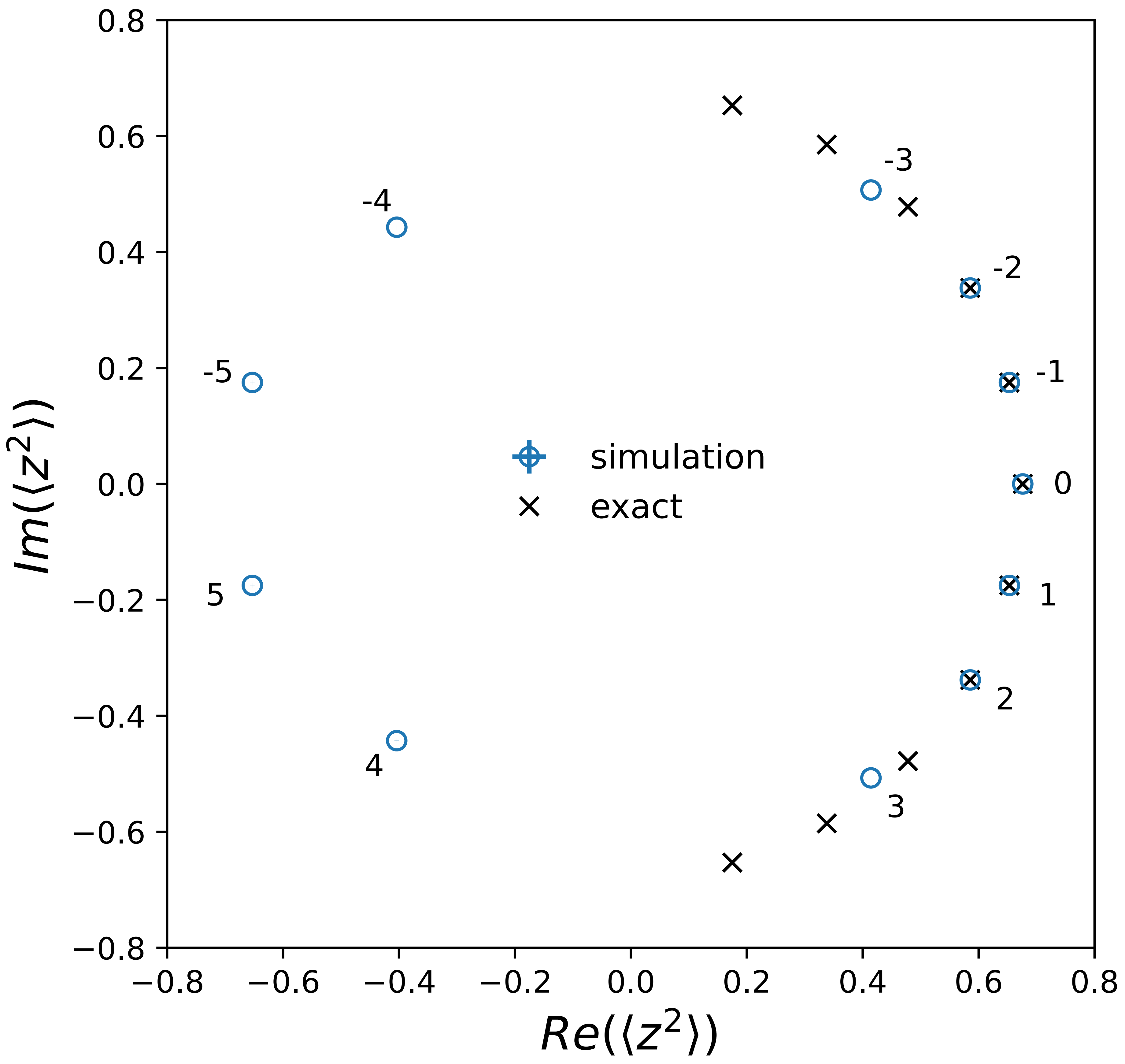

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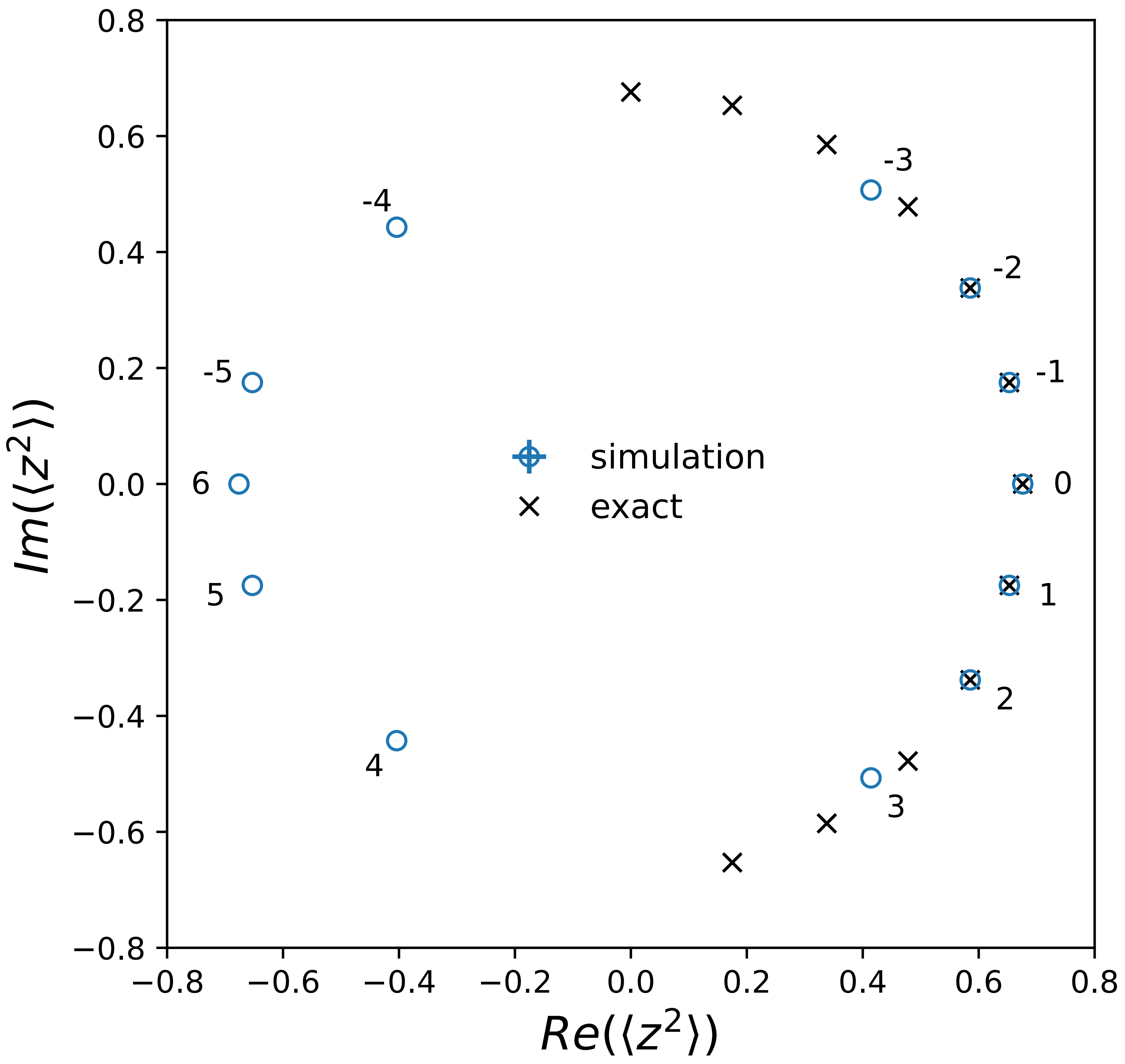

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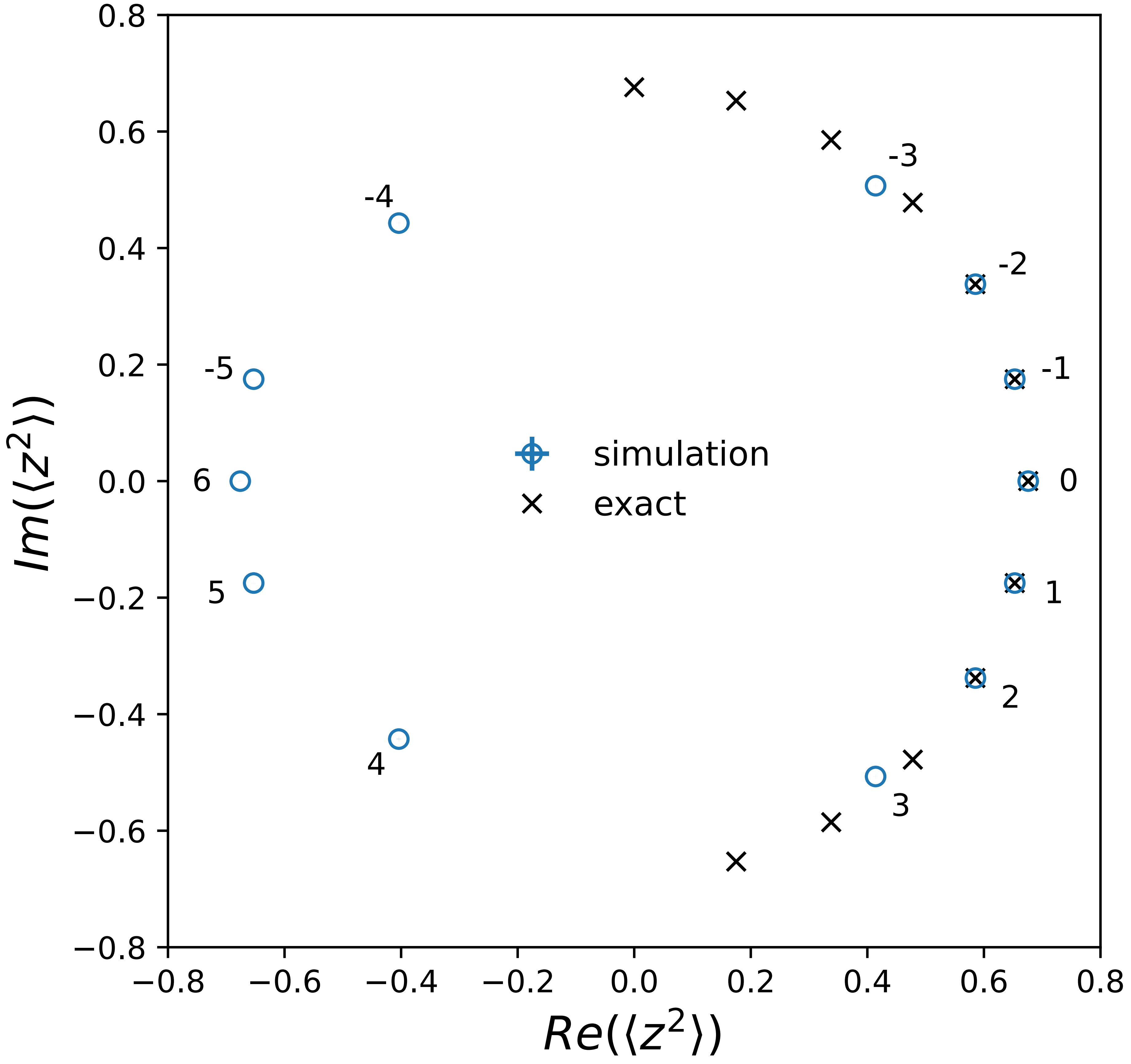

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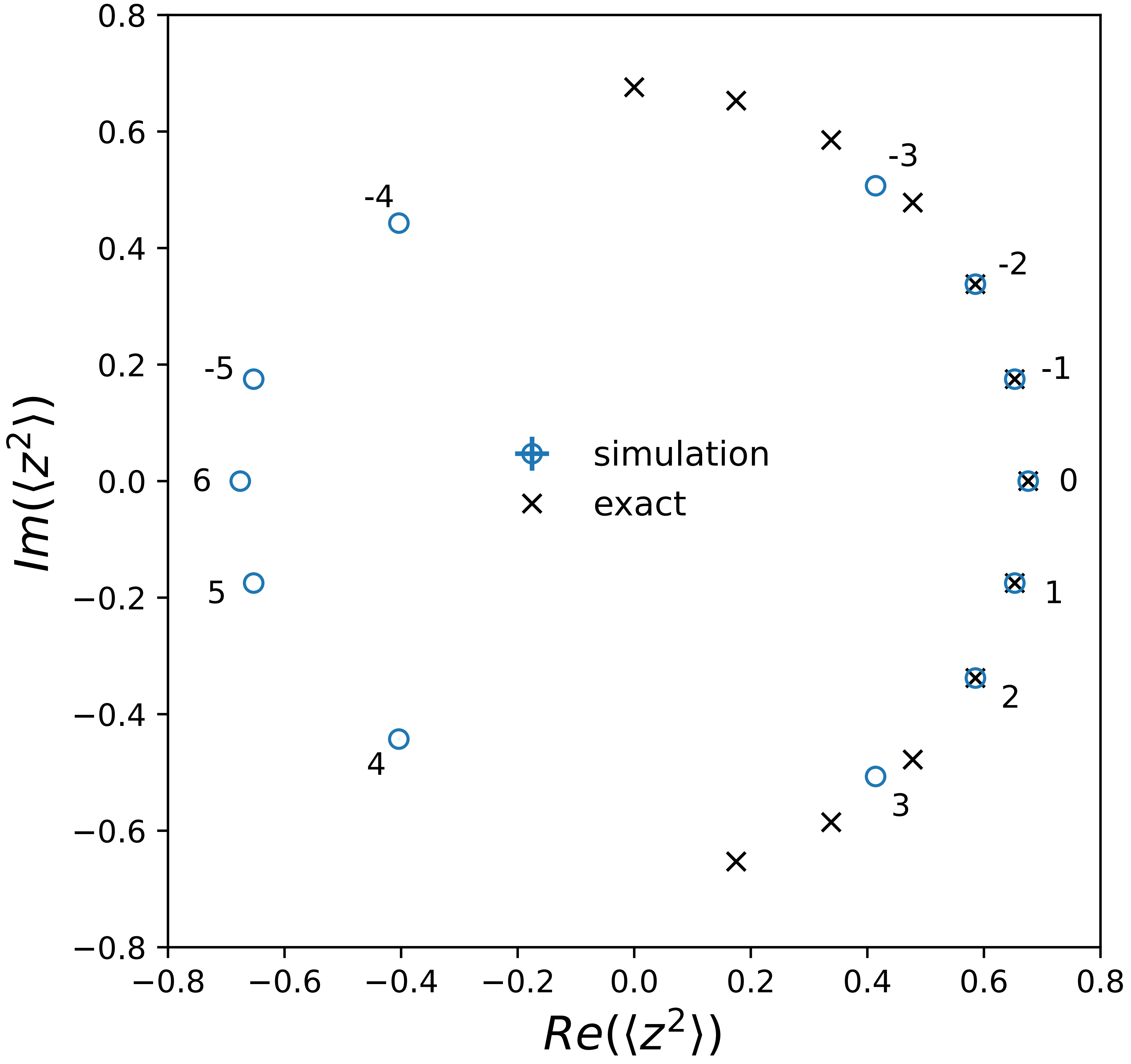
- In general, we do not know if results are correct.



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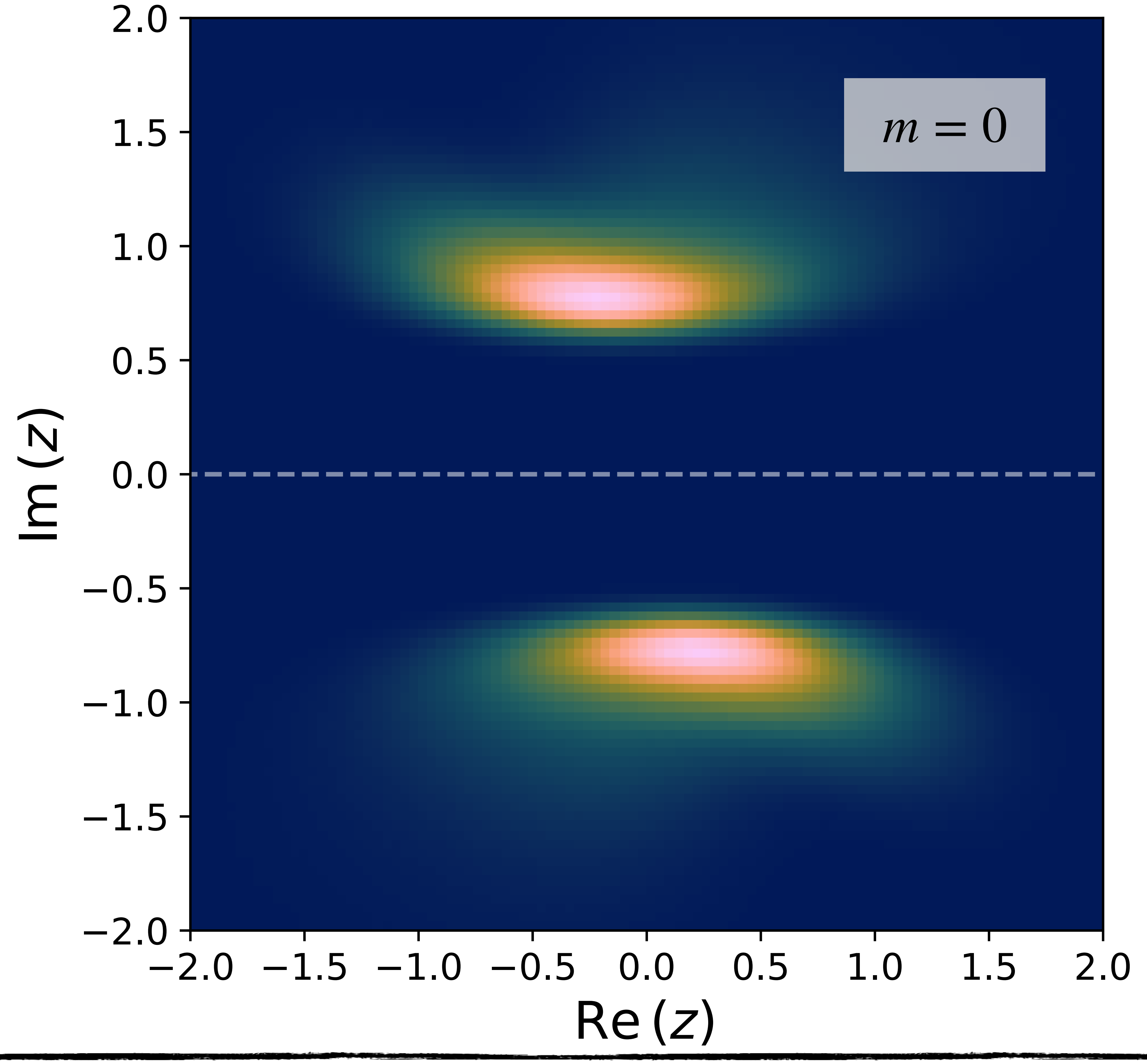

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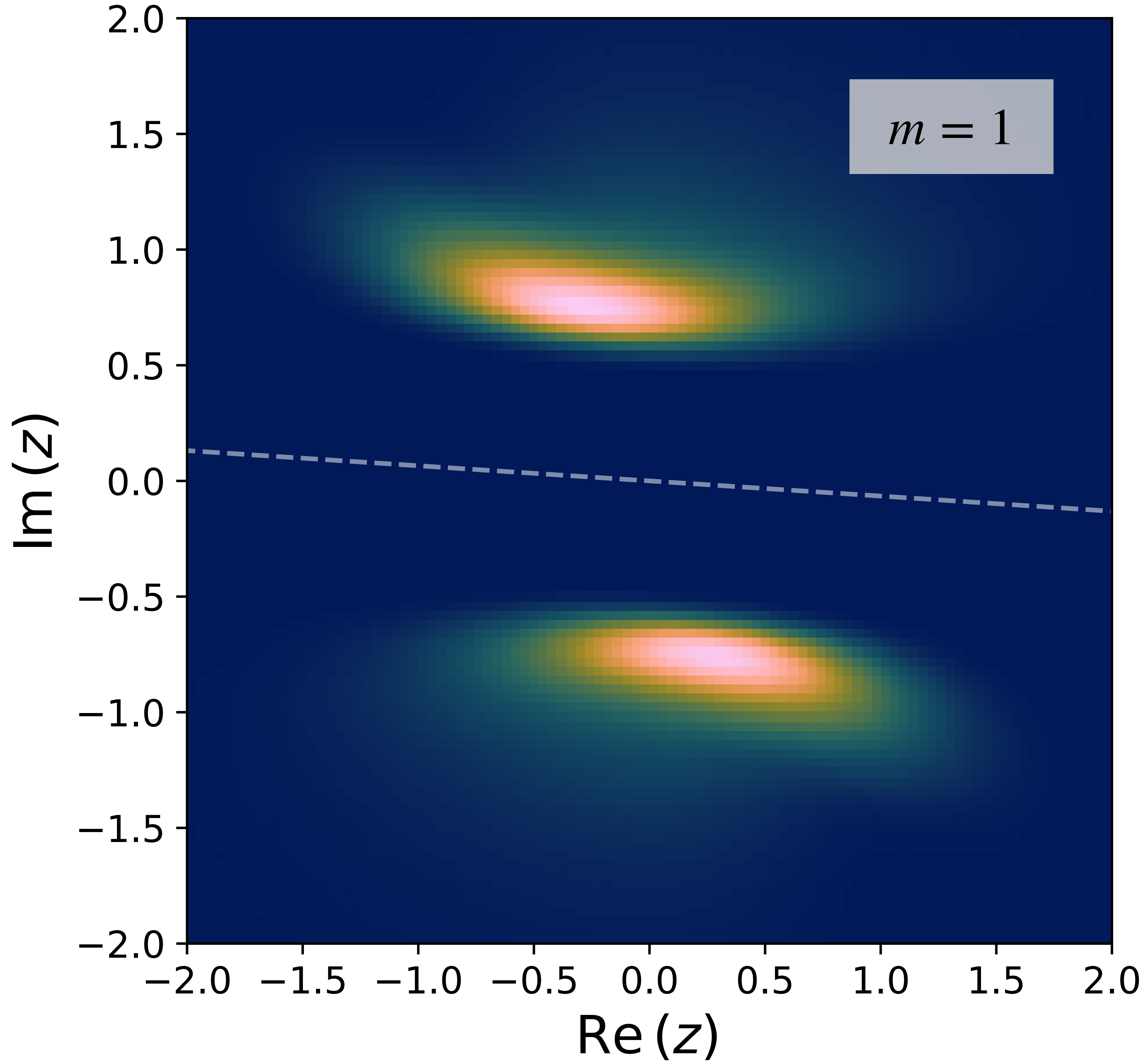

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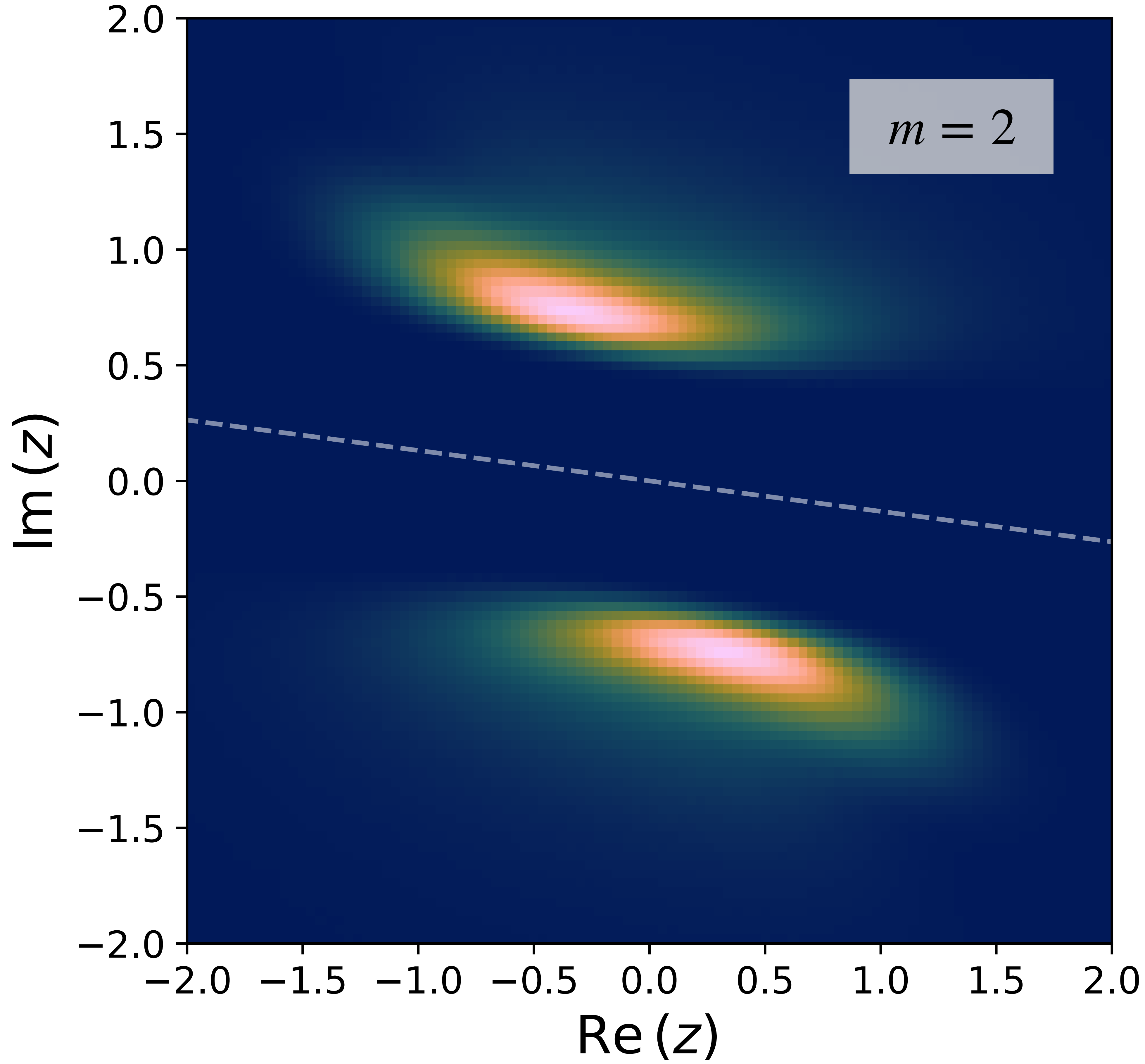

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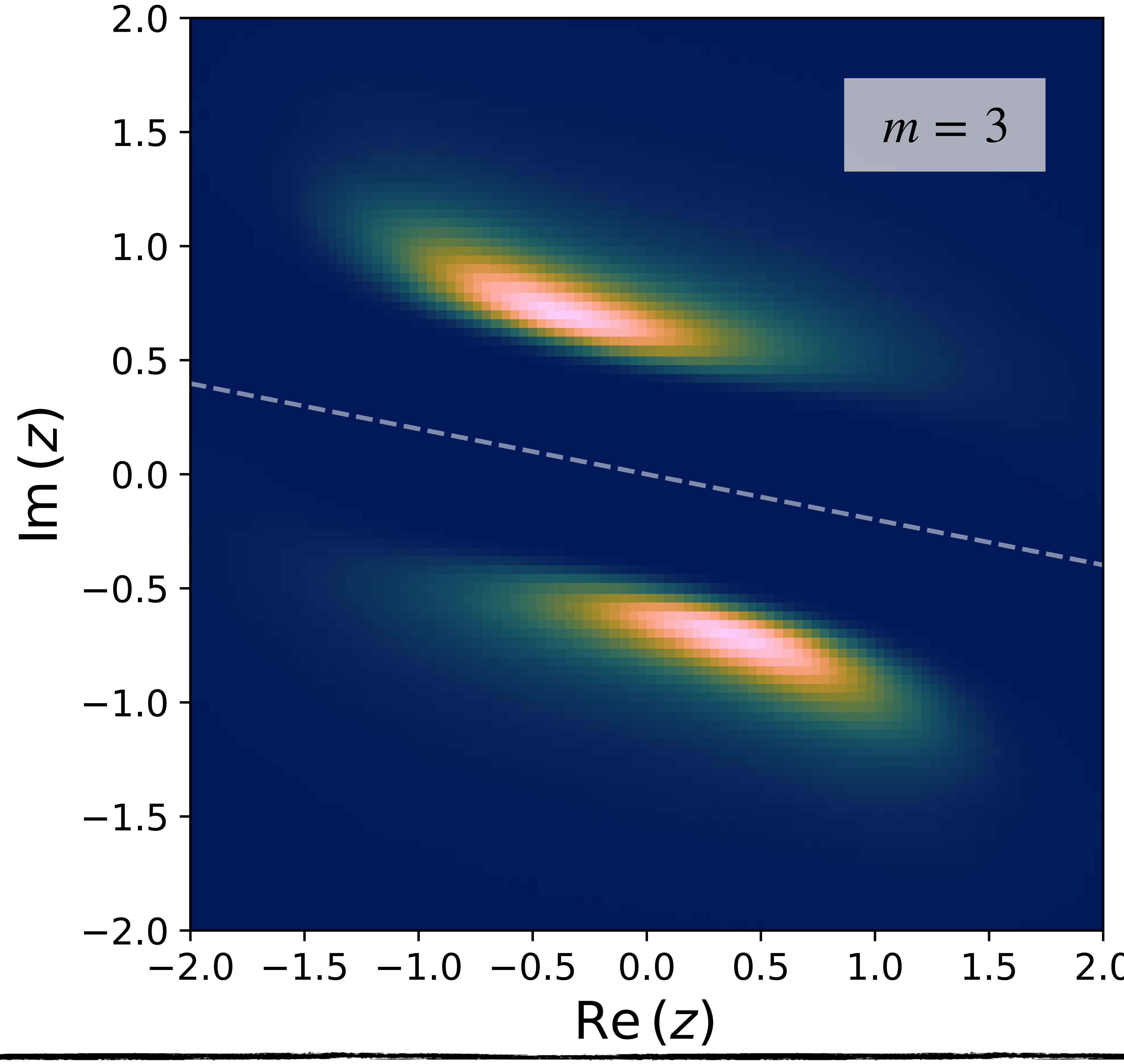

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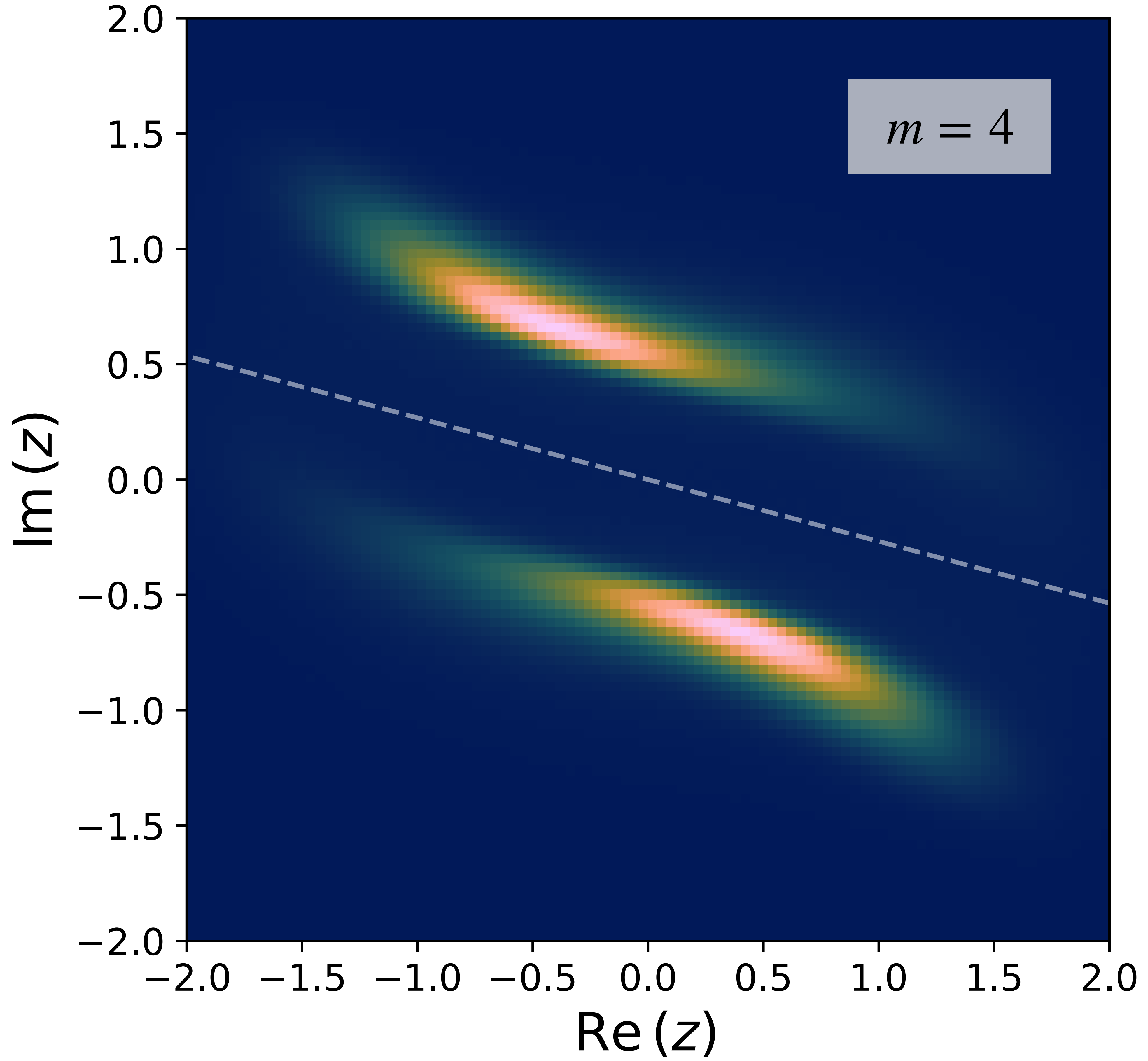

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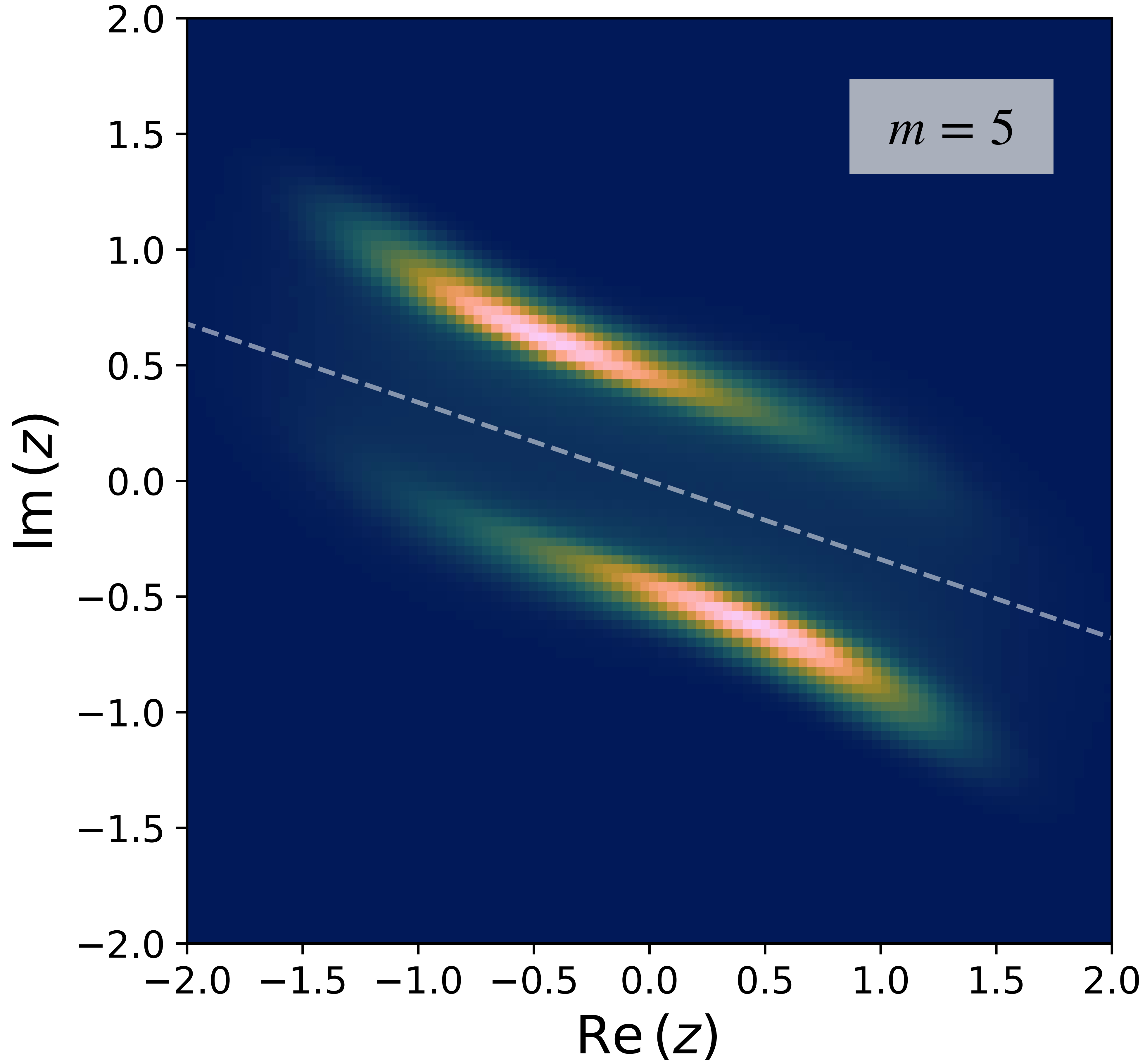

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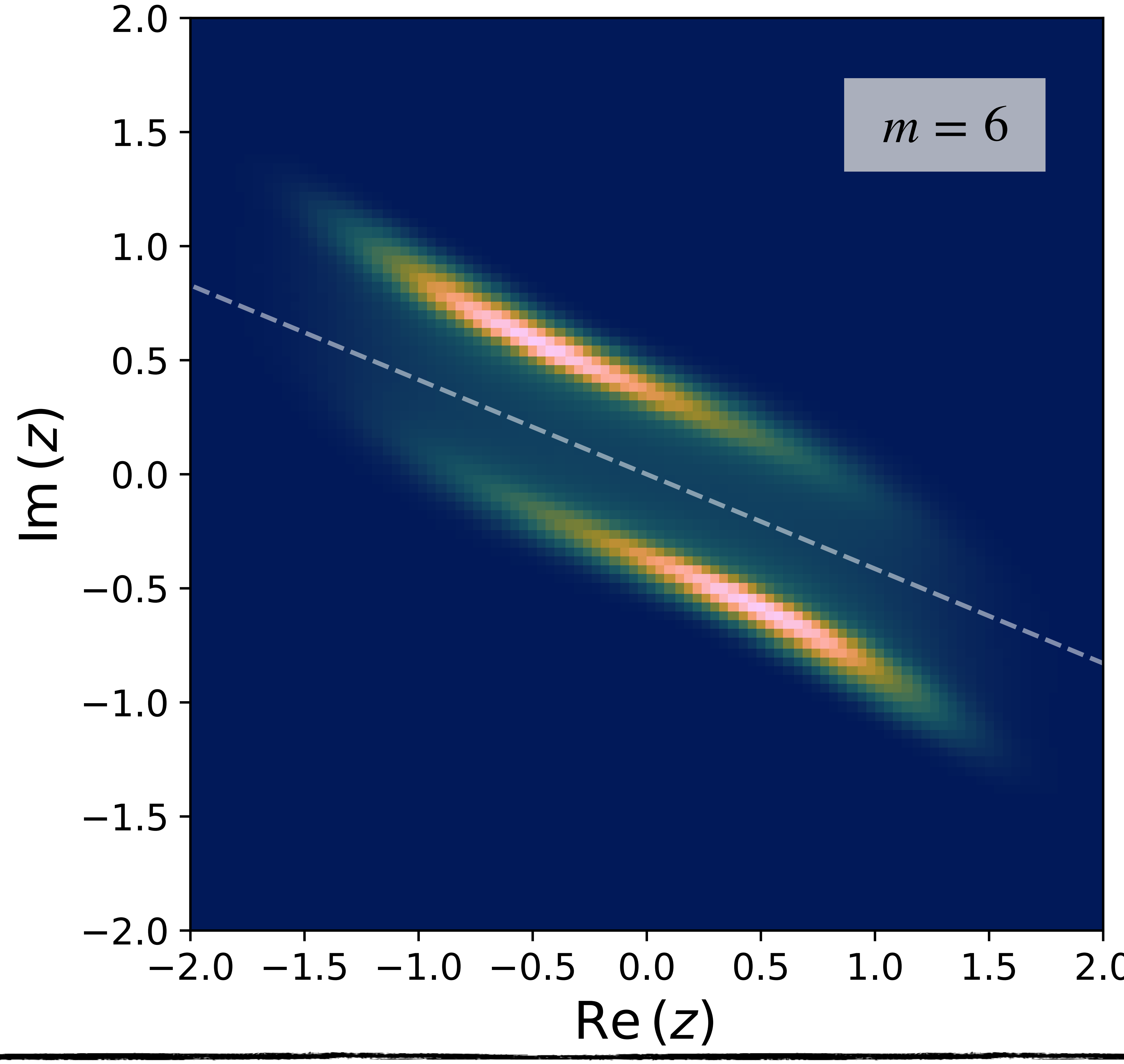

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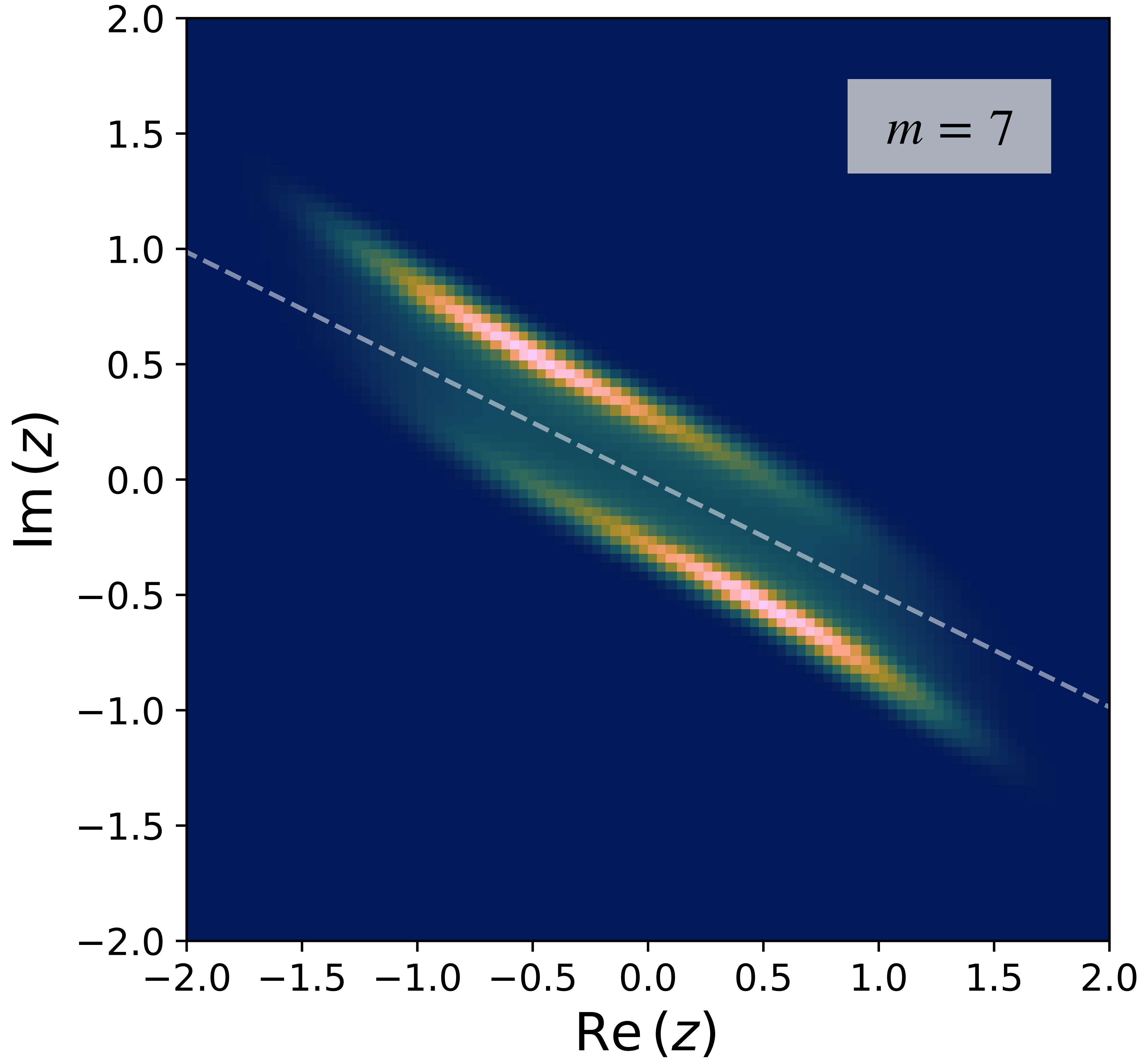

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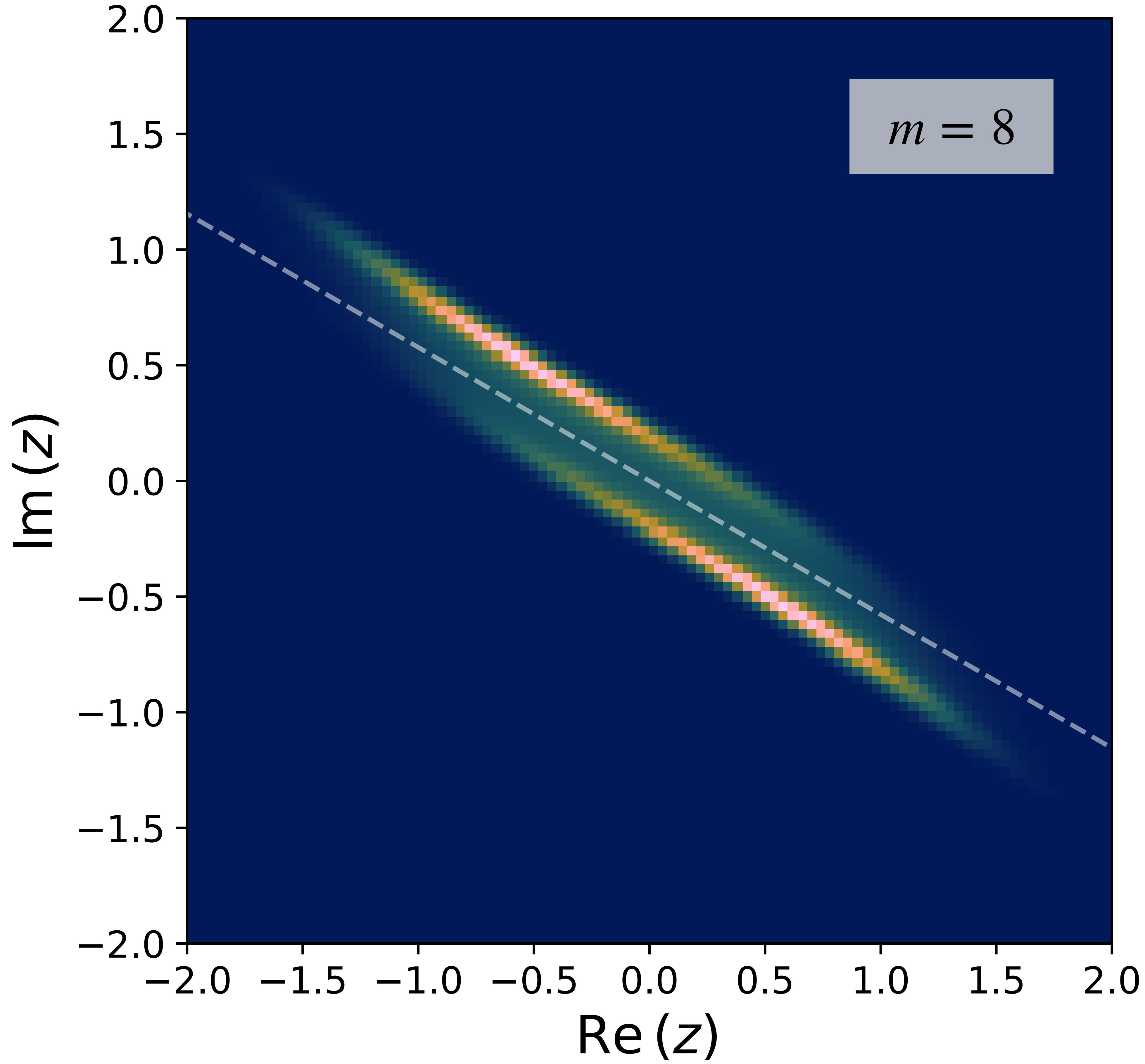

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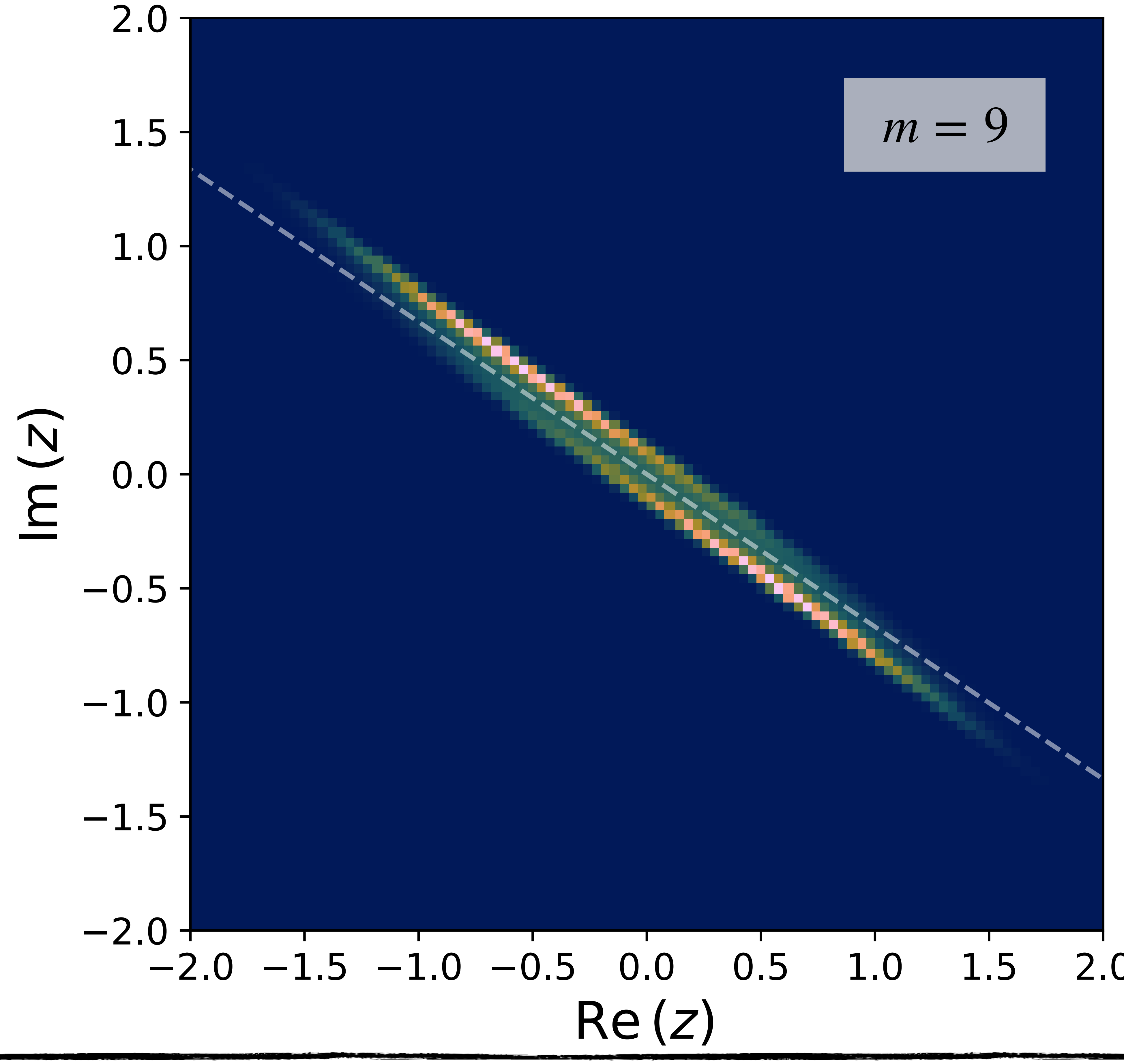

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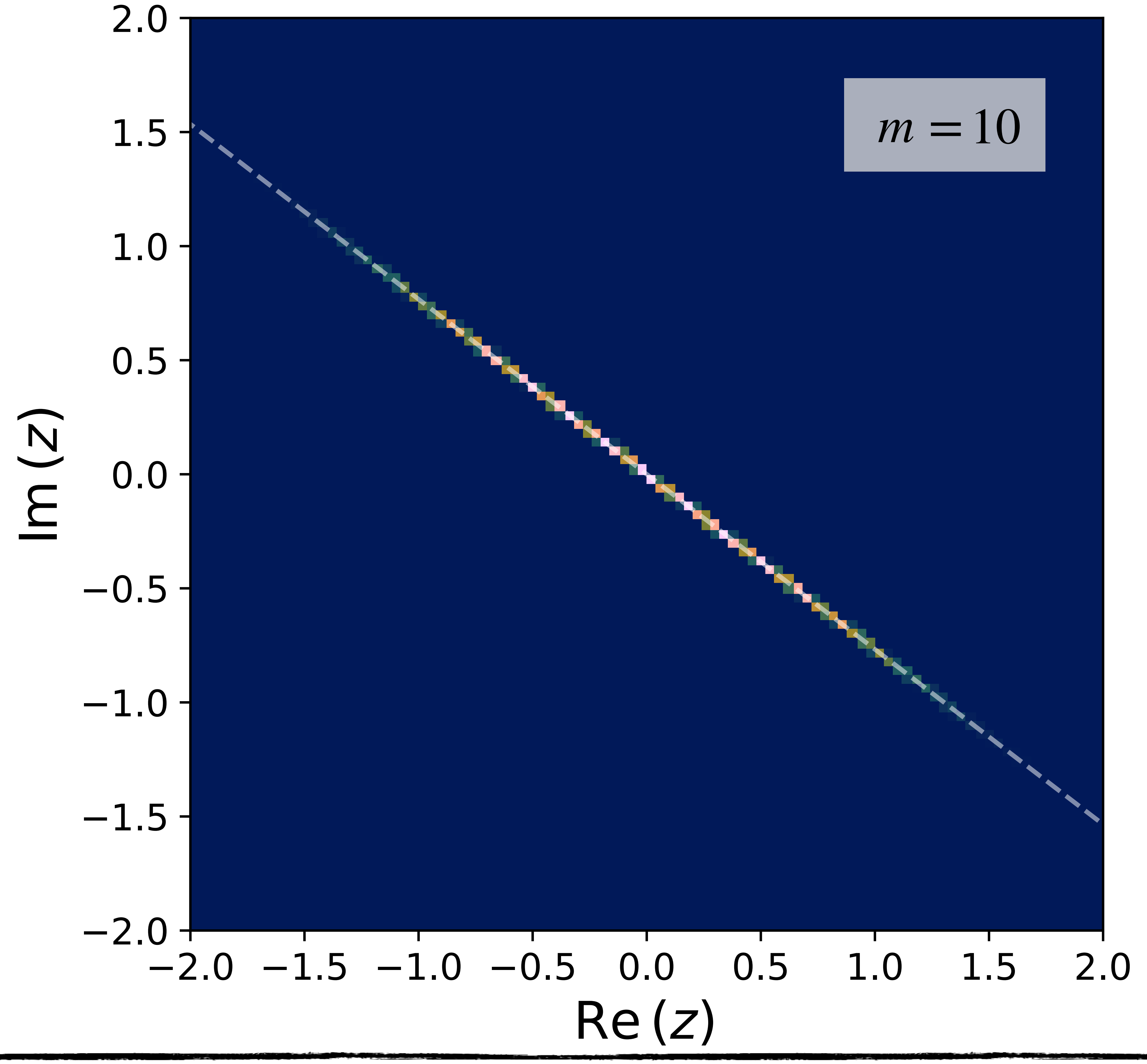

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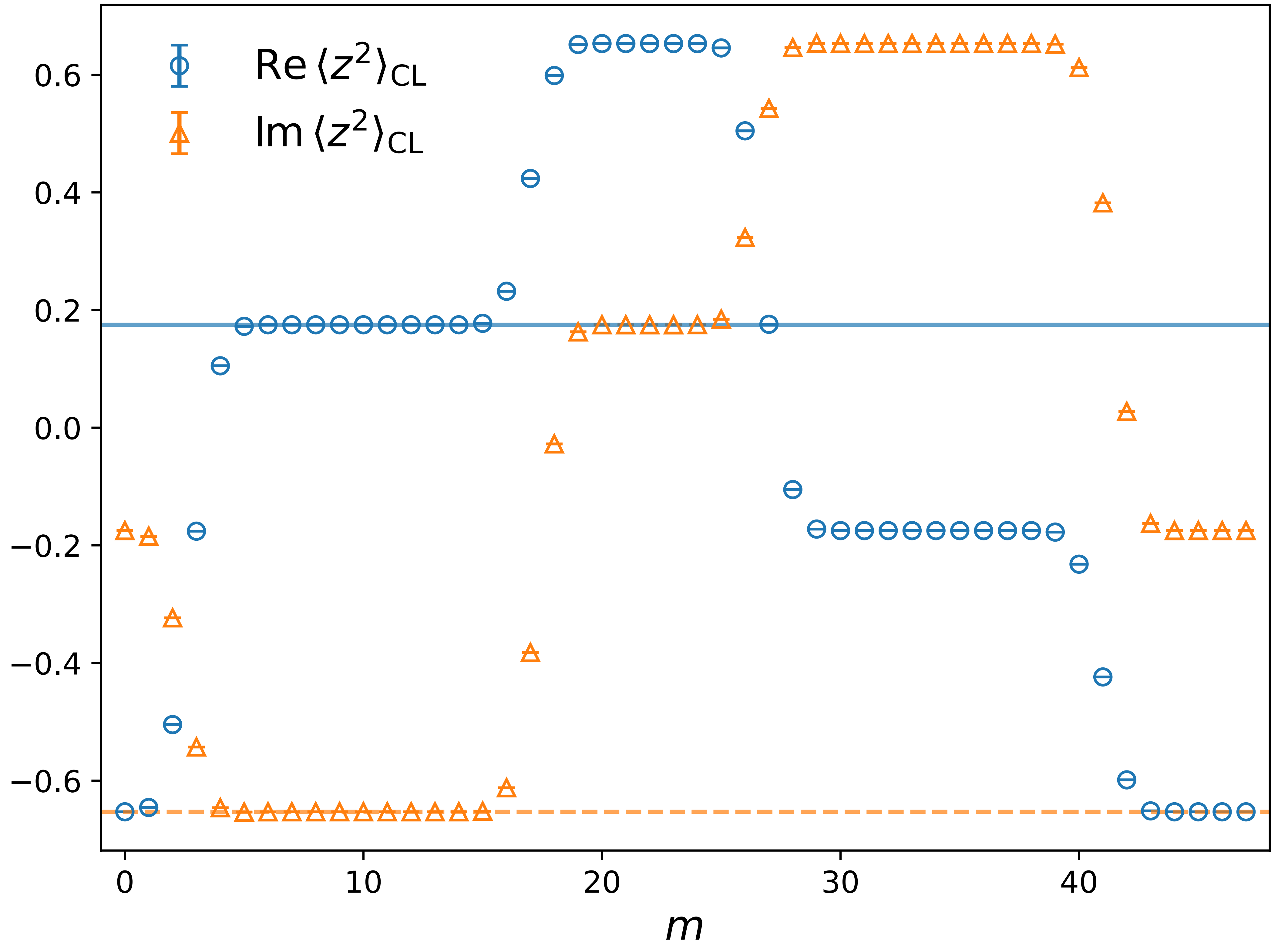

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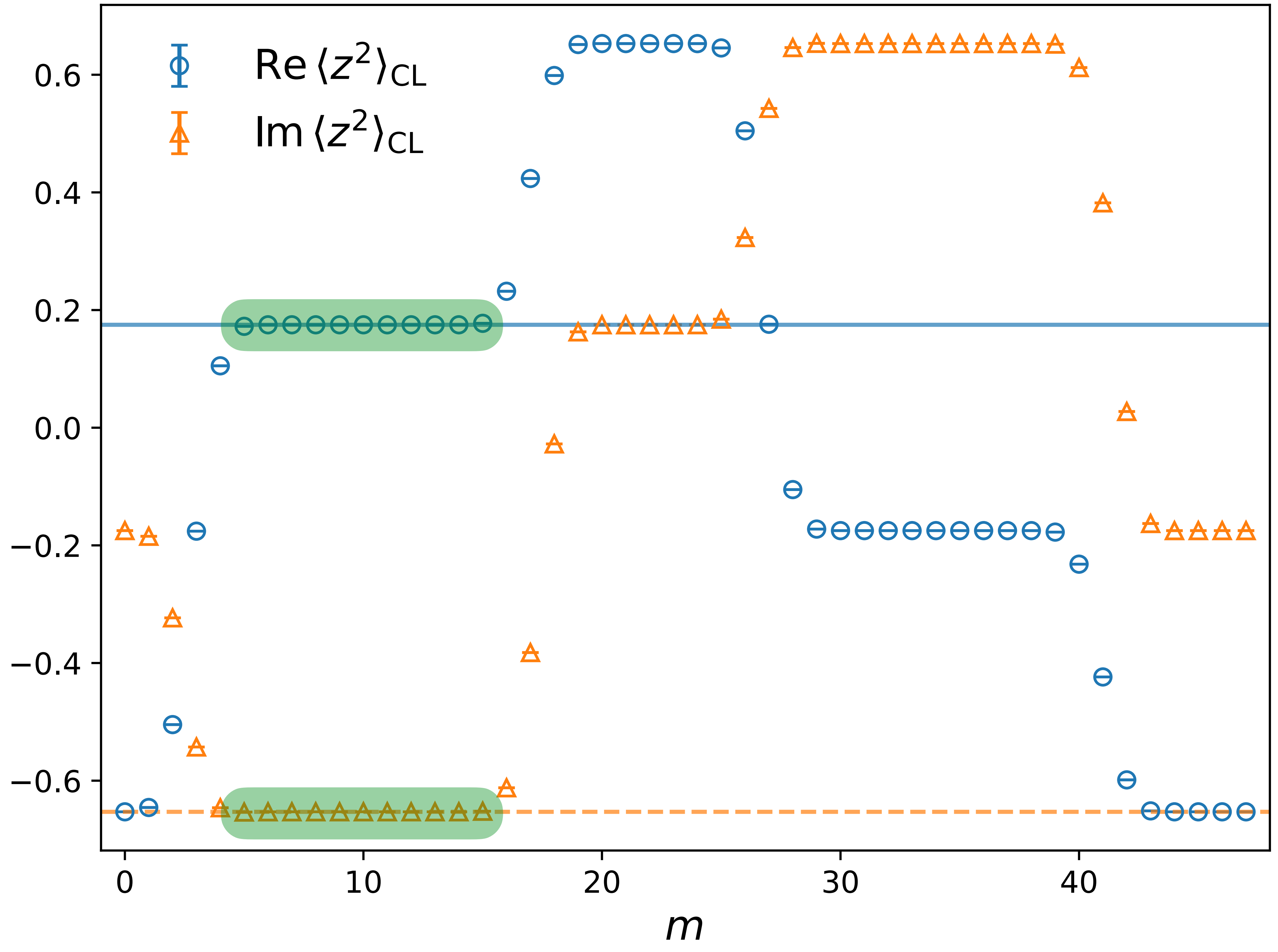

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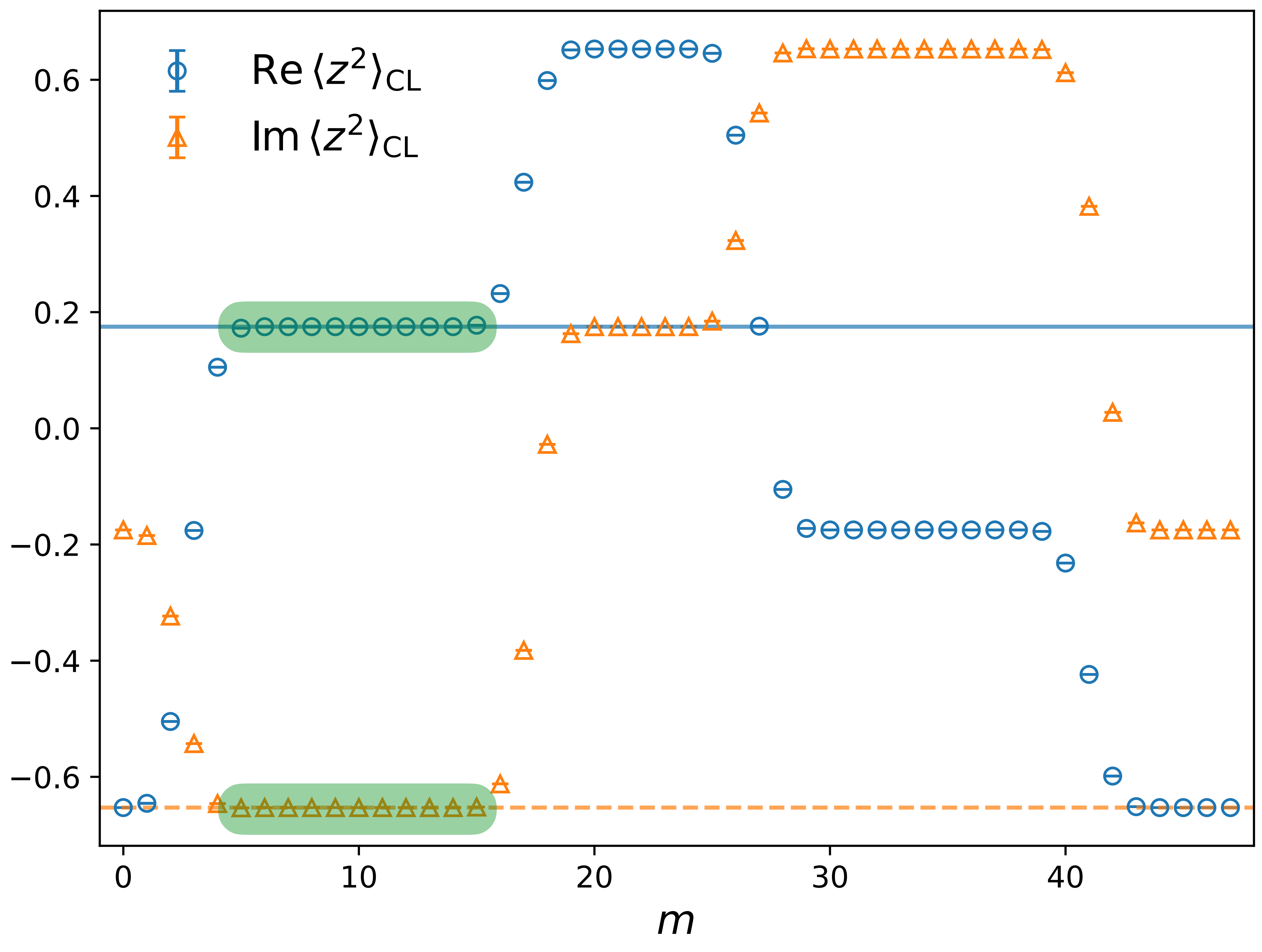

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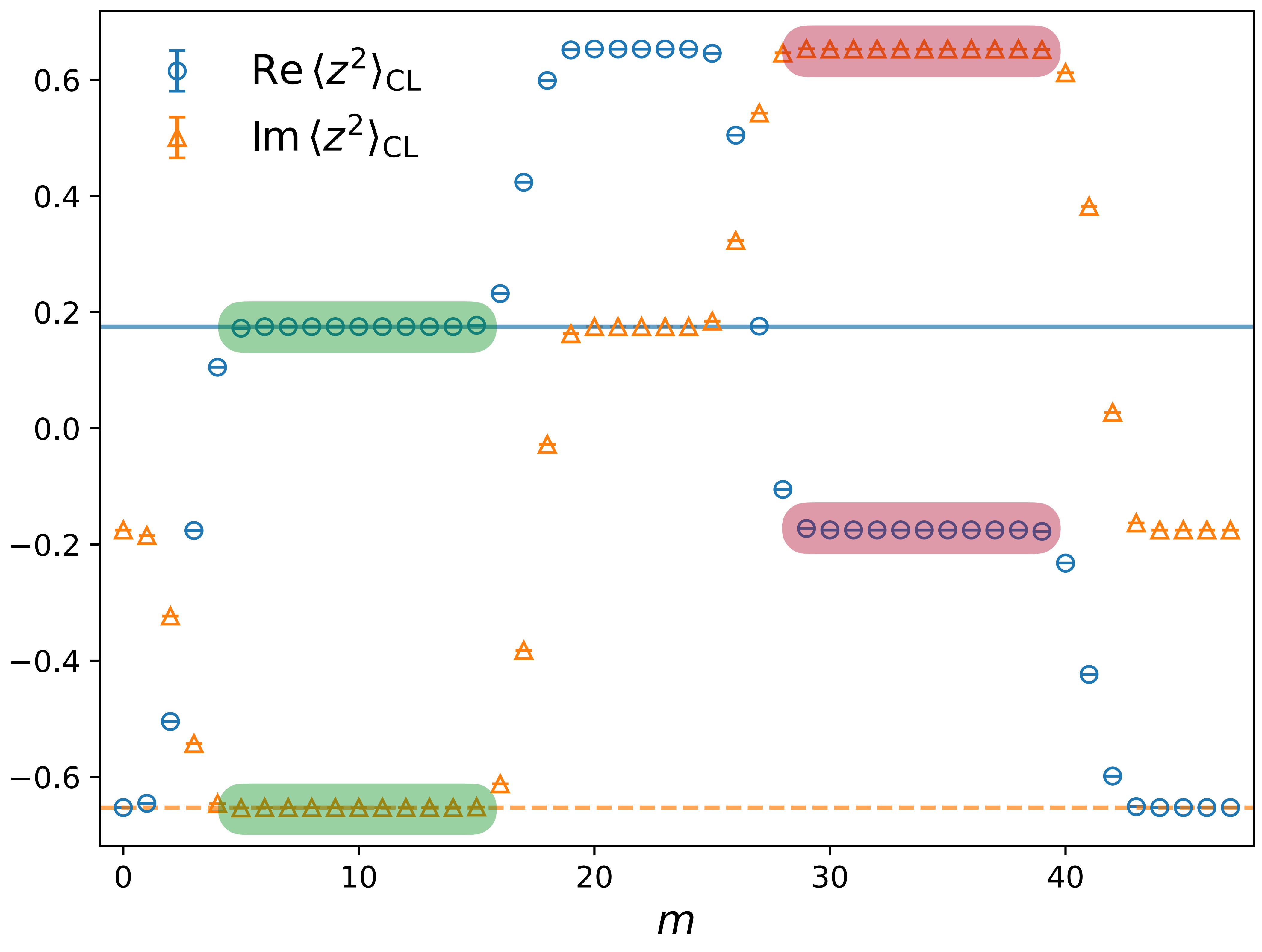

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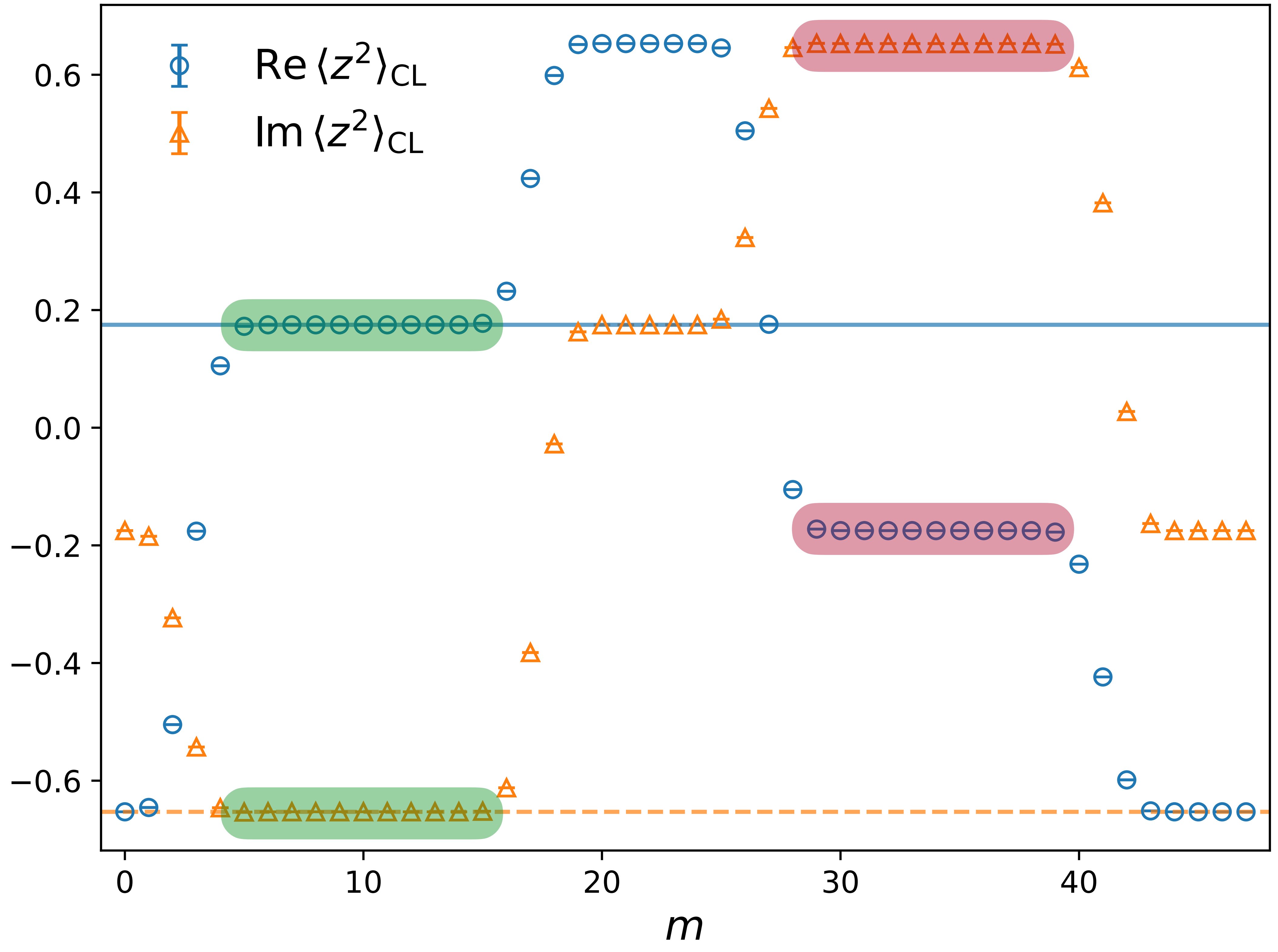

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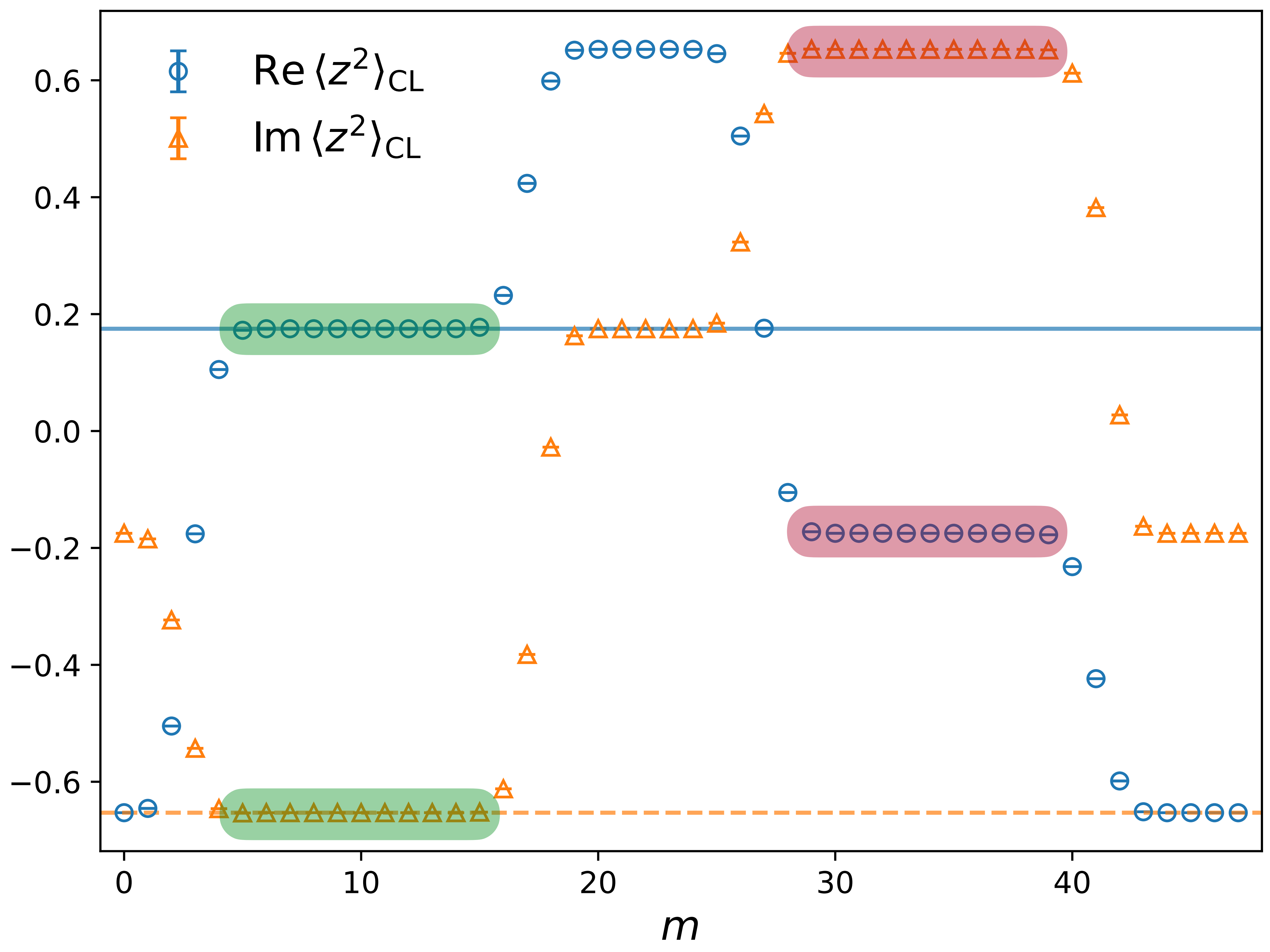

Boundary terms

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Aarts et al. '11; Scherzer et al. '19

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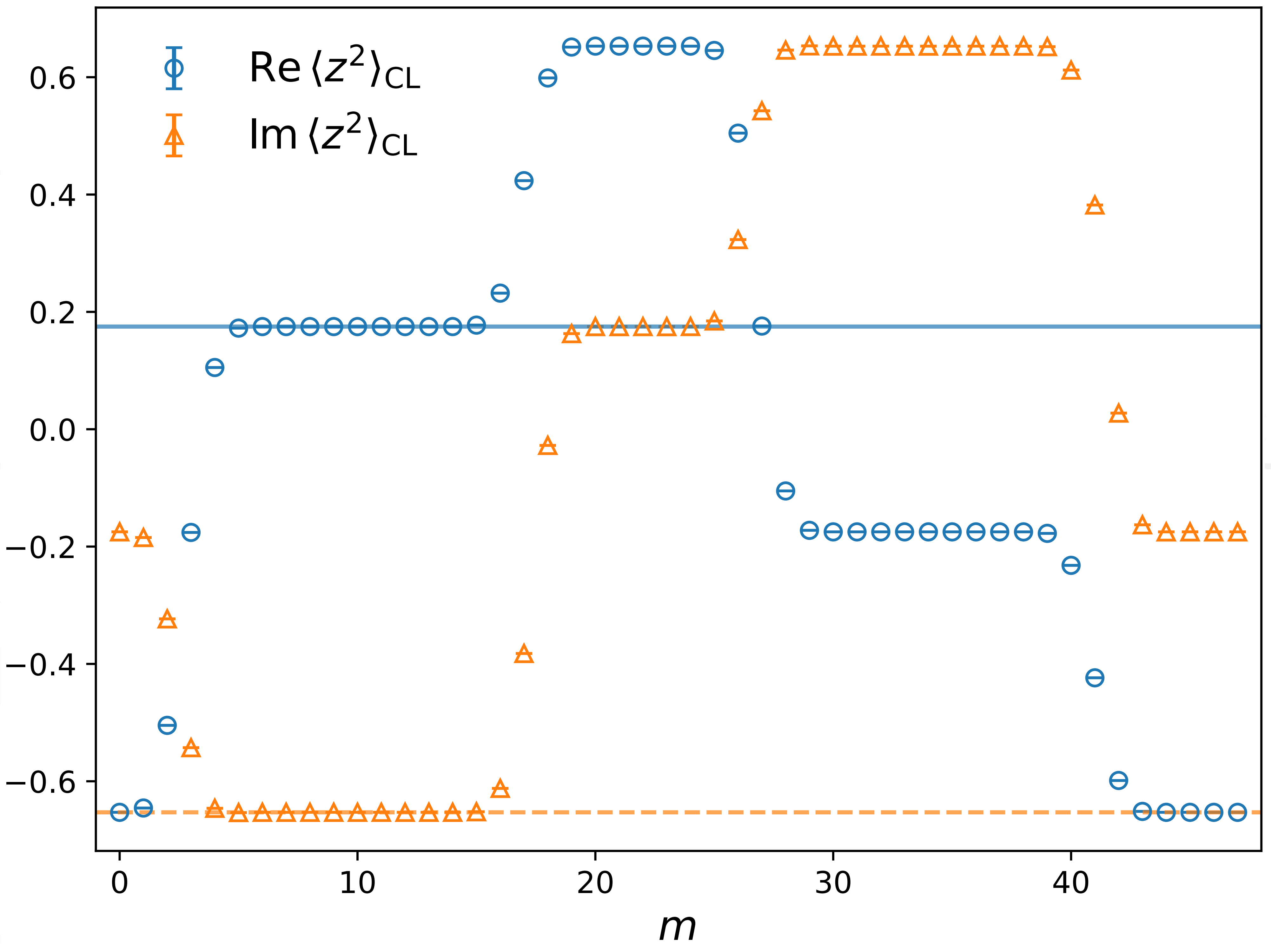

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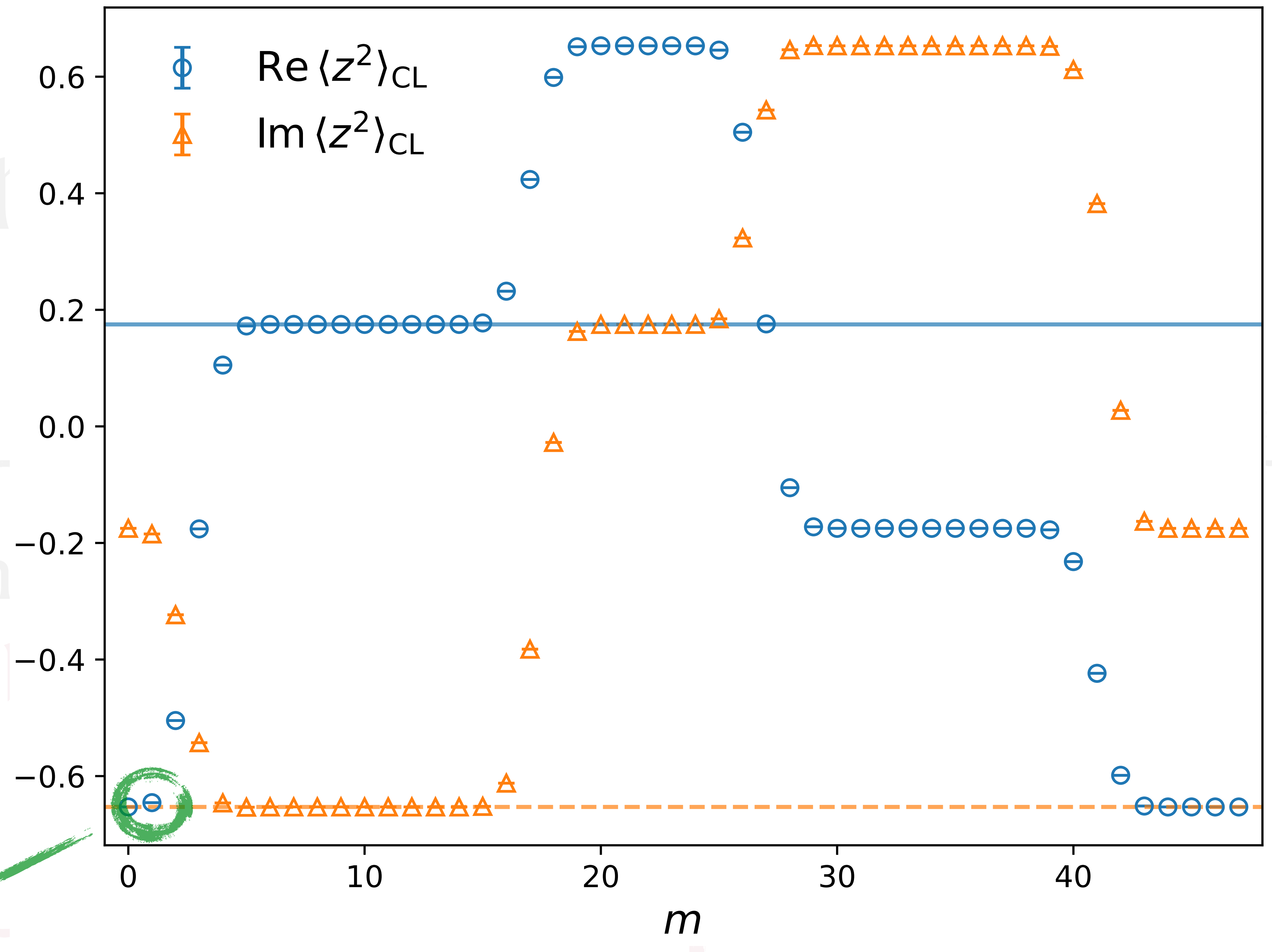

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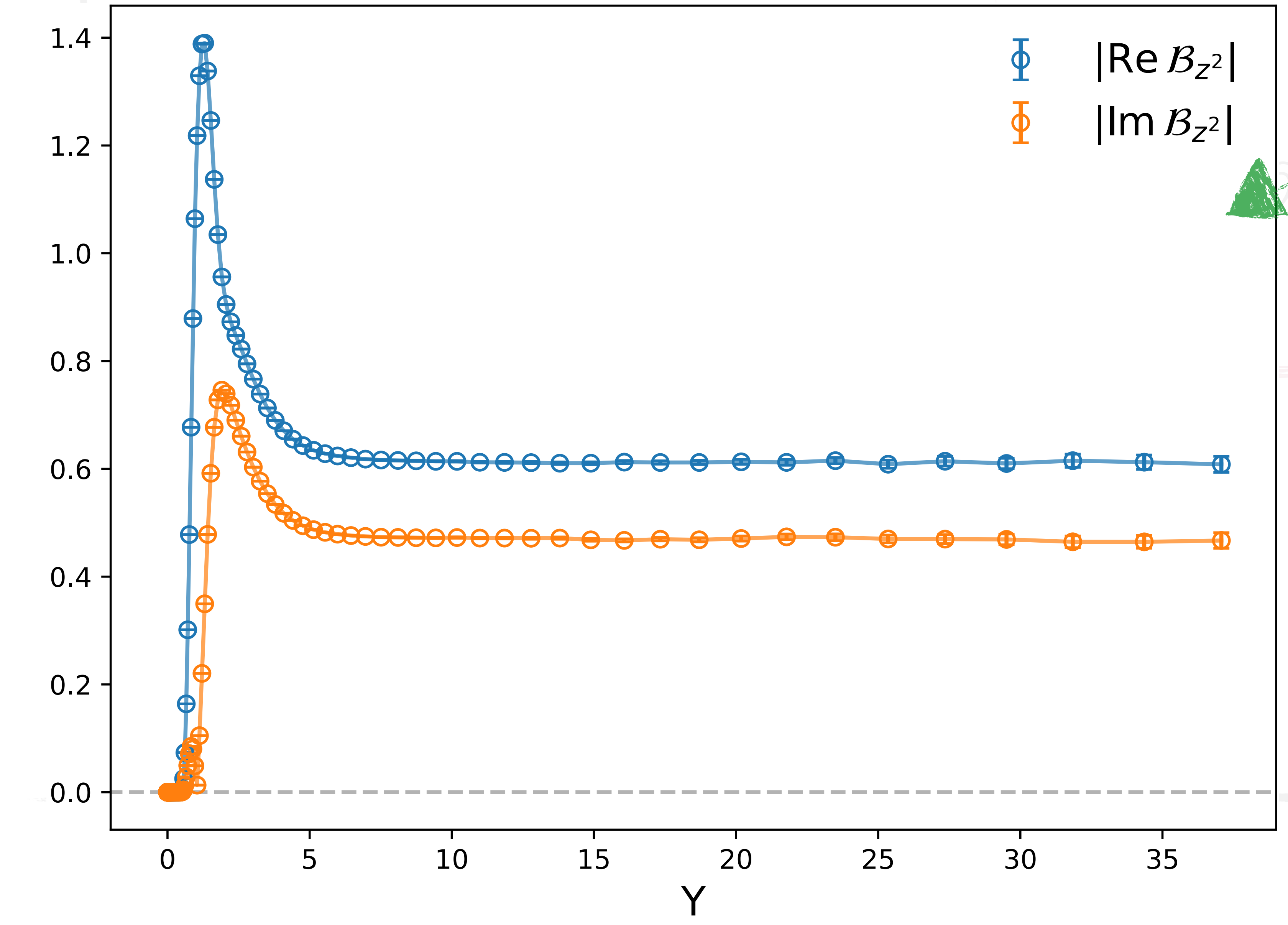

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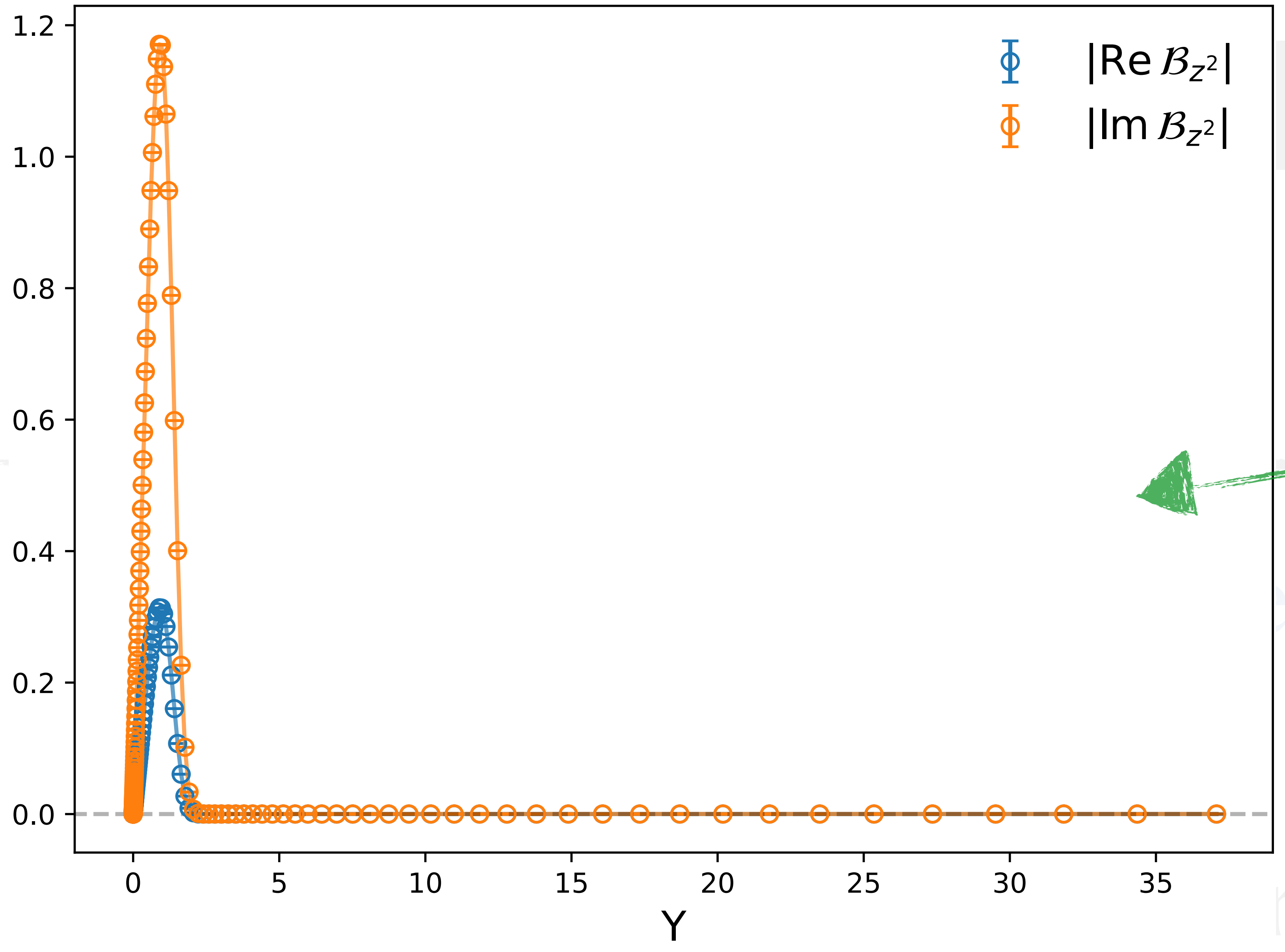

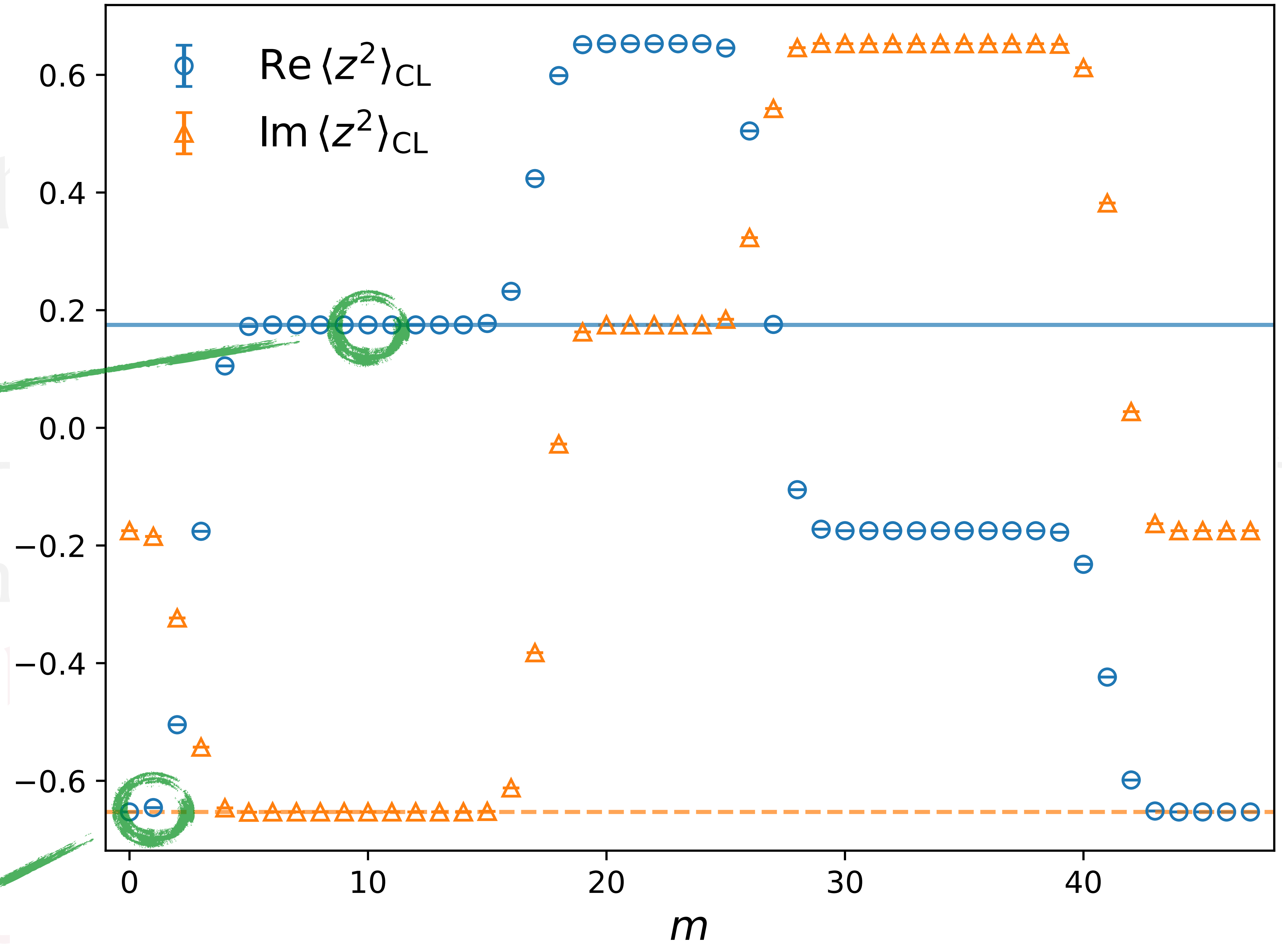

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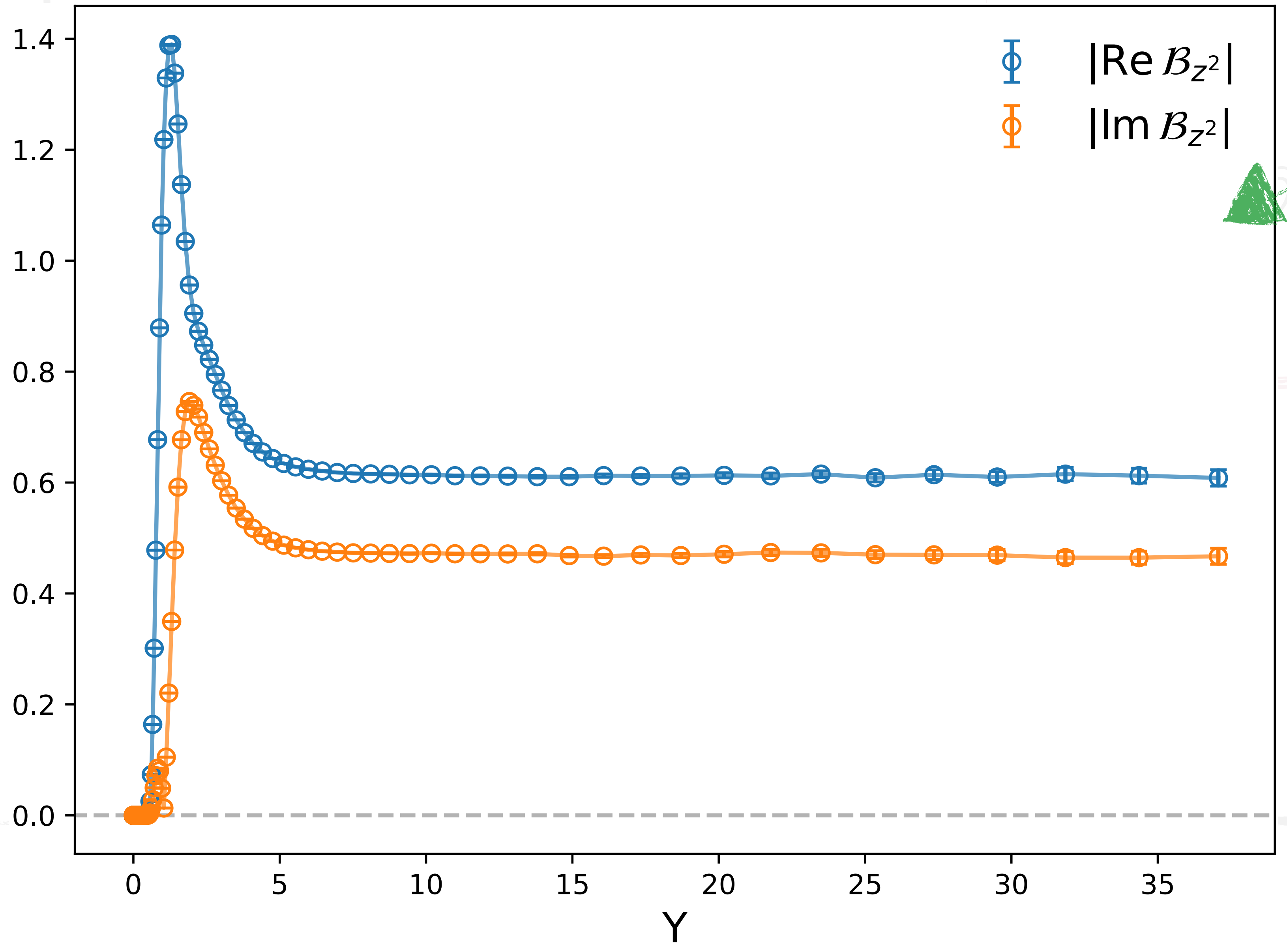

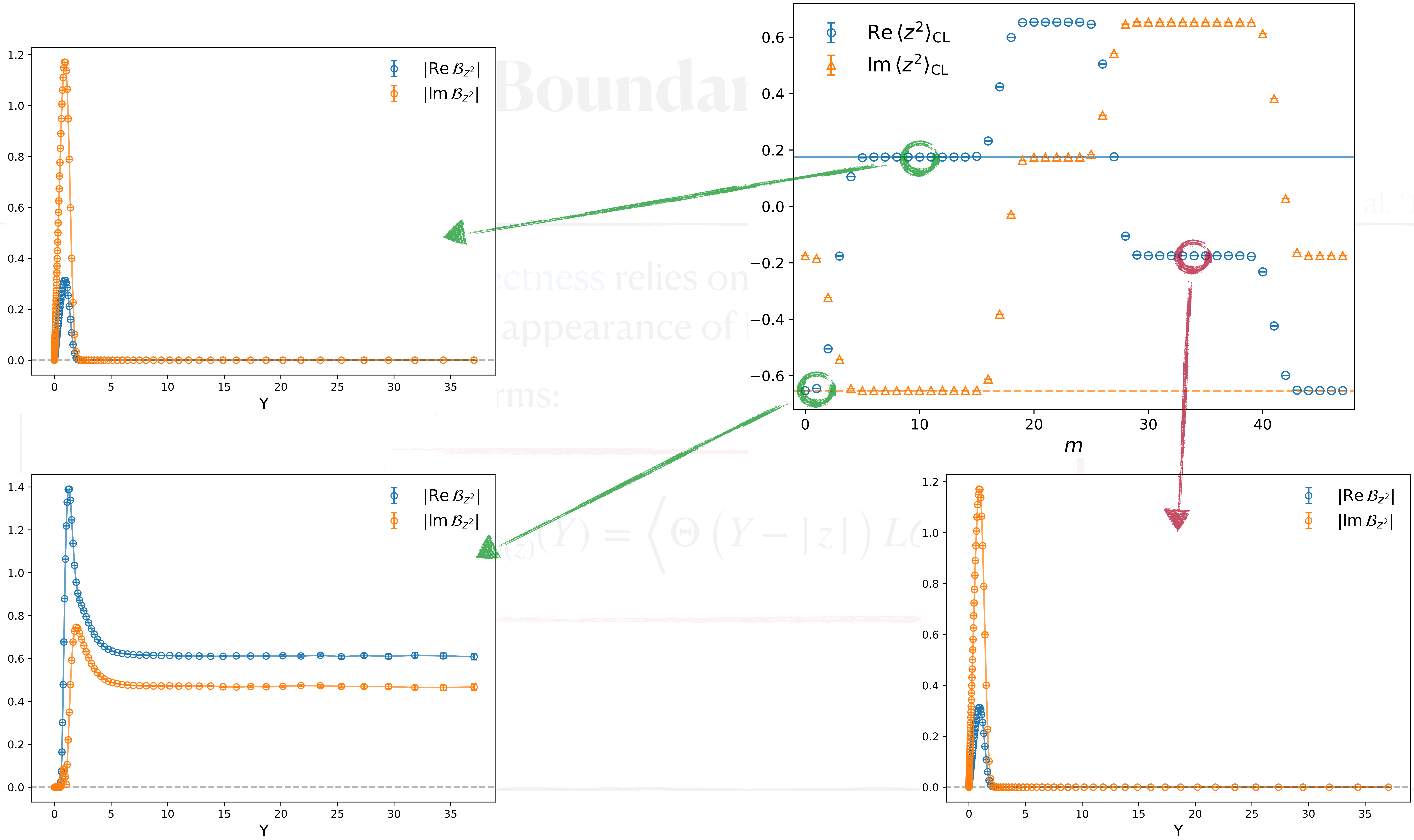

|Re $\mathcal{B}_{z^2}$|
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Y
Re $\langle z^2 \rangle_{CL}$
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m
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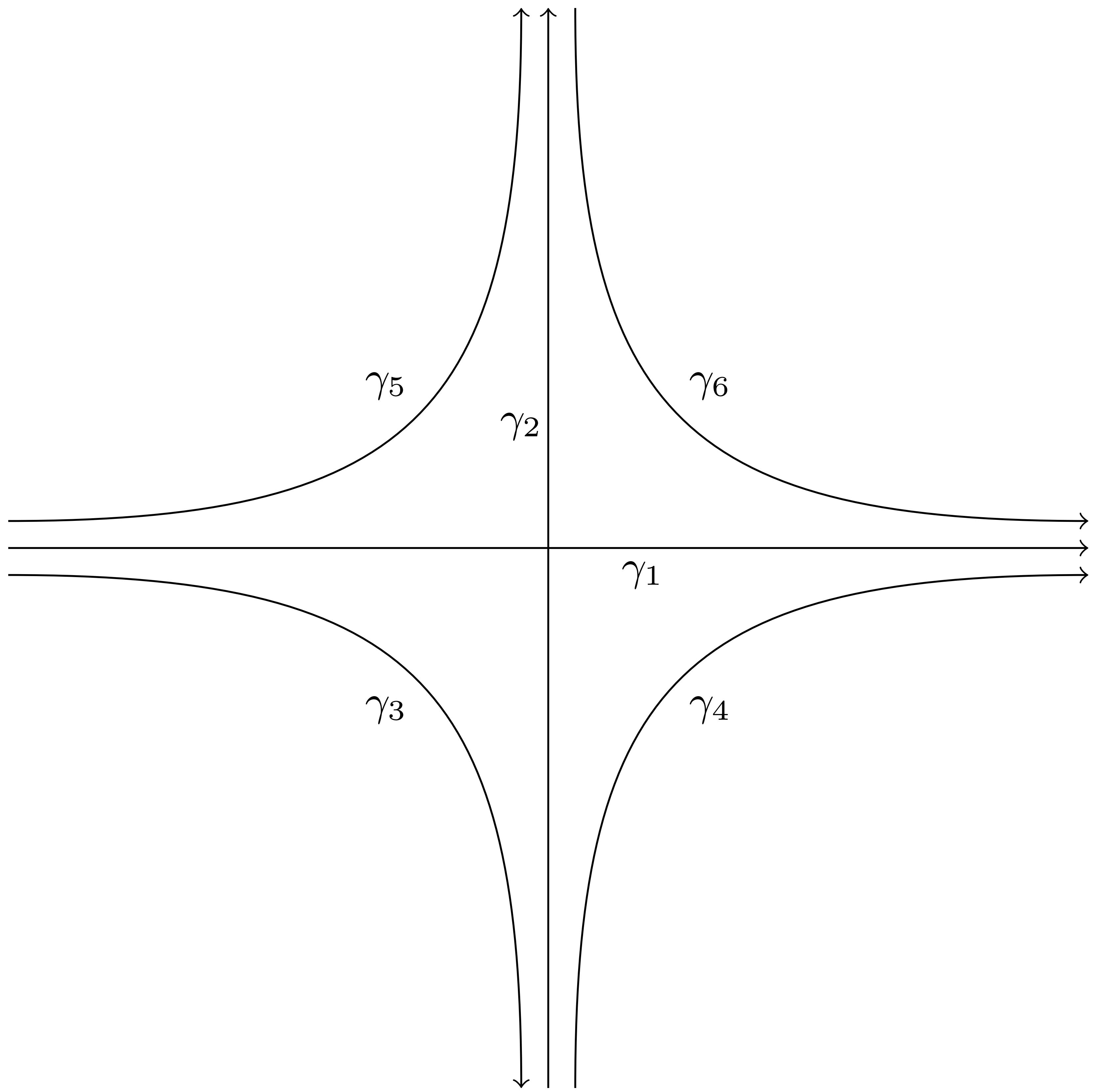

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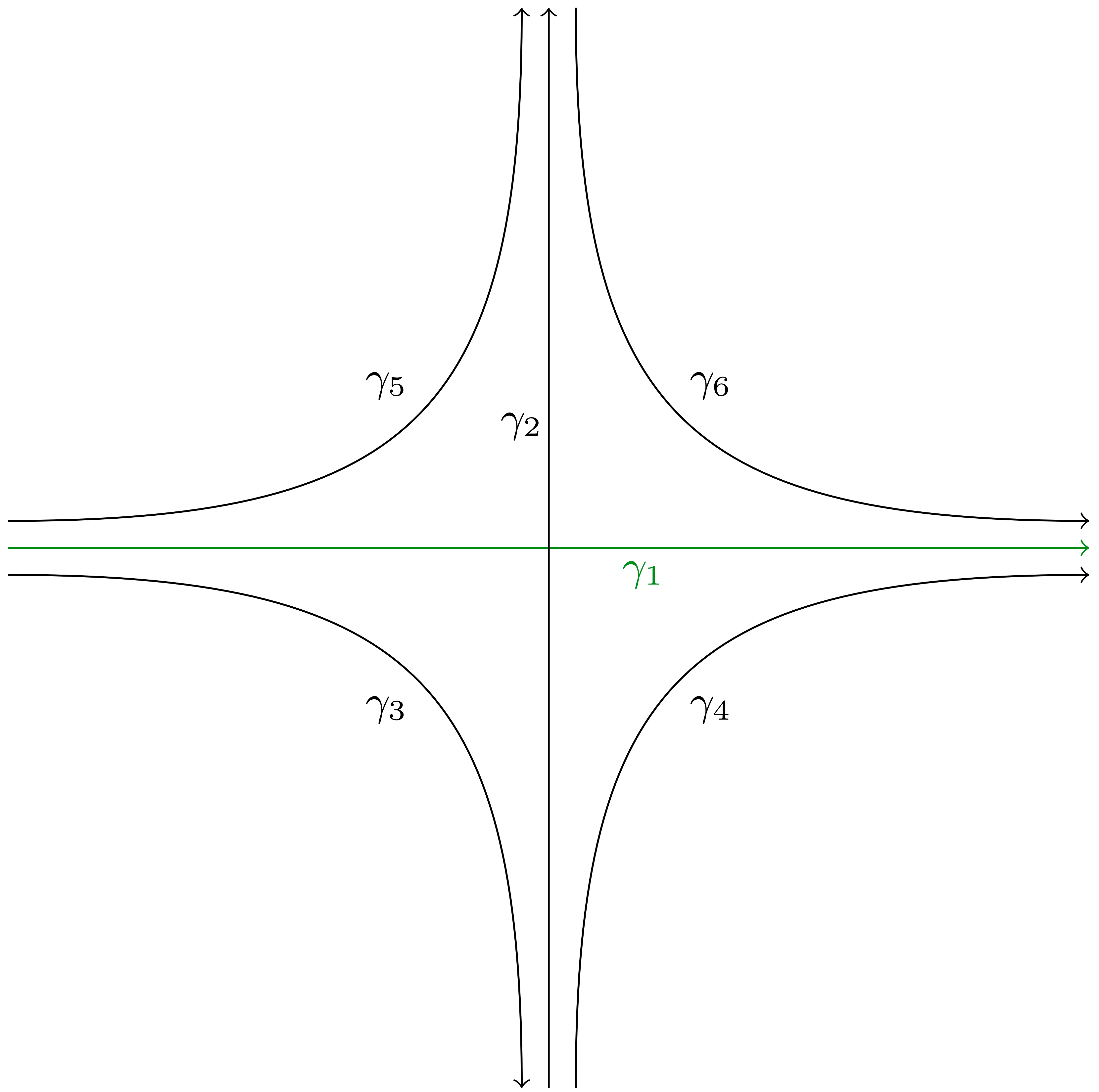

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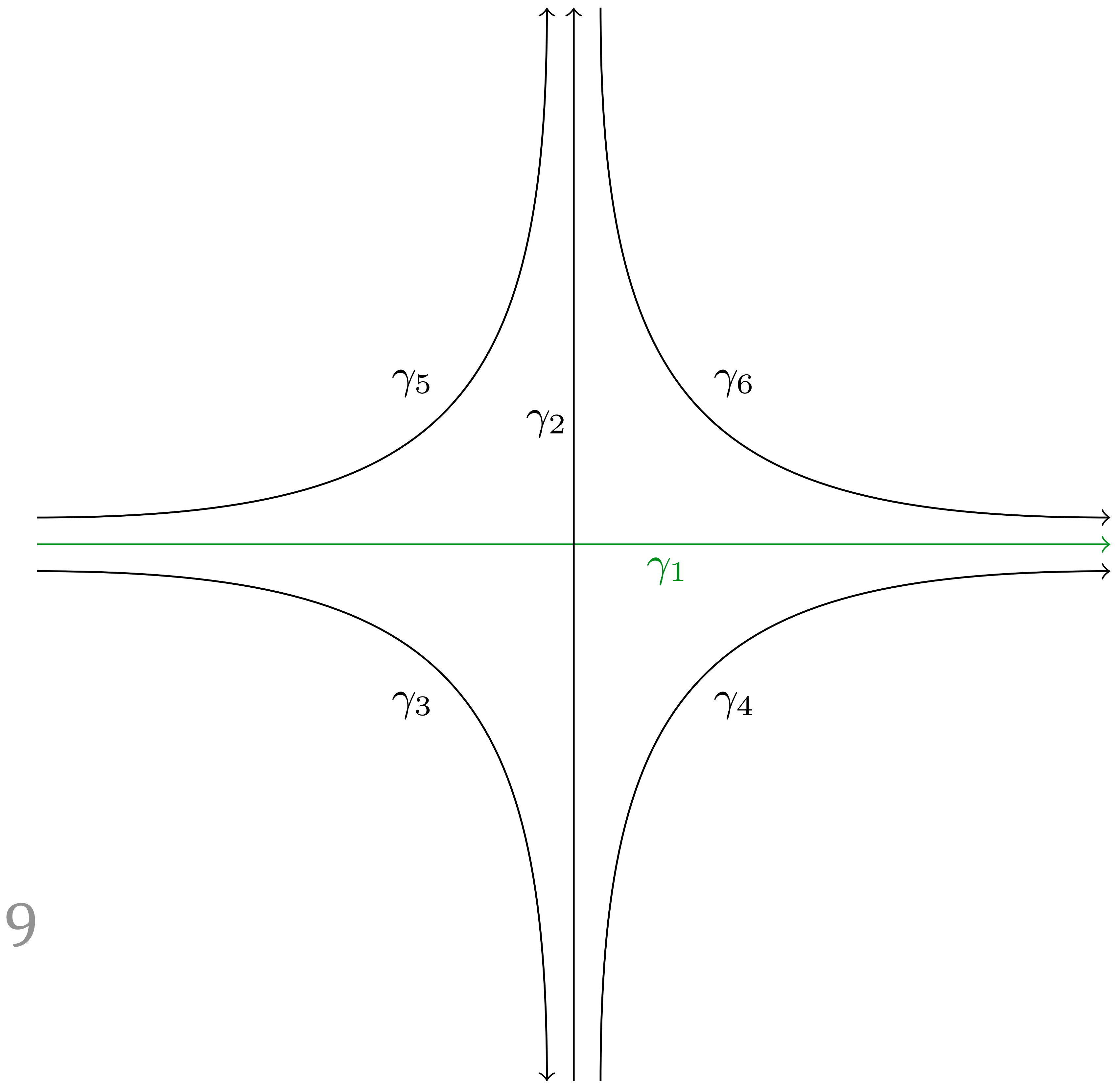
$$\langle \mathcal{O} \rangle_{\mathrm{CL}} = \sum_{i=1}^{3} a_i \langle \mathcal{O} \rangle_{\gamma_i}$$

Salcedo, Seiler '19



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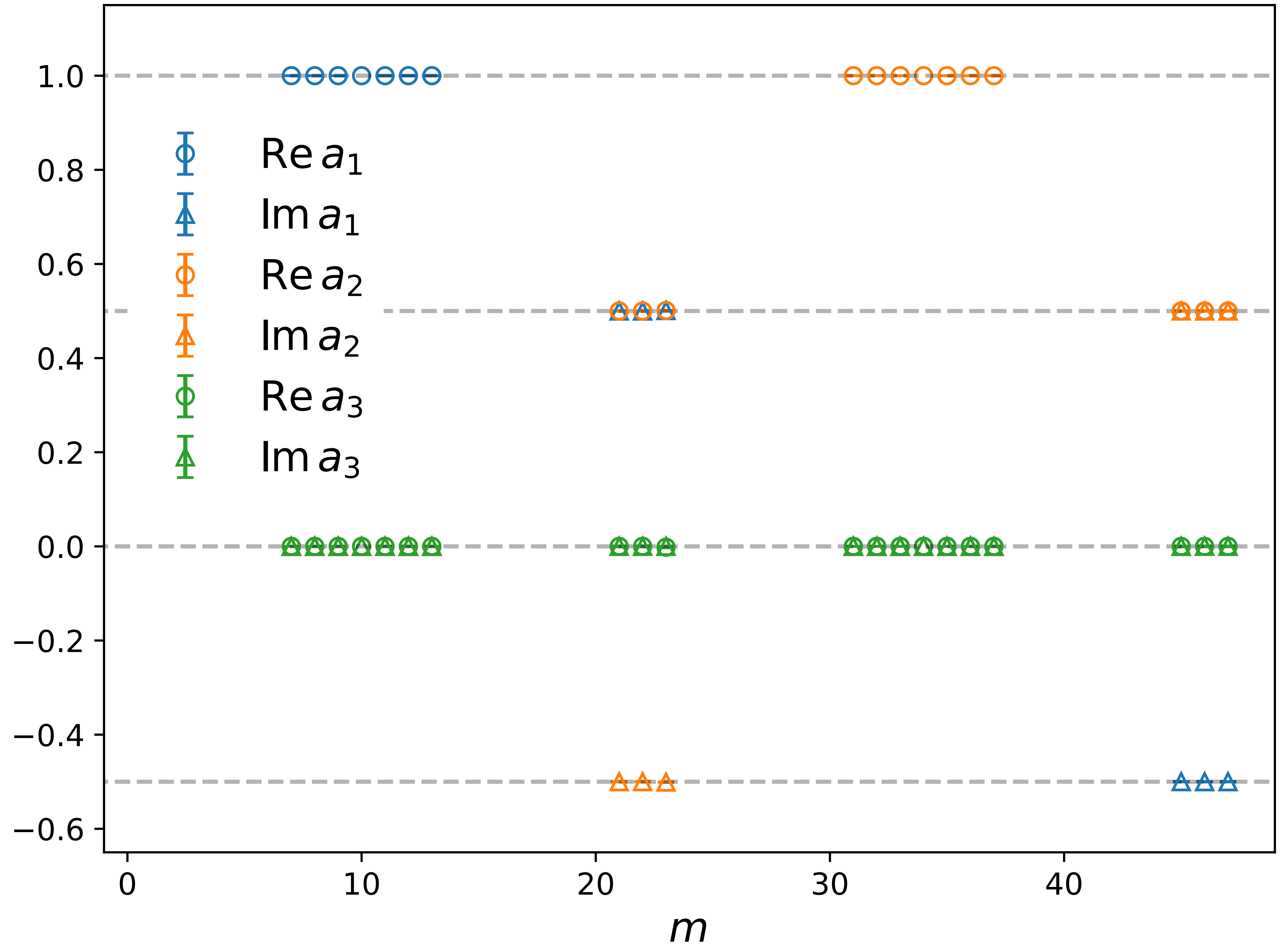

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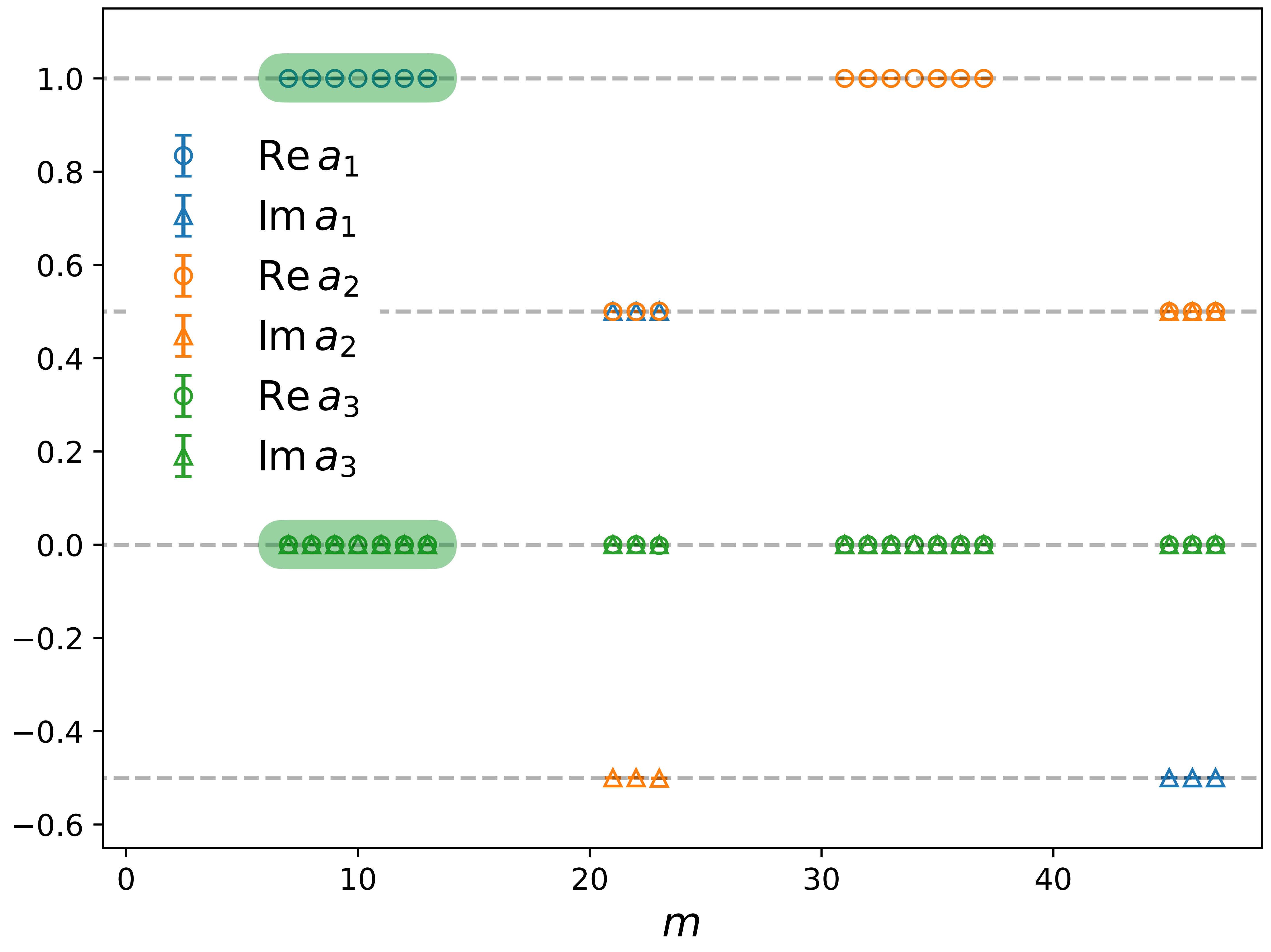

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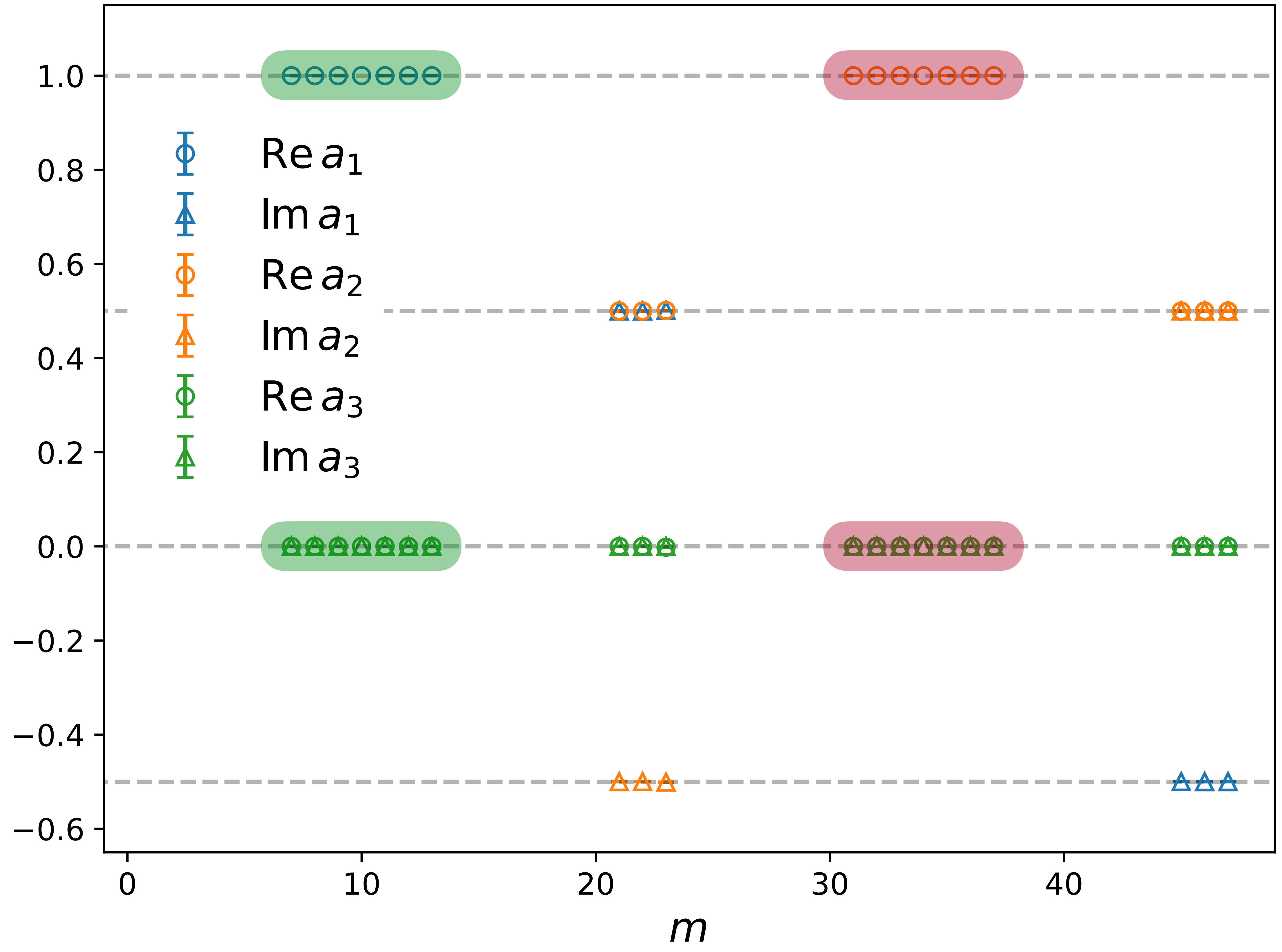

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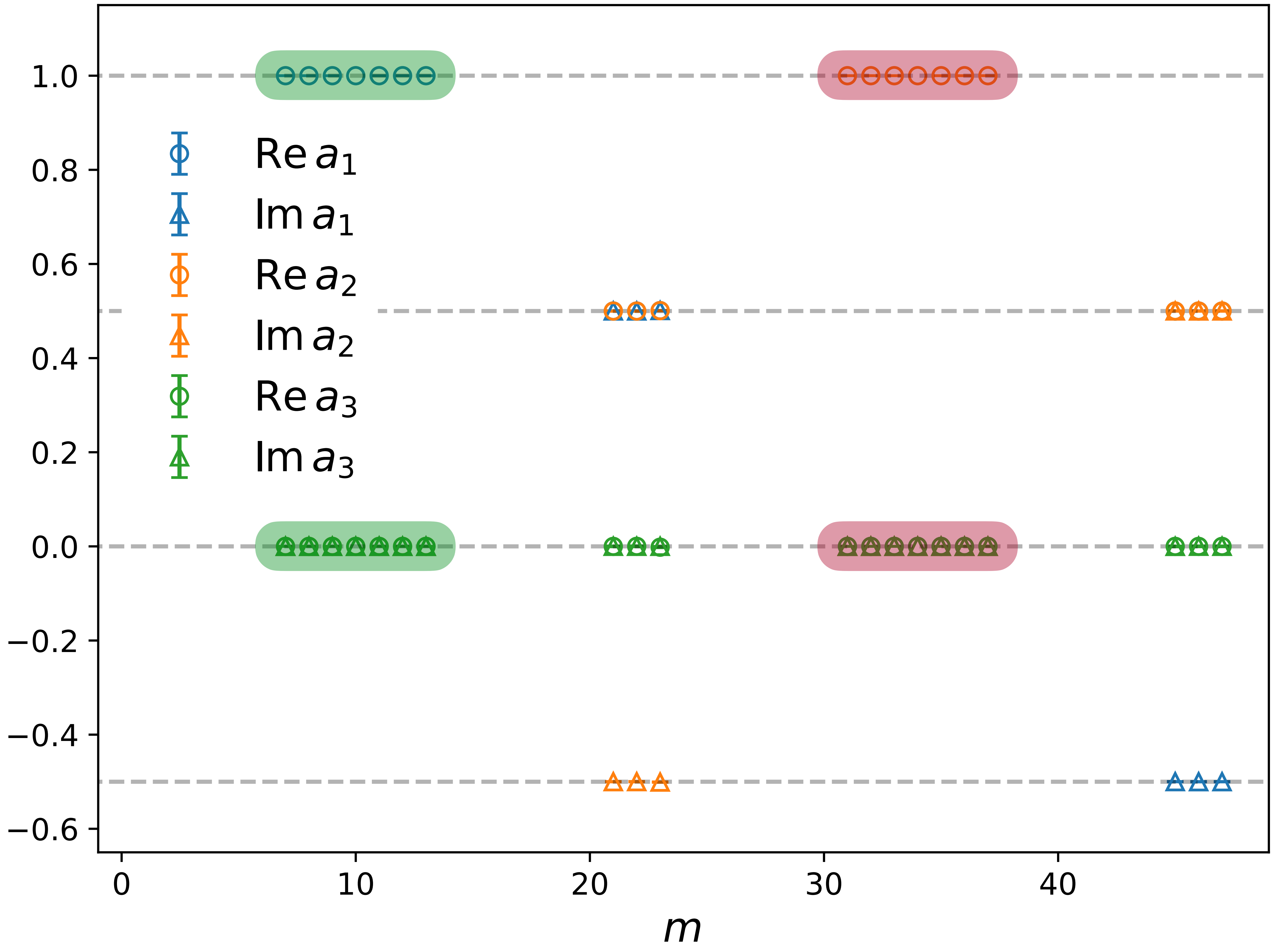

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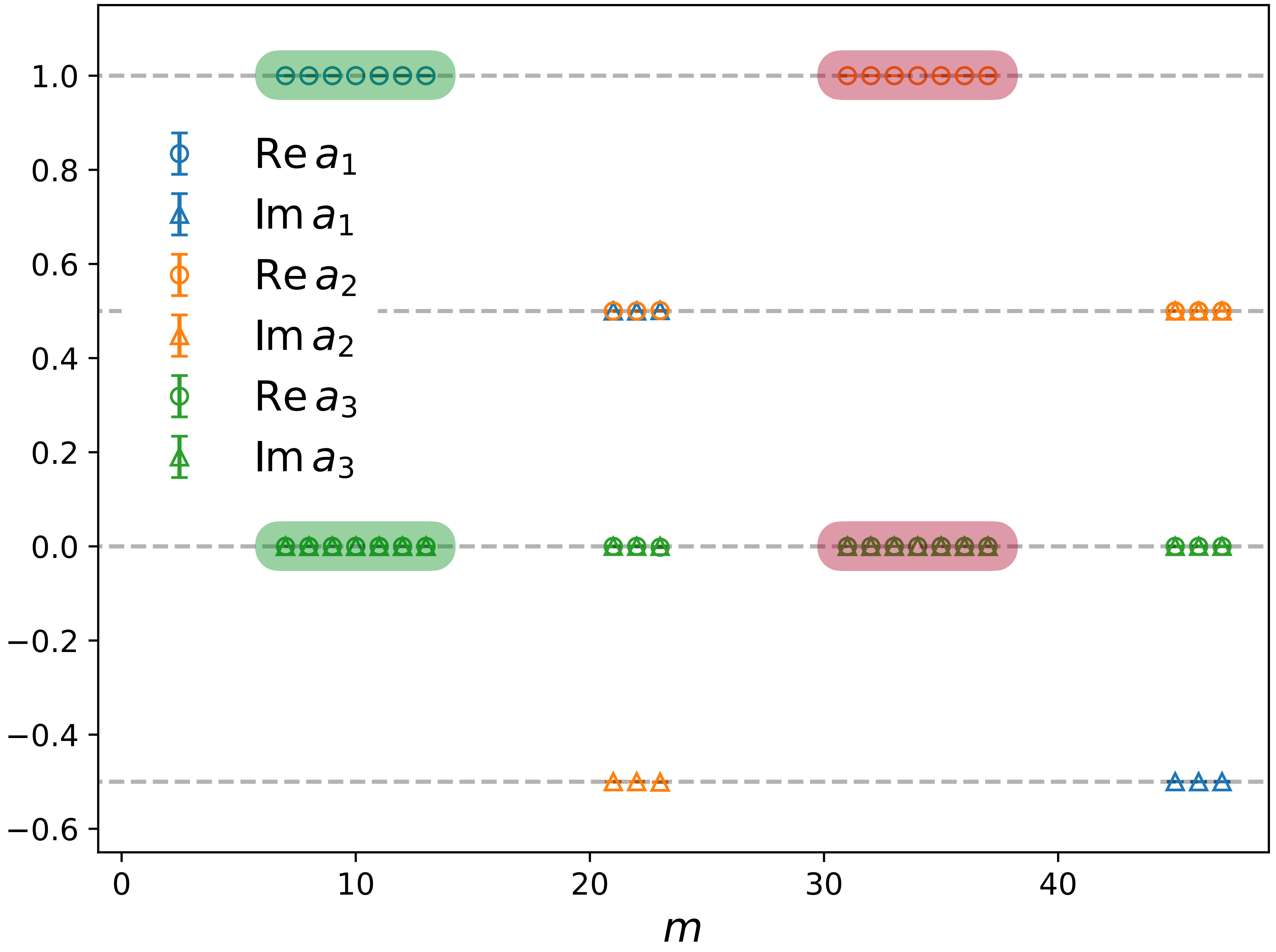

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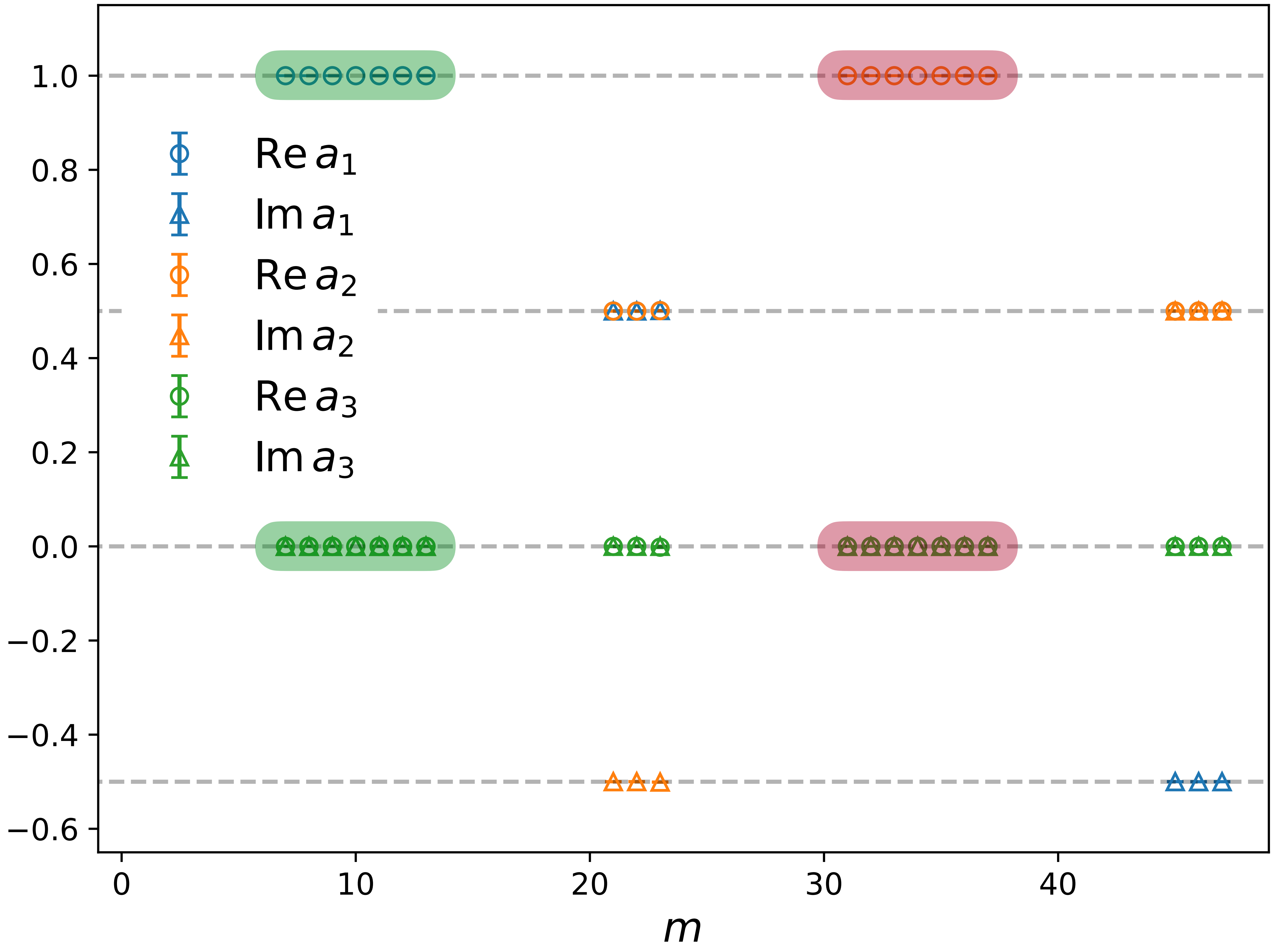

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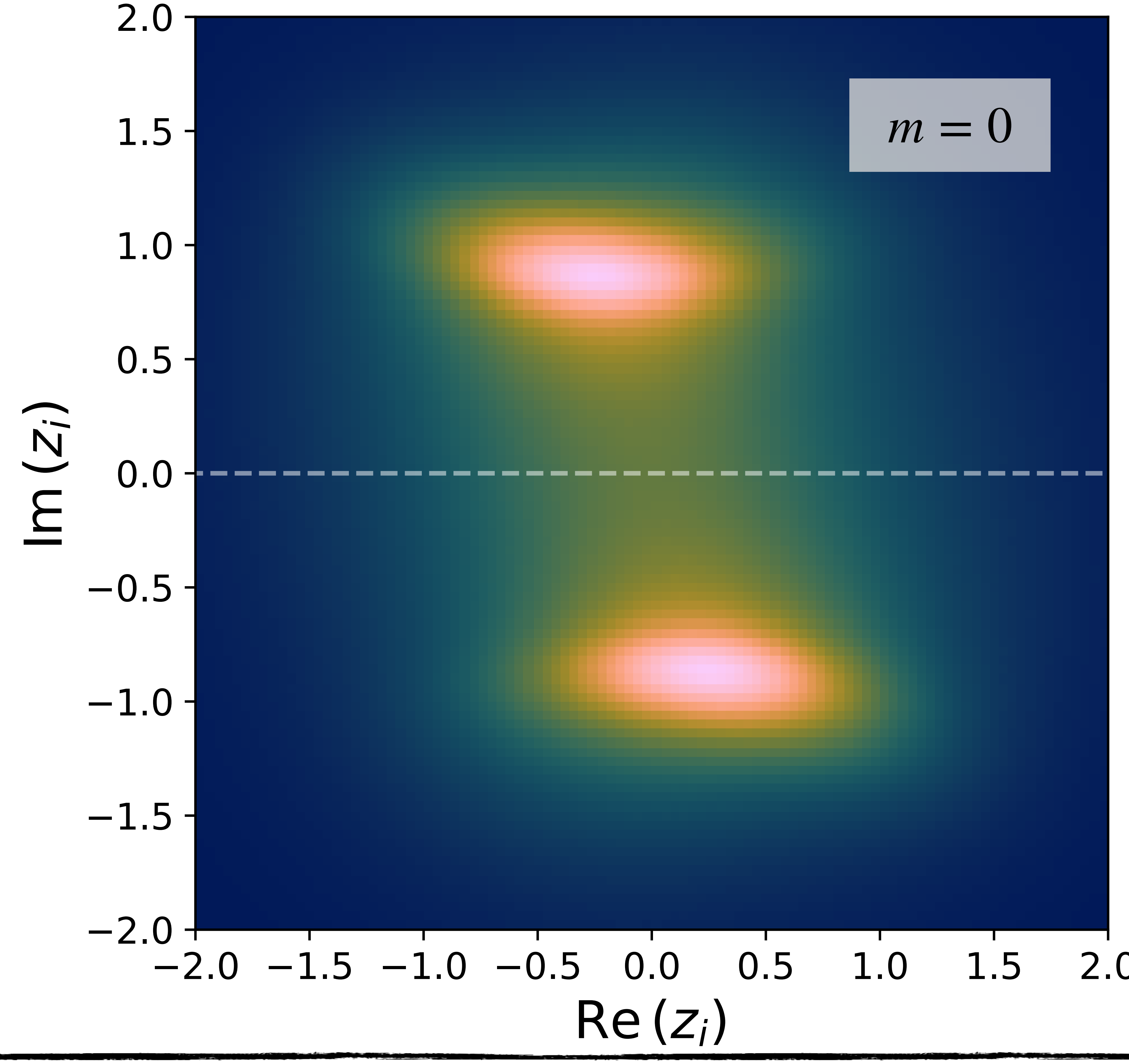

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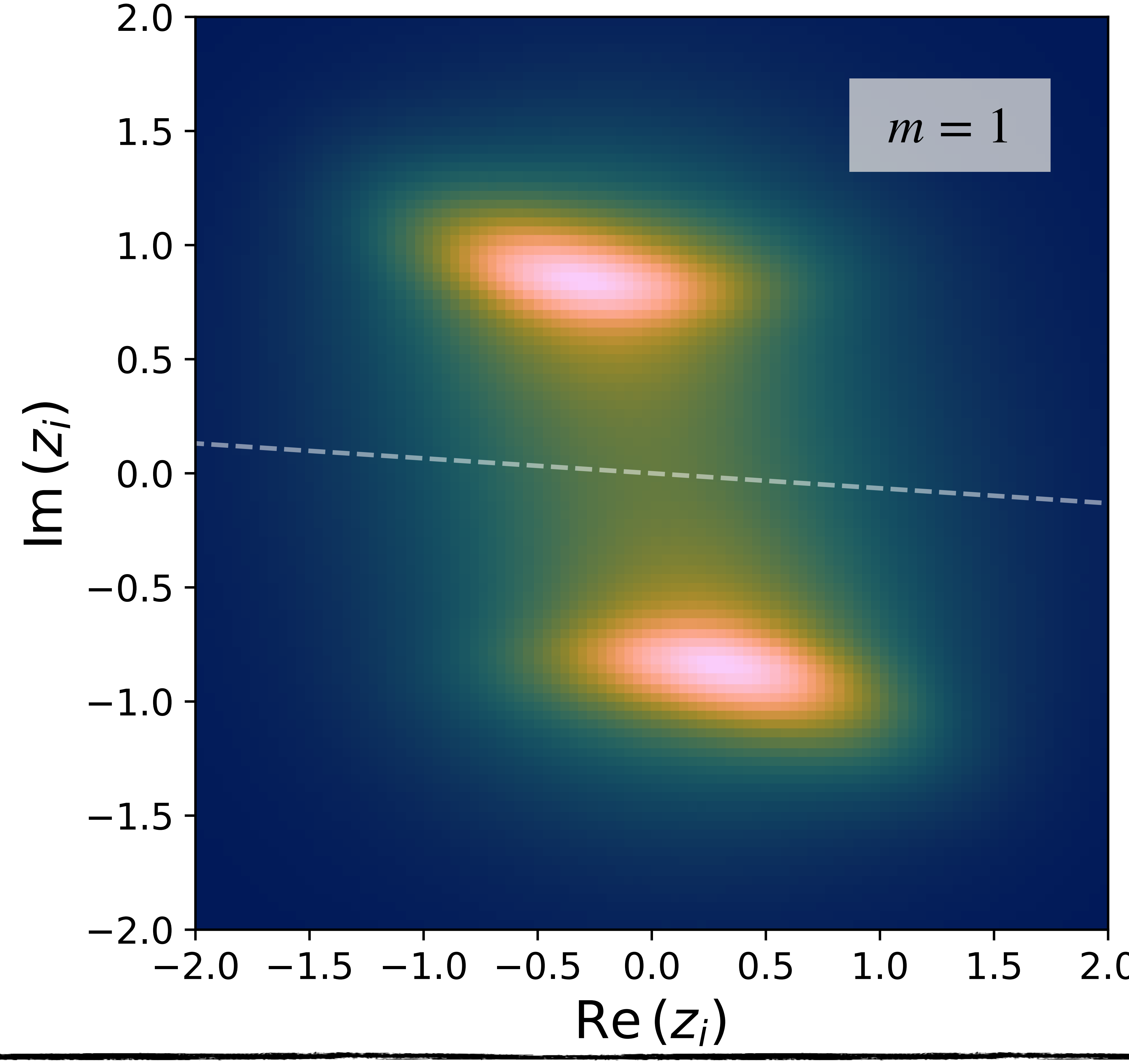

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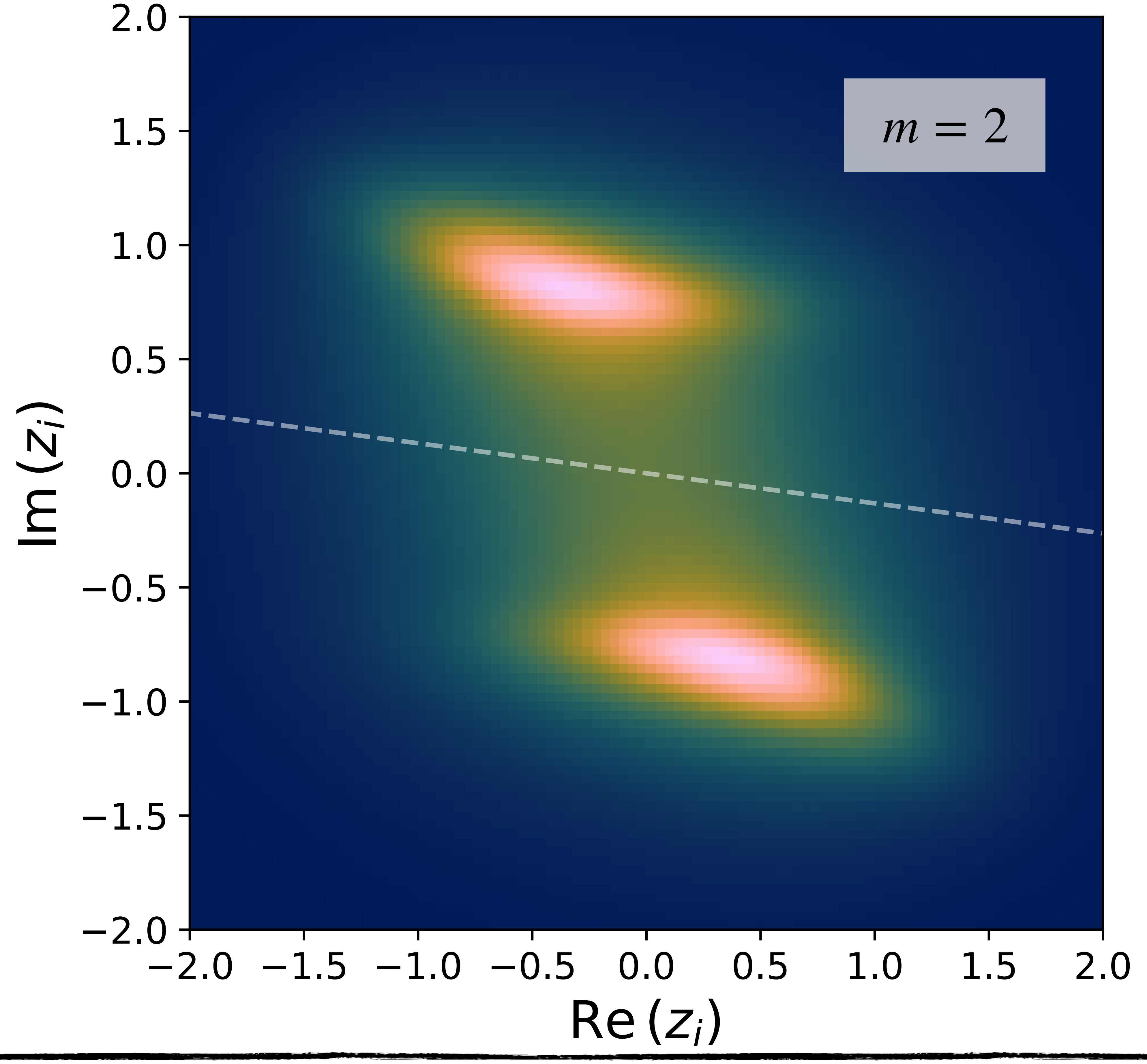

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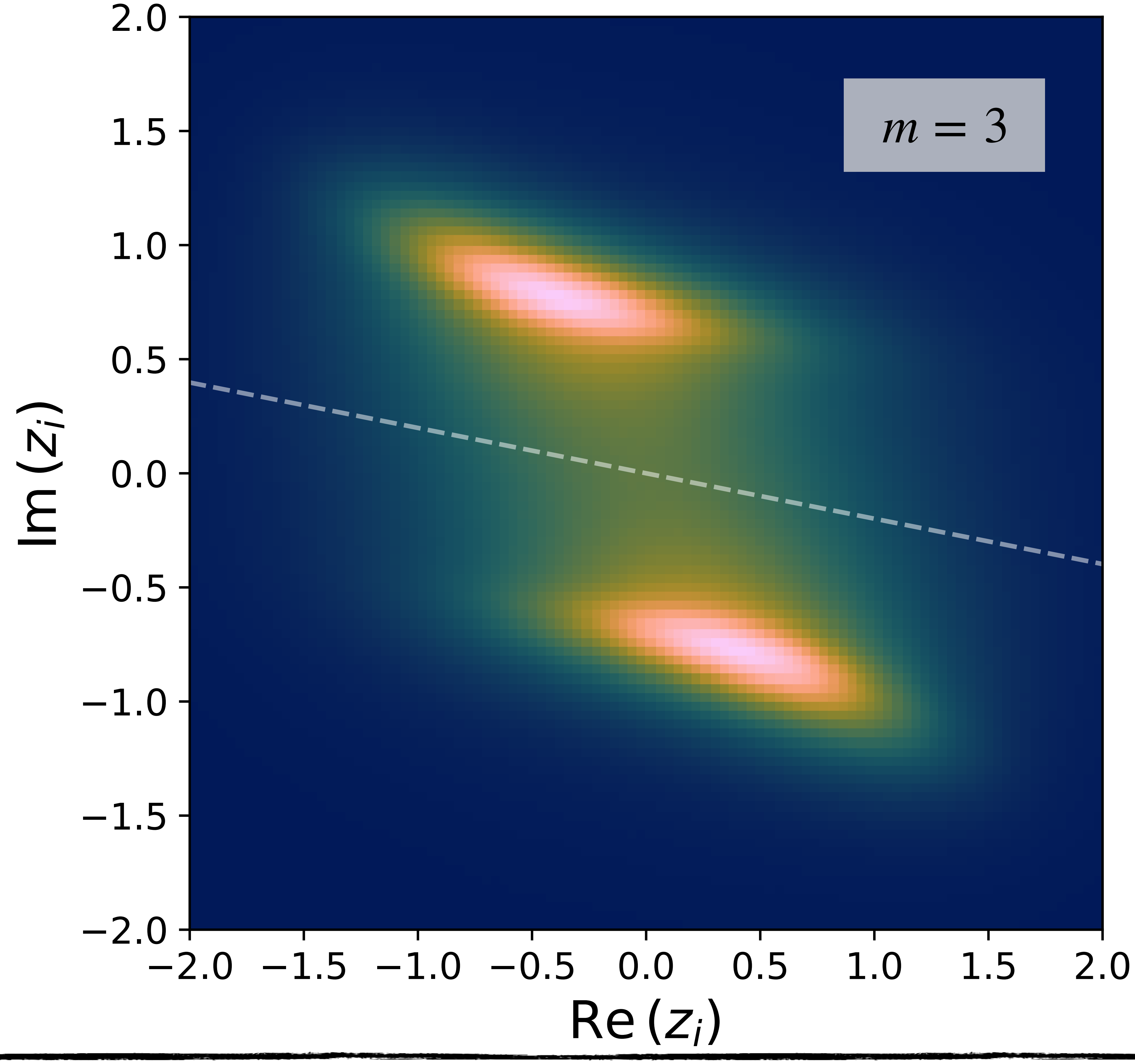

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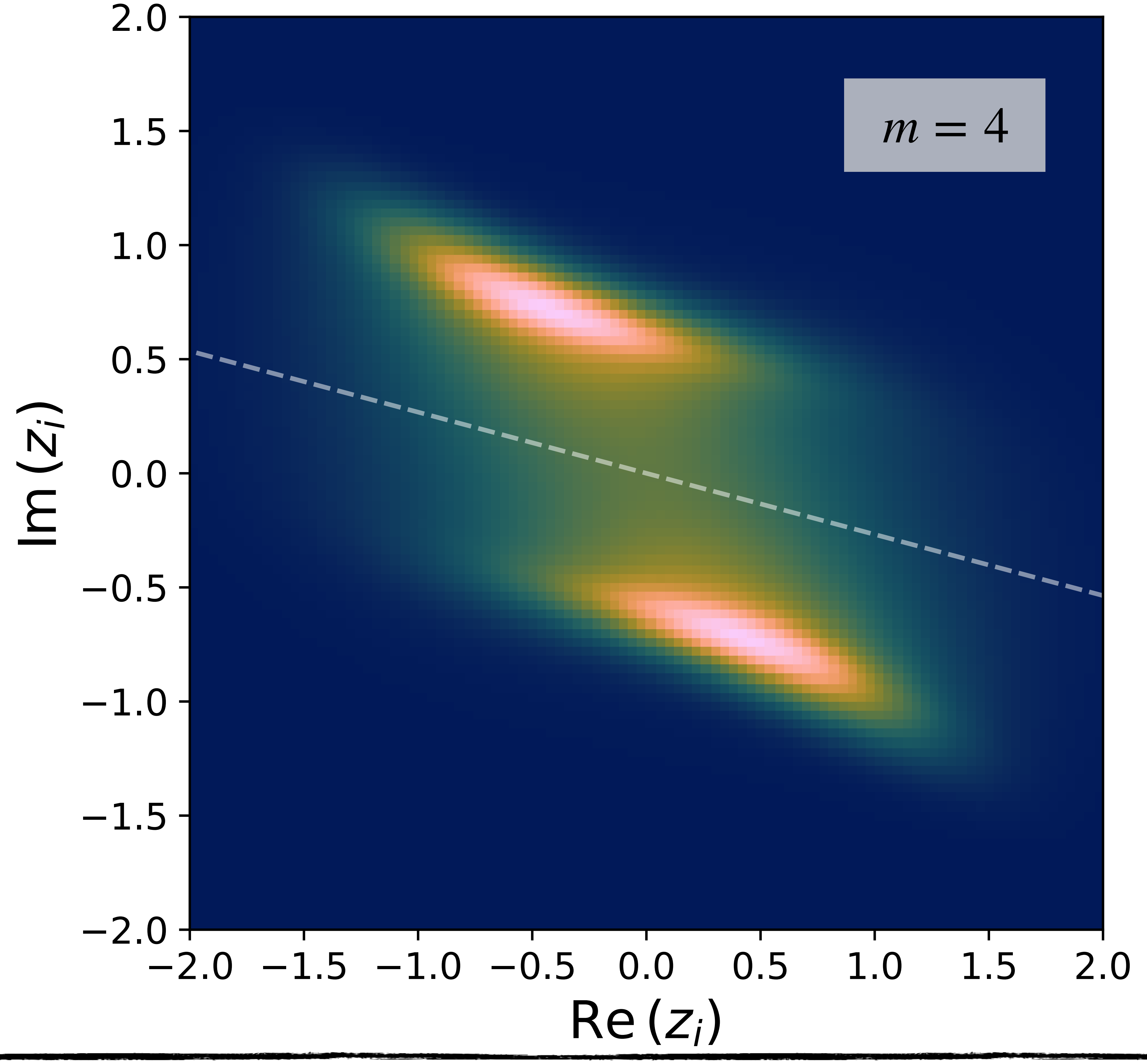

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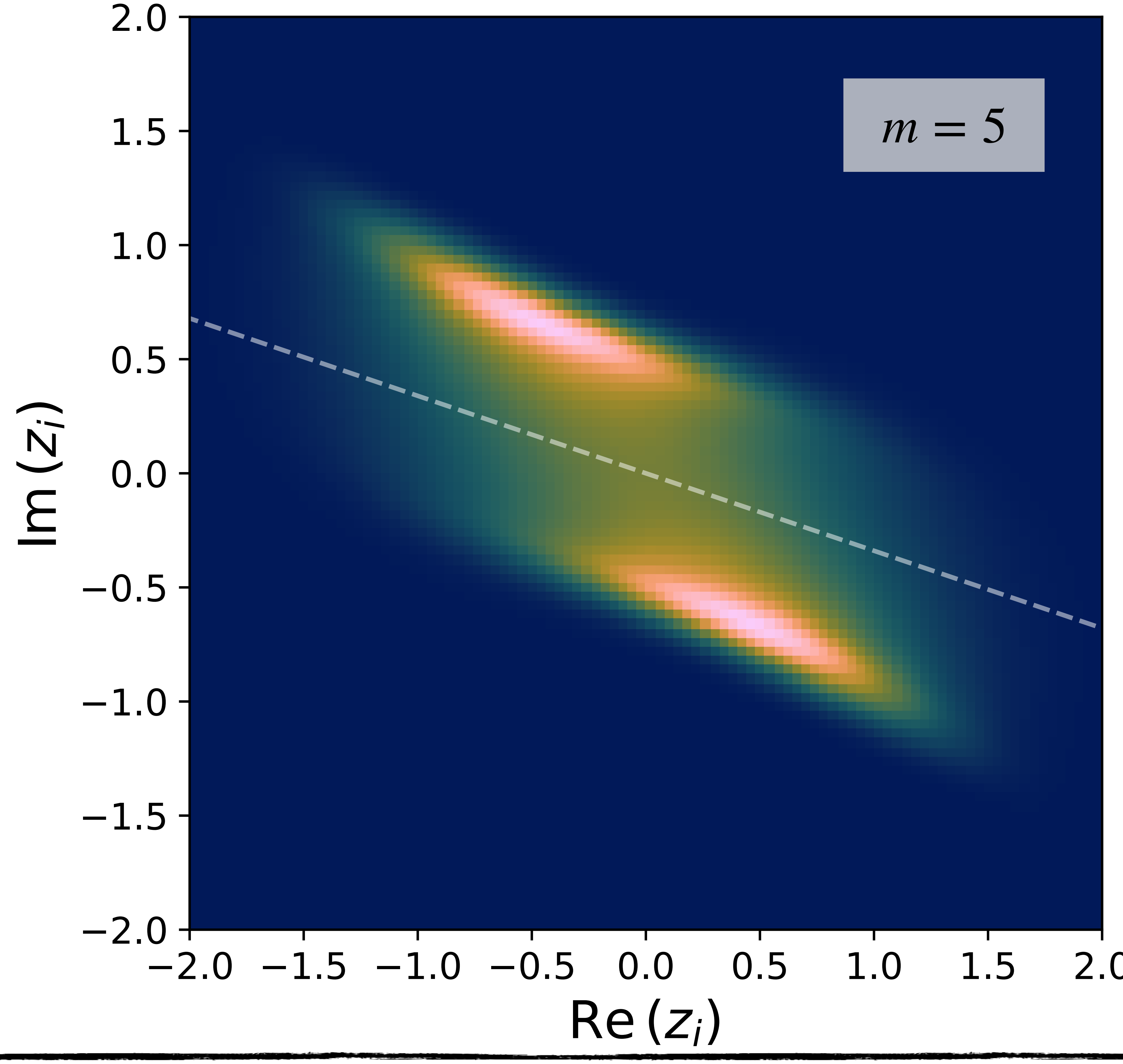

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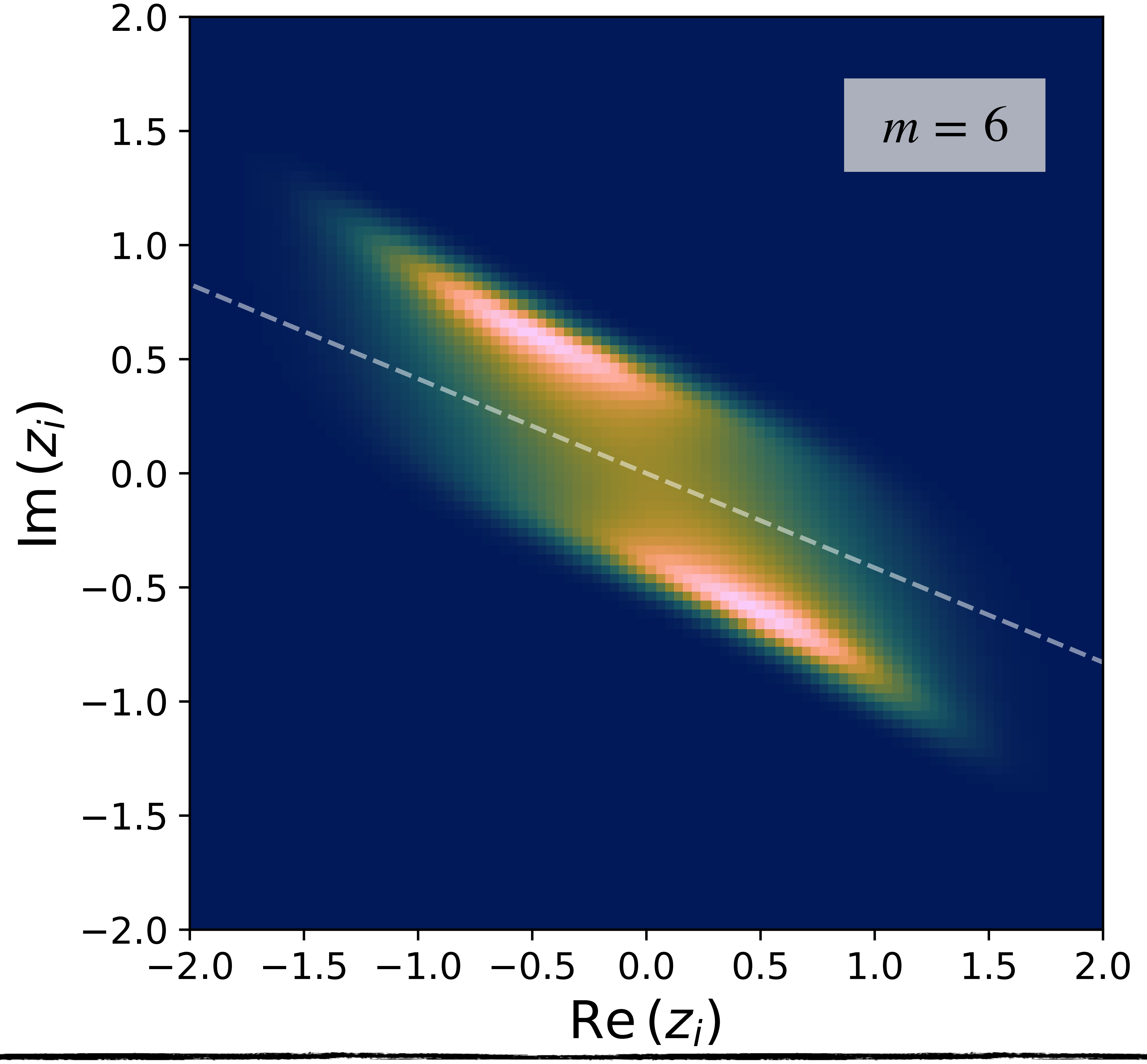

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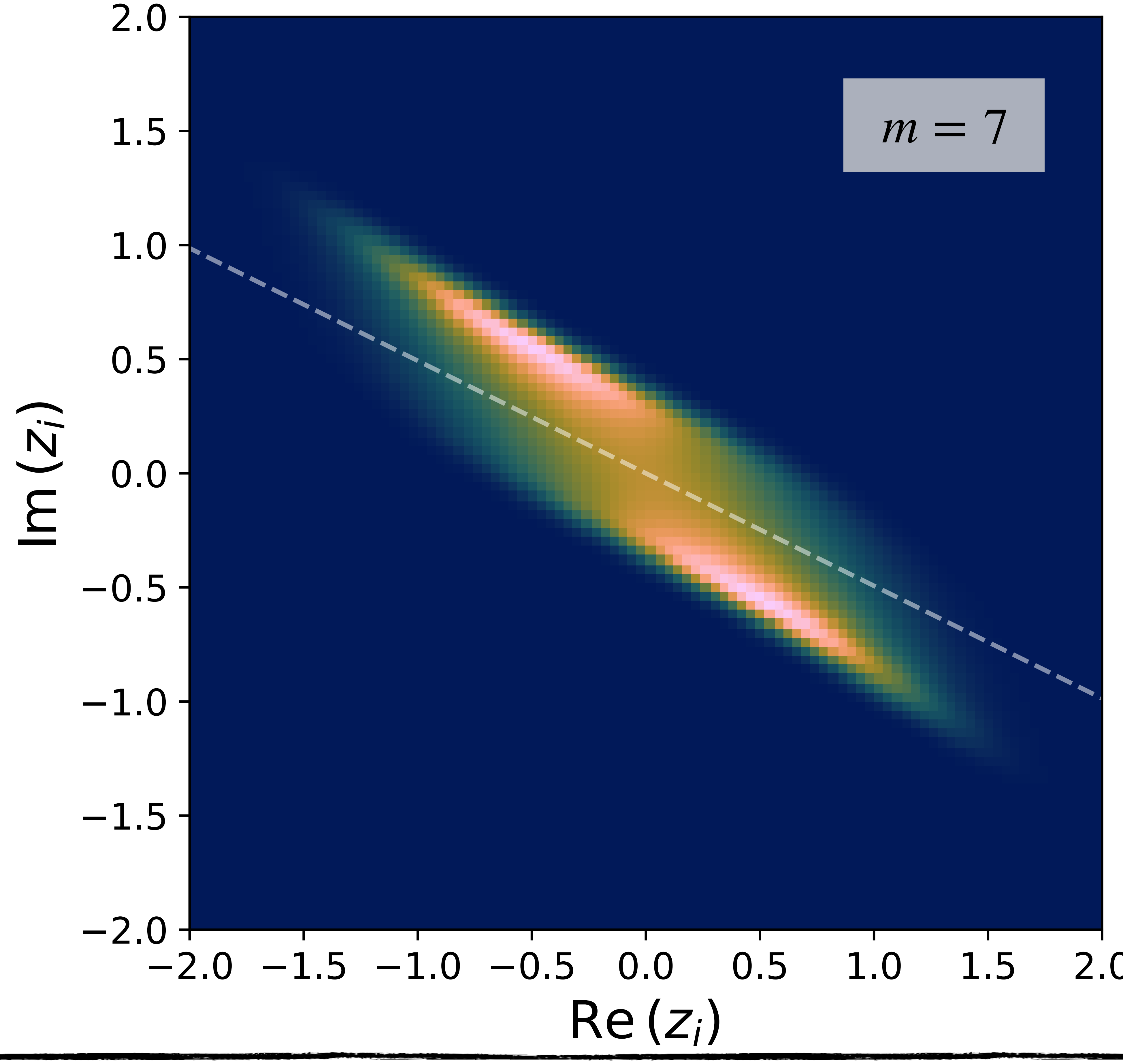

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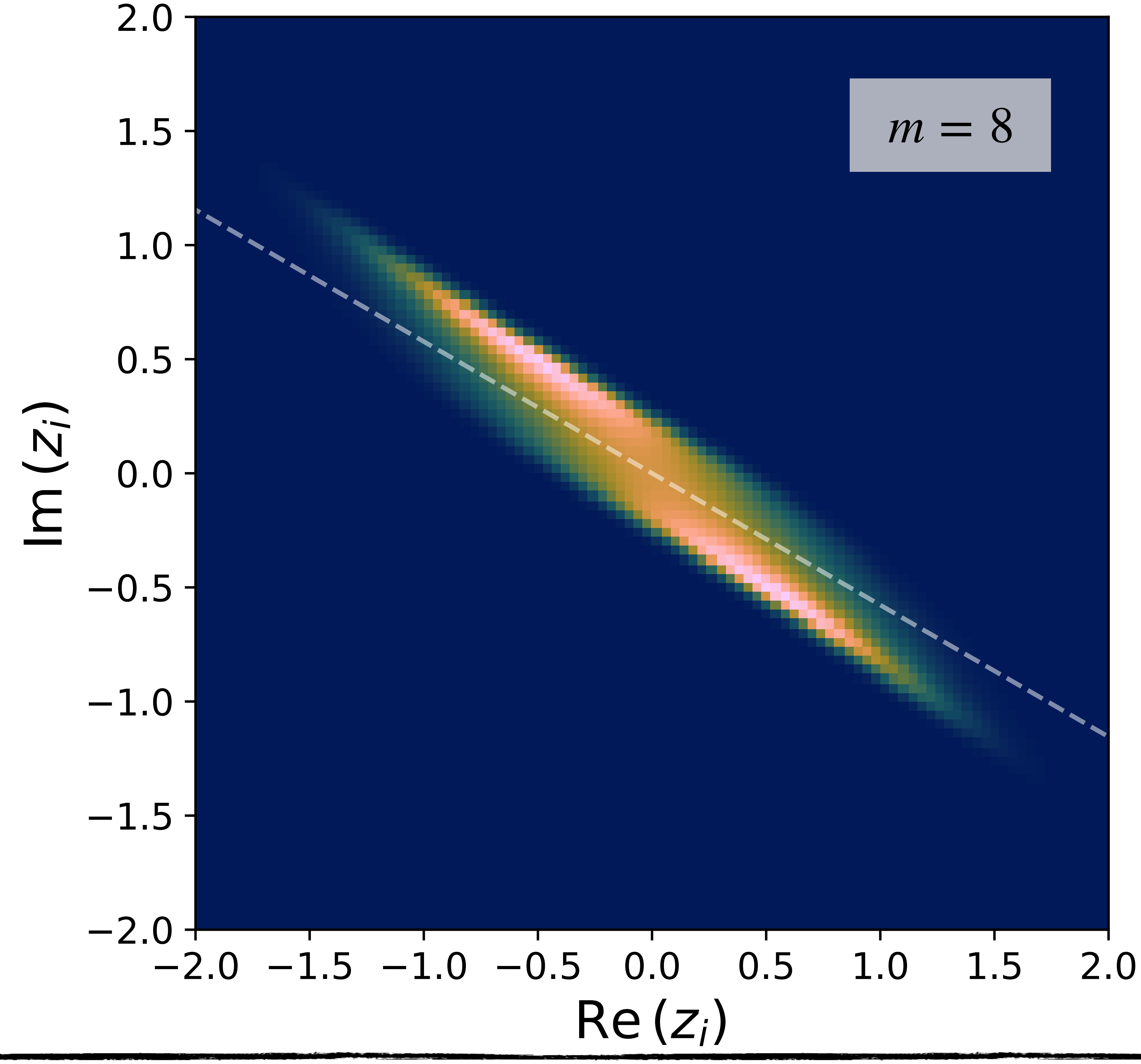

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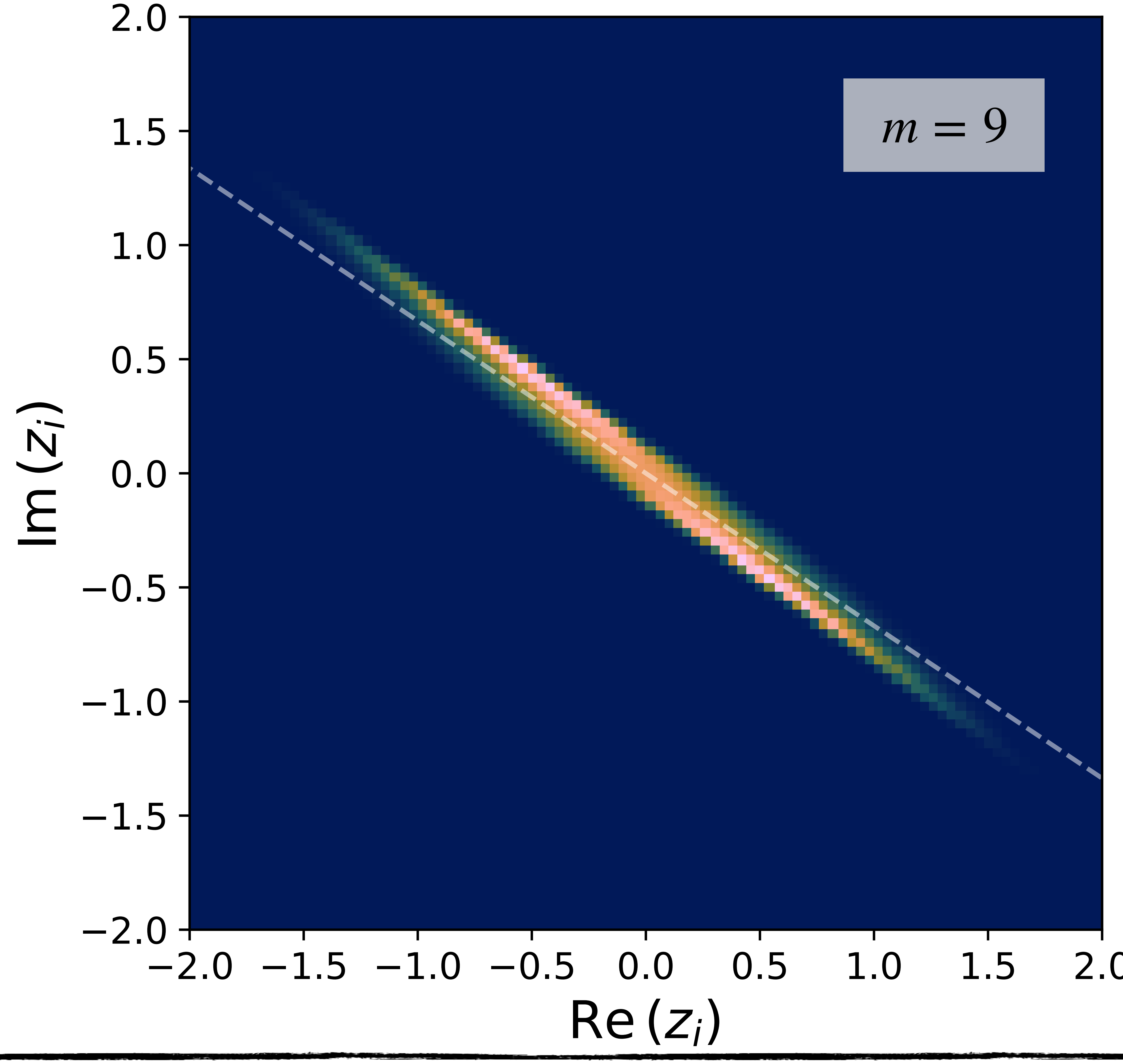

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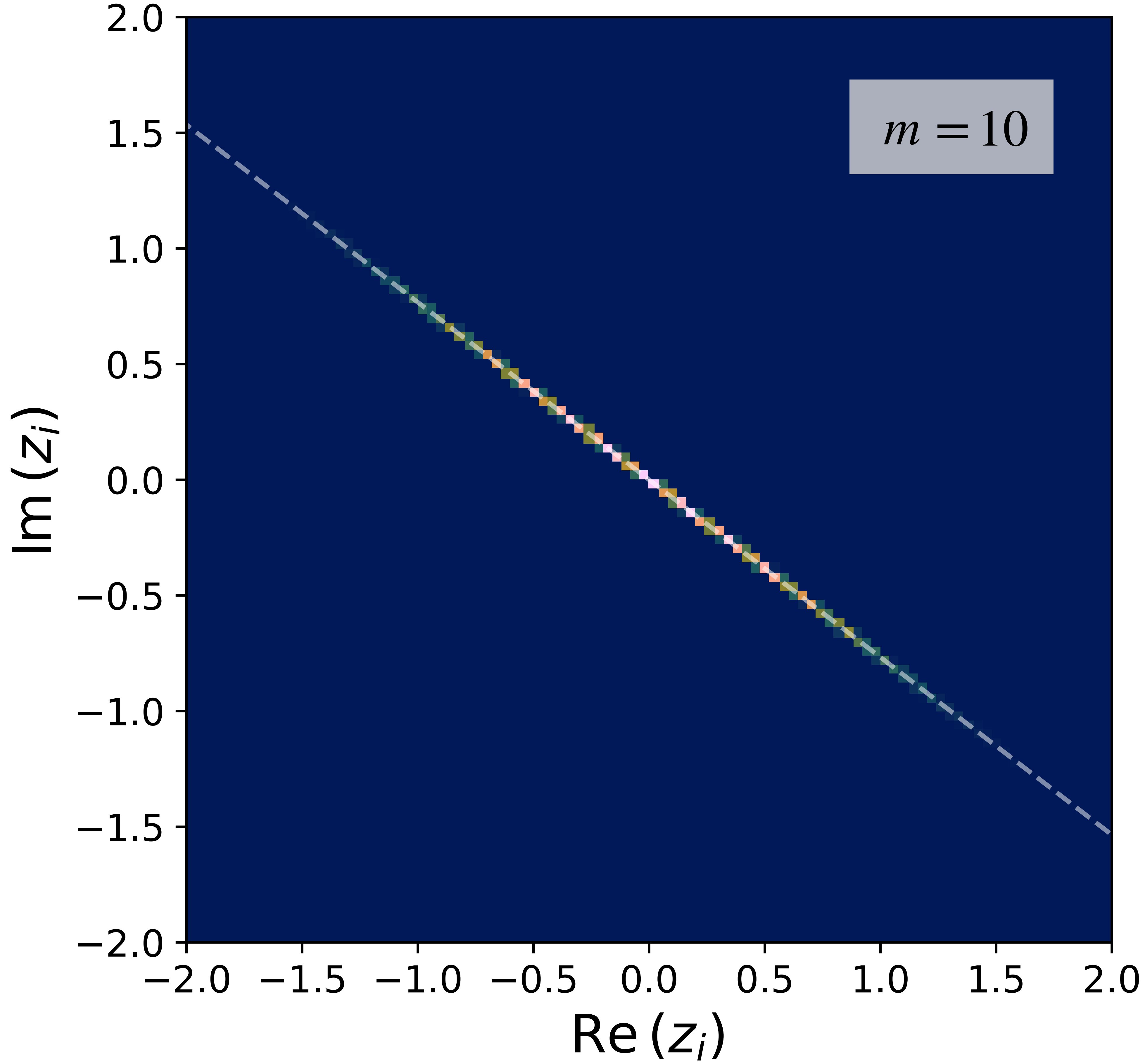

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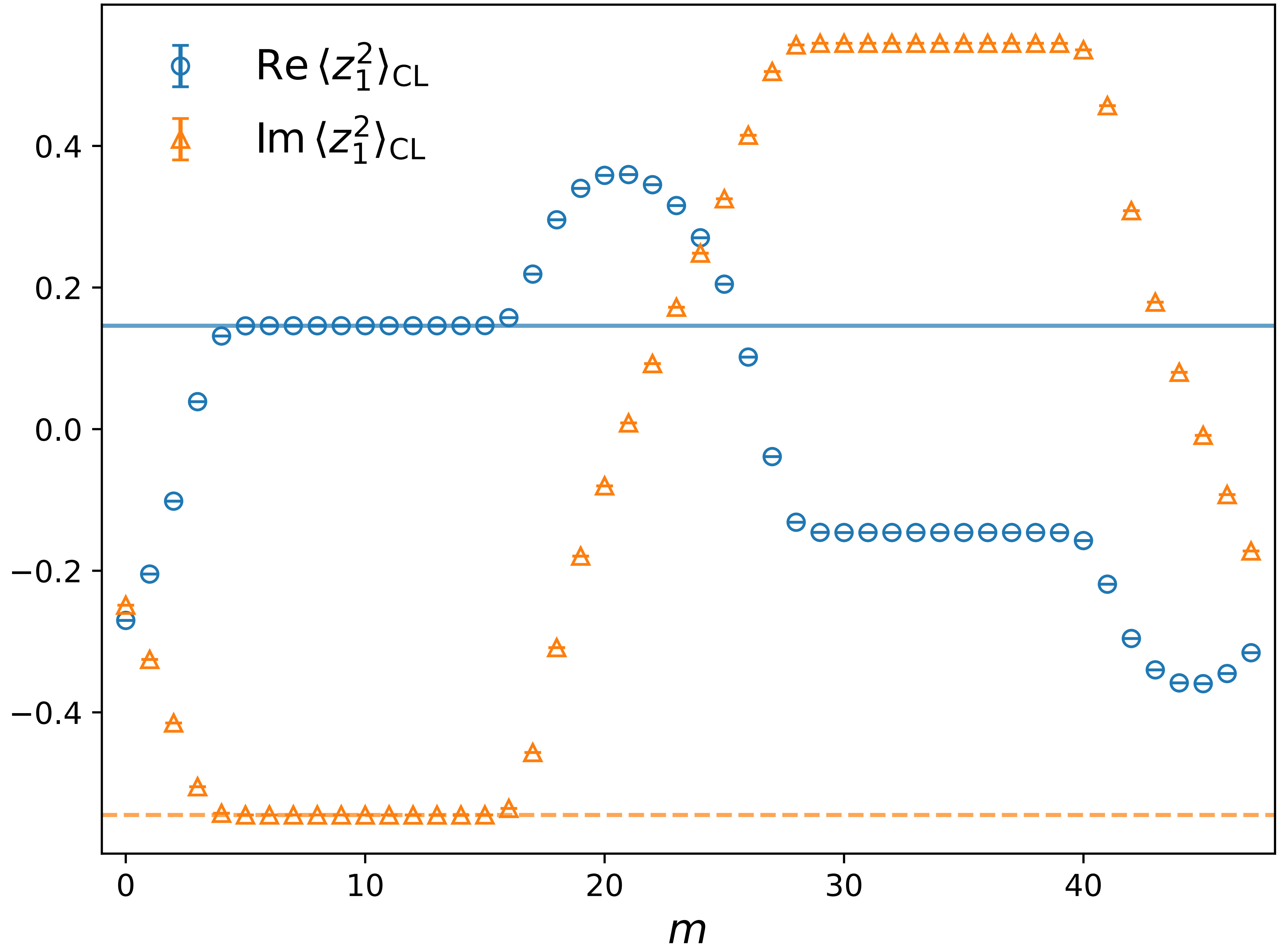

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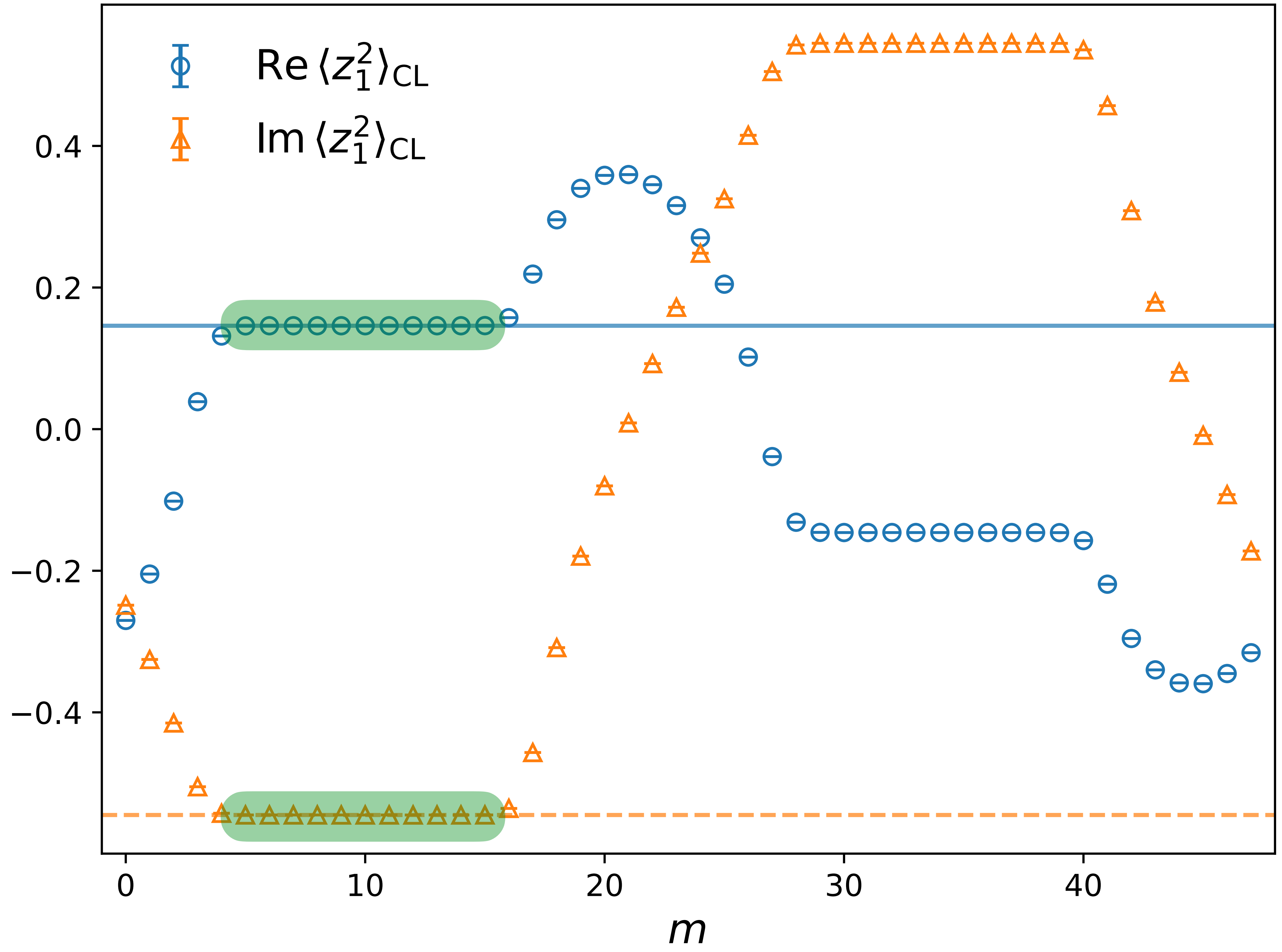

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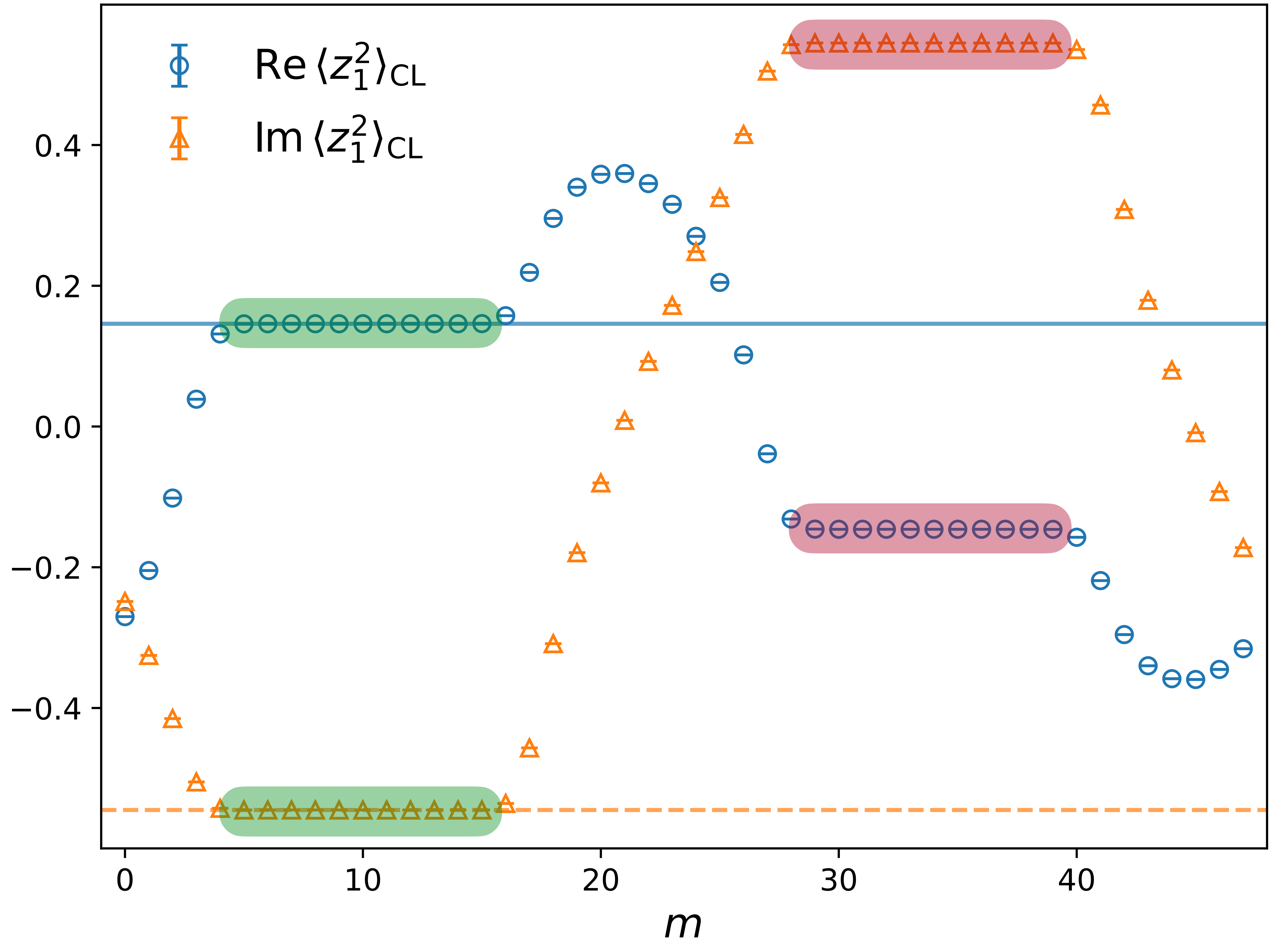

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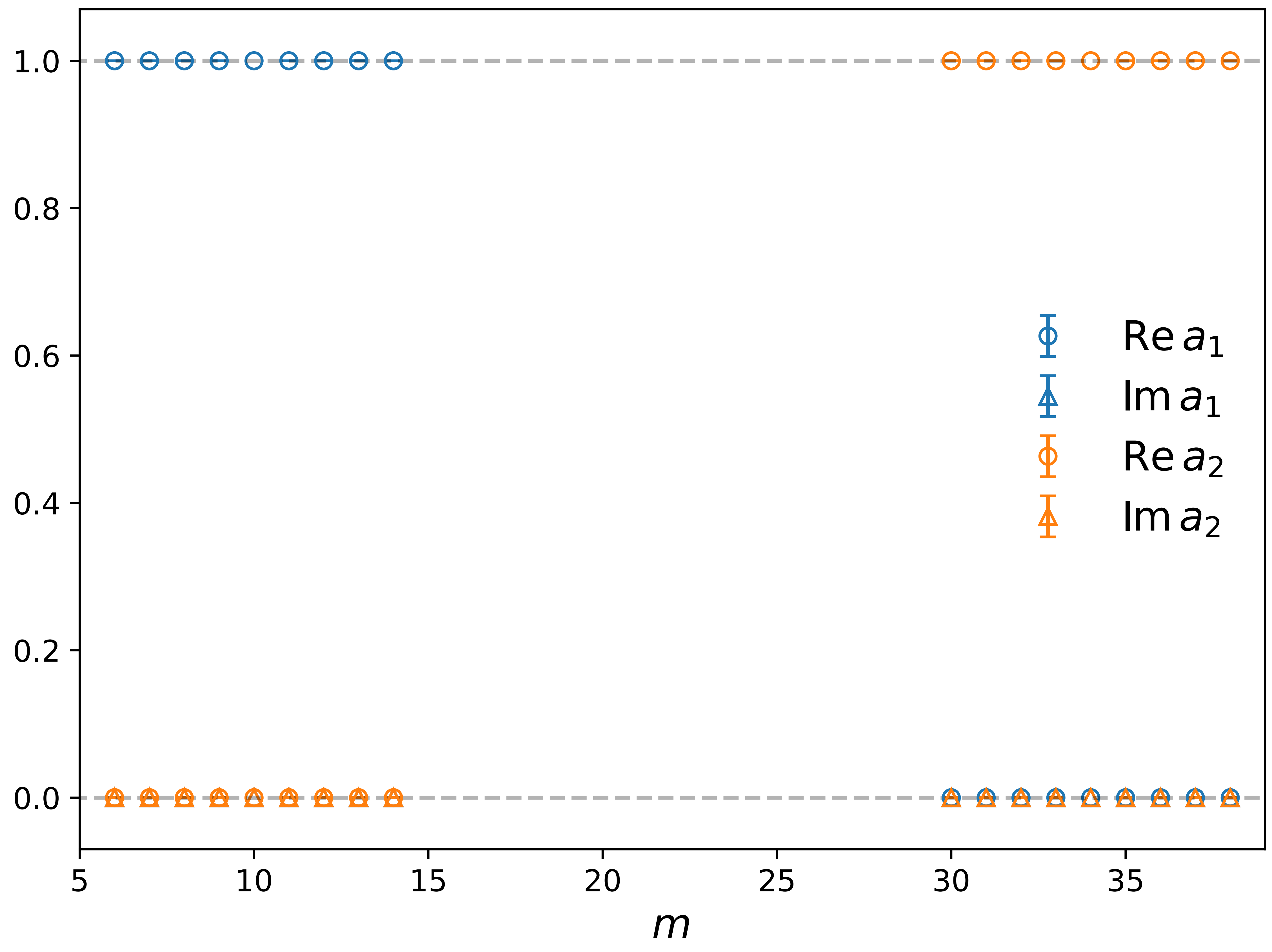

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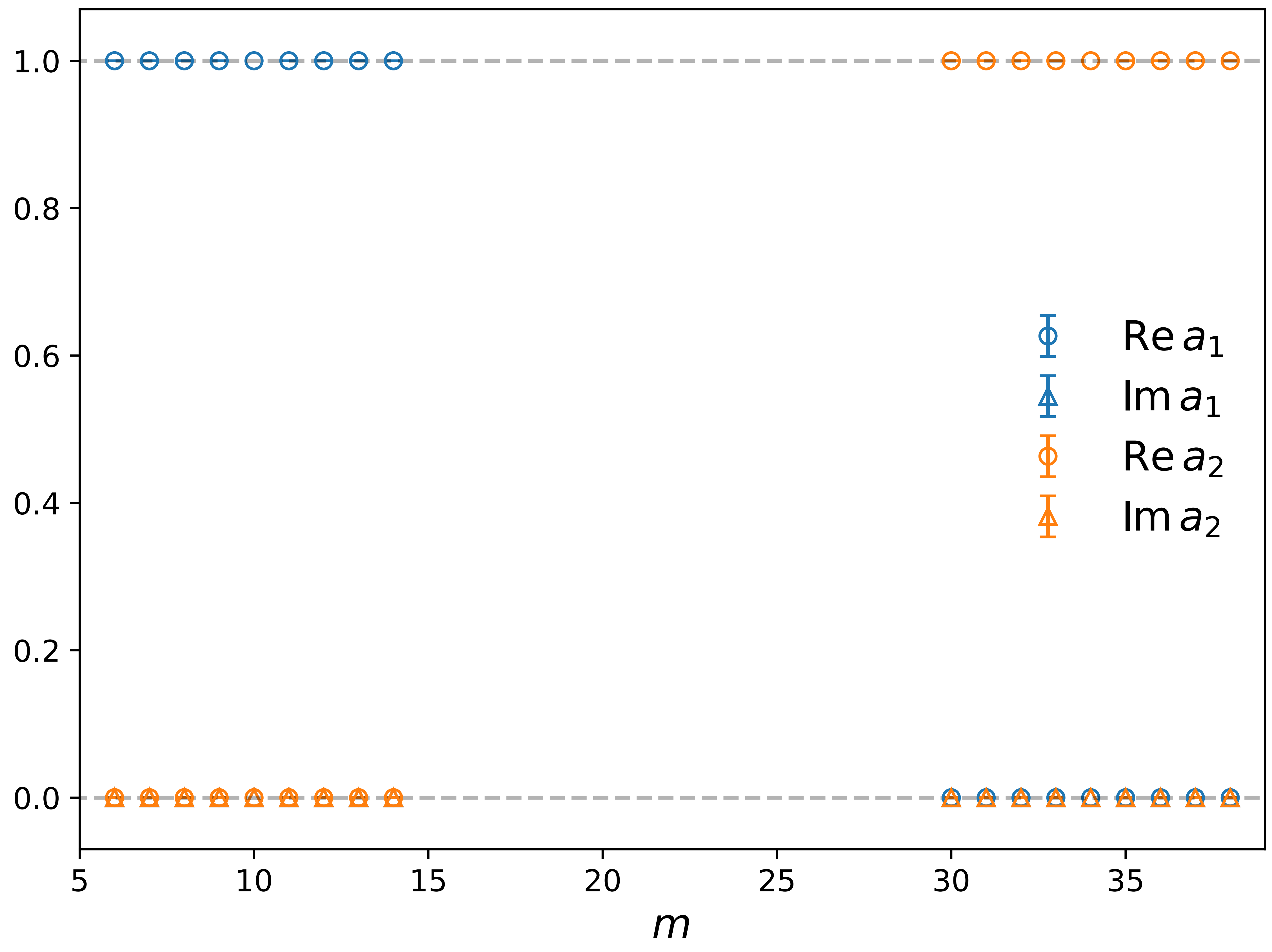

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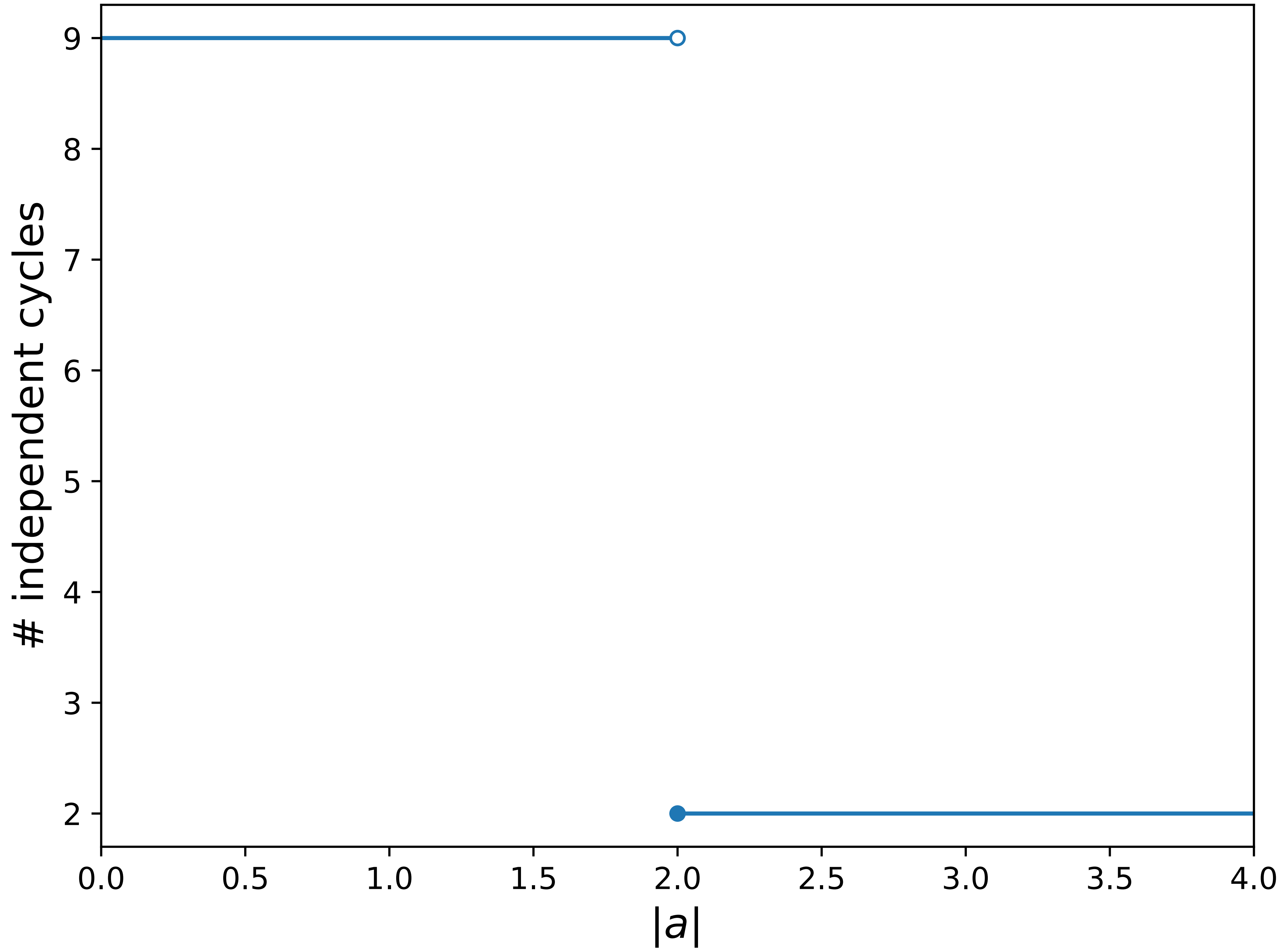

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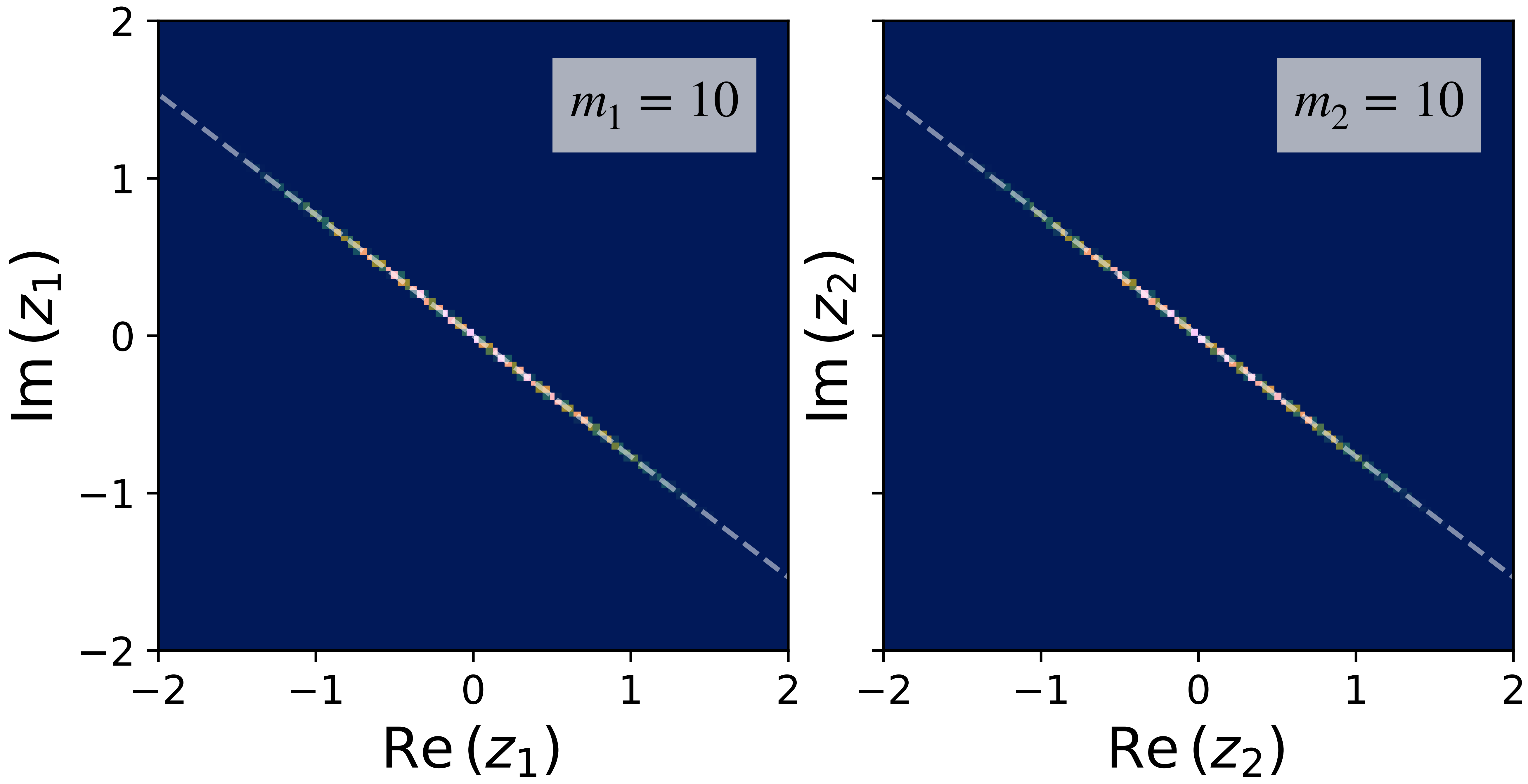

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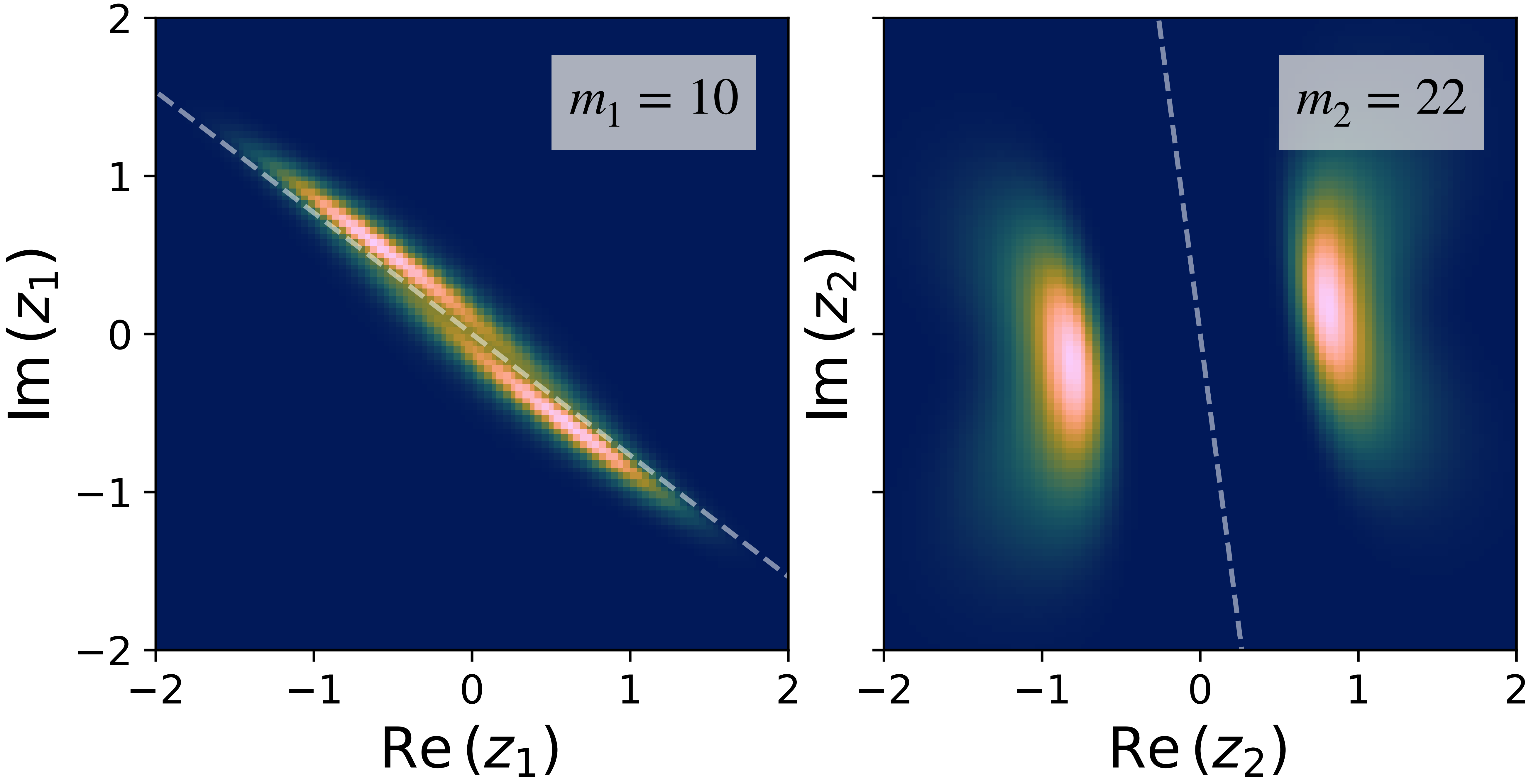

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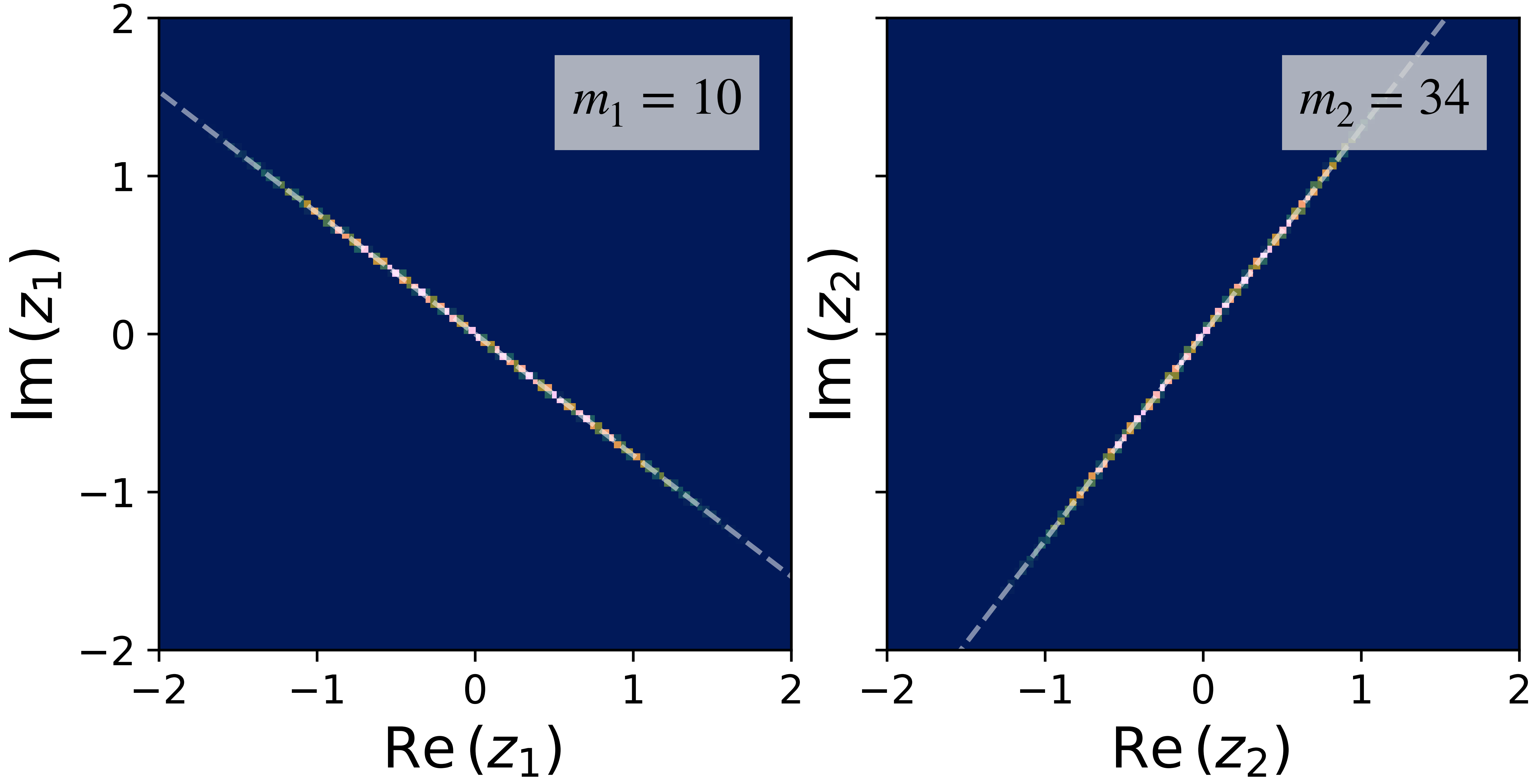

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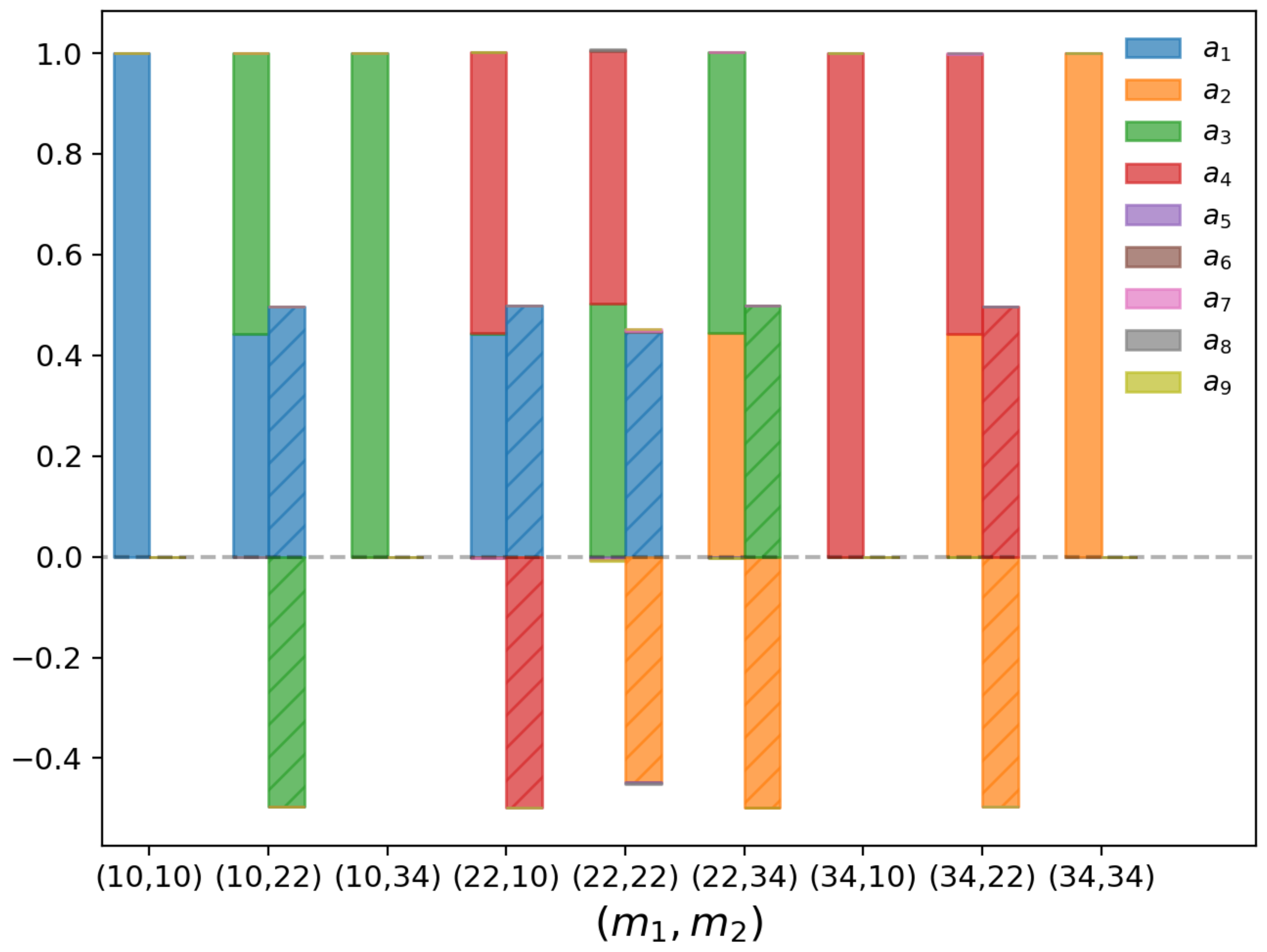

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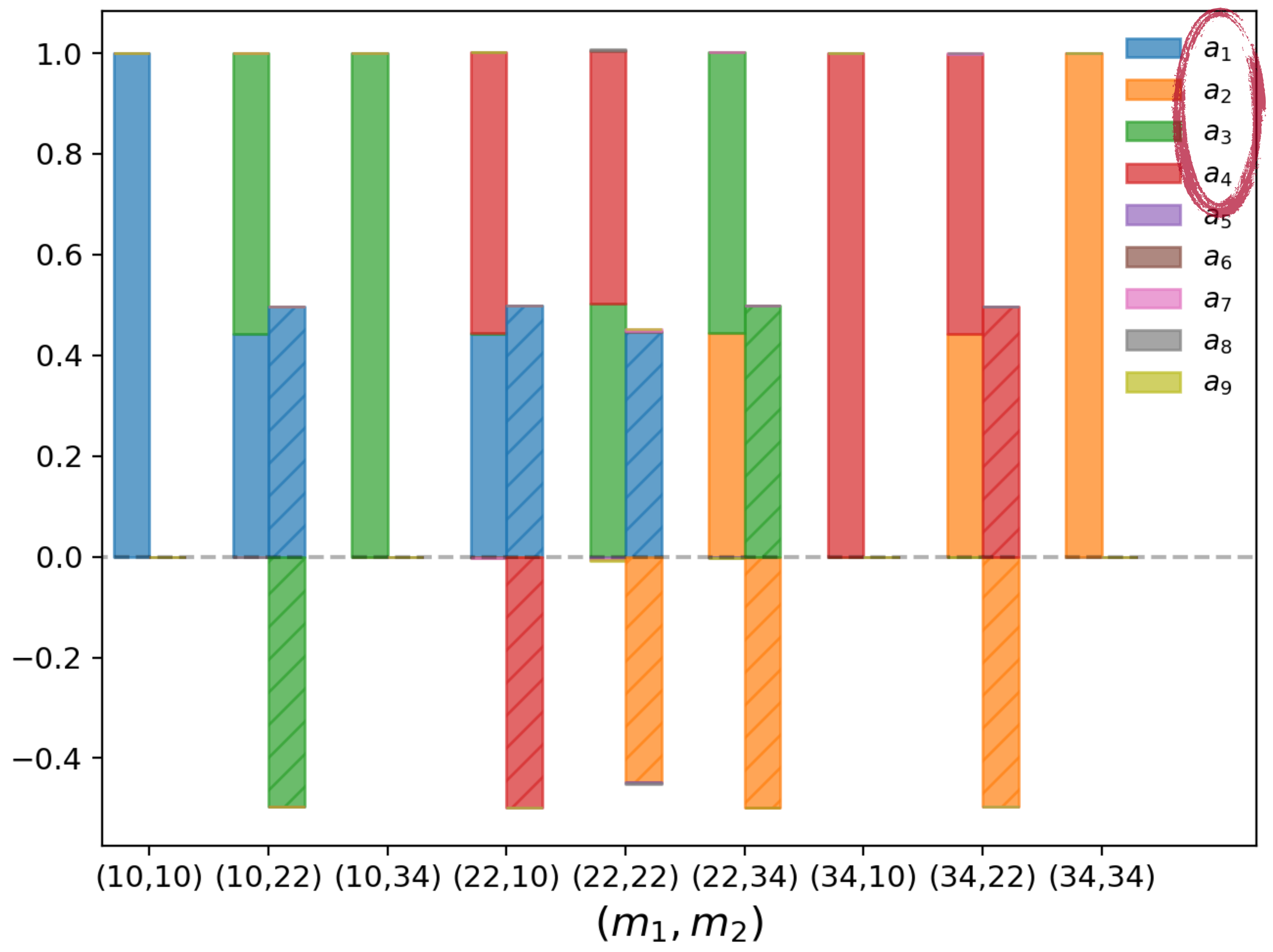

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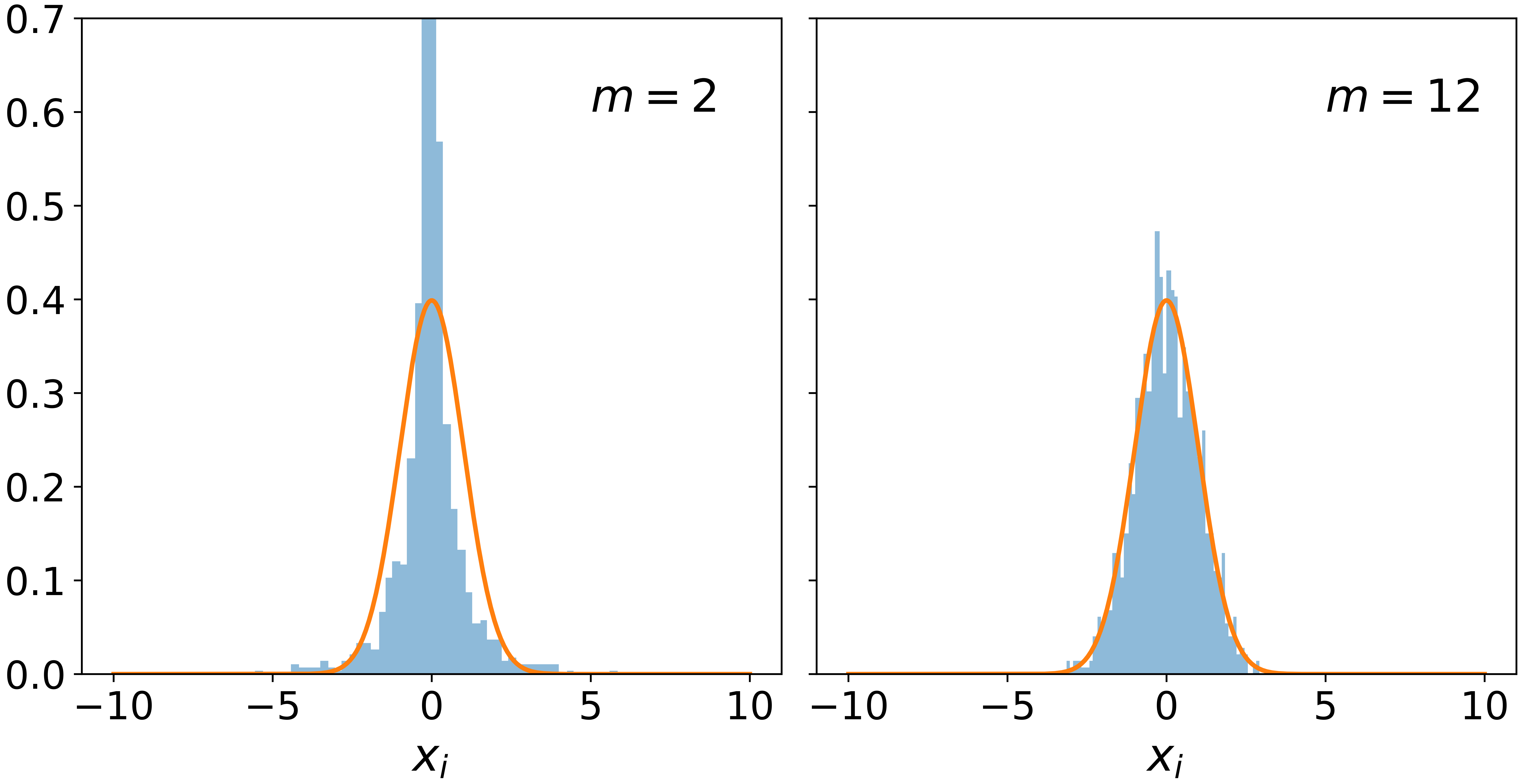
Backup

Assessing the goodness of fits



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