Prospects for Quantum Simulations of the Strong Coupling Expansion

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in collaboration with
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Based on arXiv: 2406.18721, 2308.03196, 2412.11677, 2303.01467, 2212.11328

Sign Workshop

Bern, Jan. 21, 2025

UNIVERSITÄT BIELEFELD
Faculty of Physics

CRC-TR 211
Strong-interaction matter under extreme conditions

SIGN25

Outline

Context:

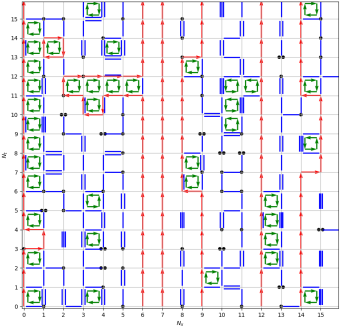
- Dual representation with staggered fermions: **effective theory of lattice QCD**
- Allow simulations at finite temperatures and densities
- Strong coupling expansion re-introduces sign problem, limiting range of validity
- Hamiltonian formulation based on dual representation may solve sign problem in quantum simulations

Content of the talk:

1. Monte Carlo simulations in dual representation including $\mathcal{O}(\beta^2)$
2. SC-LQCD on the Quantum Annealer
3. Hamiltonian approach to strong coupling lattice QCD and its generalizations
4. Towards quantum computing: strategies and prospects

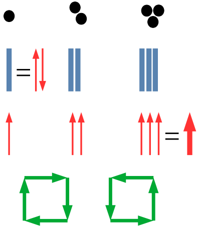
Dual Representation of Lattice QCD

- For "standard" QCD lattice action (staggered fermions, Wilson gauge action)
- But: **change order of integration:**
  - expand in $\beta = \frac{2N_c}{g^2}$
  - gauge links $\{U_\mu(x)\}$ first
  - afterwards the quarks
    $\rightarrow$ **no fermion determinant**
- "Dual" representation: via **color singlets!**
  - At $\beta = 0$: link states are **mesons** and **baryons** [Rossi, Wolff, NPB 248 (1984)]
  - At $\beta > 0$: color singlets may include gluons [Gagliardi, U, PRD 101 (2020)]



2-dim. example of configuration in terms of dual variables

- Dual degrees of freedom:
  - <u>monomers</u> m_x: quark on site, weights for quark mass
  - <u>dimers</u> $k_\mu(x)$: mesons hoppings (quark-anti-quark pairs)
  - <u>fermion world-lines</u> $f_\mu(x)$: quark hoppings, weights depend on chemical potential, form closed loops
  - <u>plaquette occupation numbers</u> n_p, $\bar{n}_p$: expansion order from gauge action

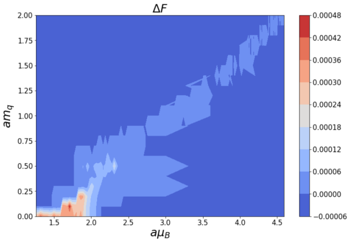


Sign Problem of Dual Representation

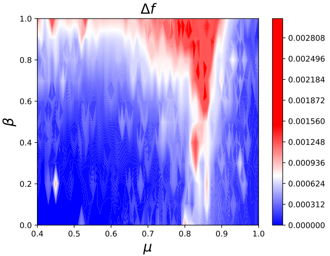
Sign of a configuration C due to **geometry** of fermionic loops:

$$\sigma(C) = \prod_{\ell} \sigma(\ell), \quad \sigma(\ell) = (-1)^{1+w(\ell)+N_-(\ell)} \prod_{\tilde{\ell}} \eta_\mu(x)$$

- -1 for each fermion loop, each backward hopping (spatial or temporal), each winding number (anti-periodic bc); product of staggered phases $\eta_\mu(x)$
- at strong coupling: mild residual sign problem $\Delta_f \simeq 10^{-5}$:
  - **baryons are heavy** (almost static)
  - color singlets closer to the **physical states** (hadrons)
  - no fluctuations from gauge fields (integrated out)!
- Sign problem in regime $\beta = \frac{6}{g^2} \lesssim 1$ **mild enough** to study full phase diagram



Δf at strong coupling as a function of m_q

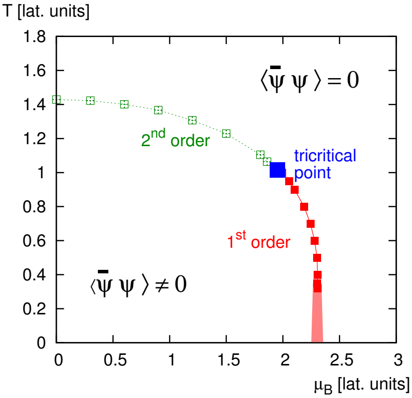


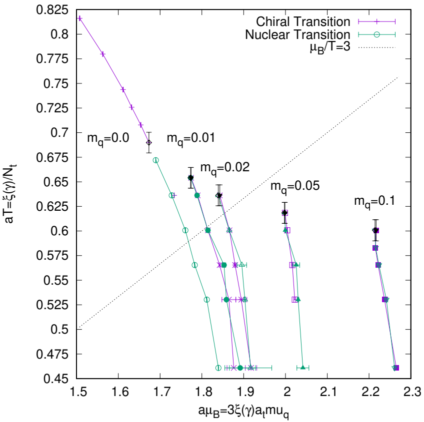
Δf in chiral limit as a function of β

The phase diagram in the strong coupling limit

Chiral and nuclear phase boundary obtained via Monte Carlo:

- at finite quark mass, the **tri-critical point** turns into Z_2 critical end point
- chiral and nuclear first order lines also match at finite quark mass





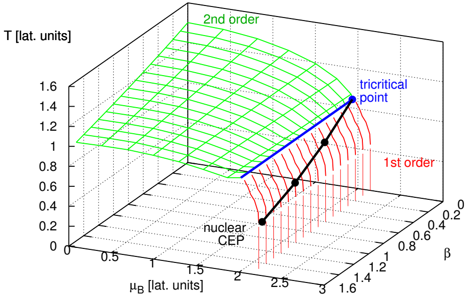
$\mu_q - T$ phase diagram in renormalized parameters [Kim & U. PoS Lattice 2016]

Goal: What does the Phase Diagram including β look like?

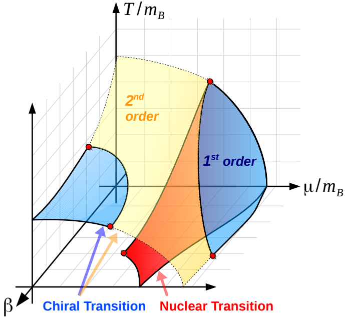
Phase Diagram in the Strong Coupling Regime and **Chiral Limit:**

- Via **reweighting** in β from $\beta = 0$: $\mathcal{O}(\beta)$ **corrections for SU(3)**

[Langelage, de Forcrand, Philipsen & U., *PRL* **113** (2014)]



[measured via Worm algorithm]



[one of serval possilbe scenarios]

Questions of interest:

- Do the **nuclear and chiral transition split?**
- Does the **tri-critical point** move to smaller or larger μ as β is increased?

Dualization of full lattice QCD

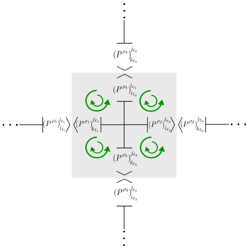
1. Combined **Taylor expansion** in the reduced gauge coupling $\tilde{\beta} \equiv \frac{\beta}{2N} = \frac{1}{g^2}$ and quark mass $\hat{m}_q$, giving rise to **dual variables**: n_p, $\bar{n}_p$, d_ℓ, $\bar{d}_\ell$ and m_x:
$$\mathcal{Z}(\beta, \mu_q, \hat{m}_q) = \sum_{\substack{\{n_p, \bar{n}_p\} \\ \{d_\ell, \bar{d}_\ell, m_x\}}} \prod_p \frac{\tilde{\beta}^{n_p+\bar{n}_p}}{n_p! \bar{n}_p!} \prod_\ell \frac{1}{d_\ell! \bar{d}_\ell!} \prod_x \frac{(2\hat{m}_q)^{m_x}}{m_x!} \mathcal{G}_{n_p, \bar{n}_p, d_\ell, \bar{d}_\ell, m_x}$$
2. Evaluate 1-link integrals in $\mathcal{G}$ in terms of **generalized Weingarten functions**
3. Decouple those integrals via a choice of orthogonal projectors:
4. Multi-indices ρ are new dual degrees of freedom: **decoupling operator indices**
5. Collect operators into a **local tensor** $T_x^{\rho^x_{-d} \cdots \rho^x_d}$ that depends on participating dual degrees of freedom $\mathcal{D}_x = \{m_x, d_{x,\pm\mu}, n_{x,\mu\nu}, \bar{n}_{x,\mu\nu}\}$
6. Final **dual partition function**:
$$\mathcal{Z}(\beta, \mu_q, \hat{m}_q) = \sum_{\substack{\{n_p, \bar{n}_p\} \\ \{k_\ell, f_\ell, m_x\}}} \sigma_f \sum_{\{\rho^x_{\pm\mu}\}} \prod_p \frac{\tilde{\beta}^{n_p+\bar{n}_p}}{n_p! \bar{n}_p!} \prod_{\ell=(x,\mu)} \frac{e^{\mu_q \delta_{\mu,0} f_{x,\mu}}}{k_\ell!(k_\ell + |f_\ell|)!} \prod_x \frac{(2\hat{m}_q)^{m_x}}{m_x!} T_x^{\rho^x_{-d}, \ldots, \rho^x_d}(\mathcal{D}_x)$$

[G. Gagliardi & W. U. PRD 101, (2020) 034509]

Truncation at $\mathcal{O}(\beta^2)$: allow for plaquette occupations $(n_p, \bar{n}_p) \in \{(1,1), (2,0), (0,2)\}$

Monte Carlo for TN-Representation via Vertex Model

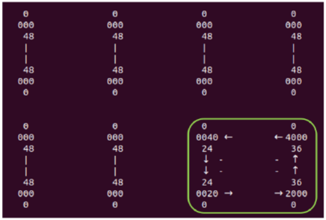
- Each tensor can be transformed into a vertex:
  DOI ρ_b^x per bond cast into integers
  vertex weight depends on directions



- Number of distinct vertices ($\mathrm{SU}(3)$, $d = 4$)

limit	$\mathcal{O}(\beta^0)$	$\mathcal{O}(\beta^1)$	$\mathcal{O}(\beta^2)$	$\mathcal{O}(\beta^3)$
all	221	3485	51125	681013
chiral	176	2960	**44672**	607792
quenched	1	1	25	137

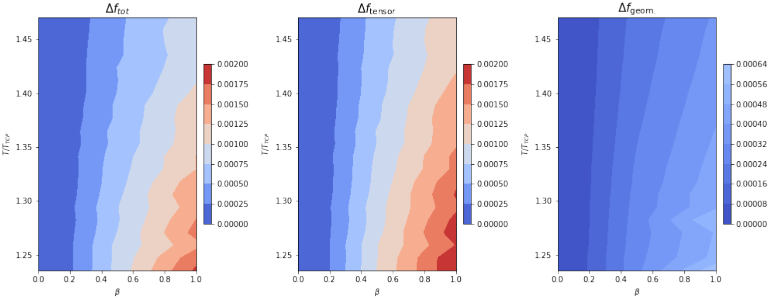
- compare to 8-vertex model for SC $\mathrm{U}(1)$, $d = 2$ [U. Wenger PRD 80 (2009) 071503]
- Some vertices have negative weight, but most configurations are positive
- Use **heatbath algorithm** for to modify vertices along closed contours; has been parallelized; Worm algorithm not yet applicable beyond $\mathcal{O}(\beta)$
  [P. Pattanaik & U. PoS Lattice (2023)]



- At $\mu_B = 0$: crosschecked with HMC
- Lattice Setup: 8^3x4, 12^3x4 and $16^3 \times 4$, SC, and $\mathcal{O}(\beta)$, $\mathcal{O}(\beta^2)$ for $\beta = [0.0, \ldots, 1.0]$ and at $T = 0.8, 0.85, 0.9, 0.95, 1.0$, all for chiral limit

Sign Problem of Vertex Formulation

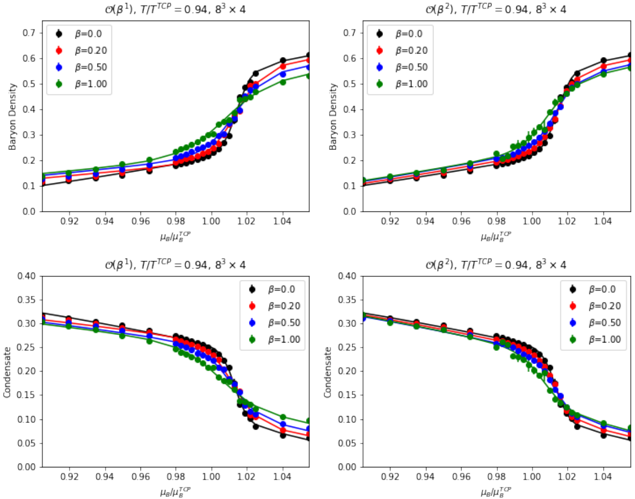
- In addition to the geometric fermionic sign $\sigma_{\mathrm{geom}}(C)$, also the tensors T_x^ρ/vertices v_x can be negative: $\sigma_{\mathrm{tot}}(C) = \sigma_{\mathrm{tensor}}(C)\sigma_{\mathrm{geom}}(C)$, $\sigma_{\mathrm{tensor}}(C) = \prod_x v_x^i$
- This particularly worsens the sign problem for $\mathcal{O}(\beta^2)$, where non-trivial decouling operator indices are present
- Additional tensor sign results in larger statistical errors for $\mathcal{O}(\beta^2)$ up to $\beta \lesssim 1$
- Finite size scaling for $\mathcal{O}(\beta^2)$ remains difficult



Decomposition: $\Delta f_{\mathrm{tot}} = \Delta f_{\mathrm{tensor}} + \Delta f_{\mathrm{geom.}}$, *data for* $\mu_B/\mu_B^{\mathrm{TCP}} = 0.75$, $8^3 \times 4$

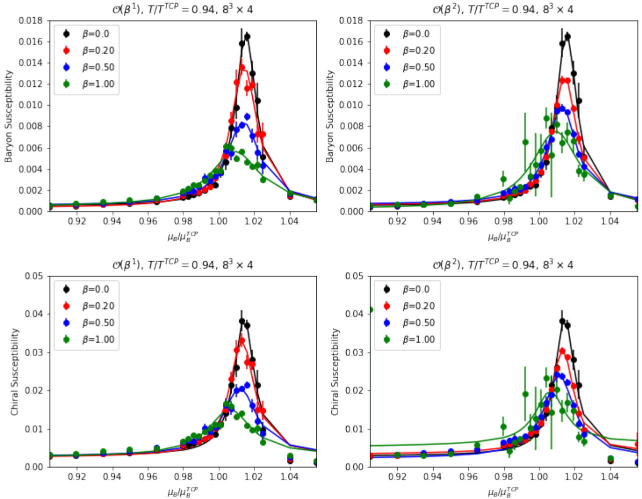
Results: Baryon Density and Chiral Condensate

- All results relative to the location of the strong coupling tricritical point: $T_{N_\tau=4}^{TCP} = 0.85$, $\mu_B{}_{N_\tau=4}^{TCP} = 1.99$



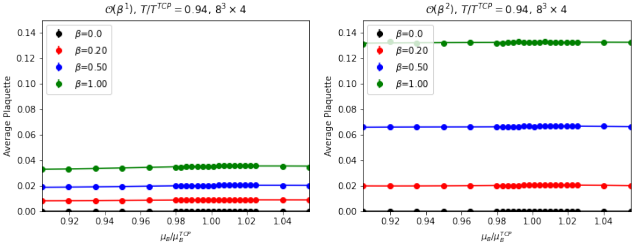
Results: Baryon Susceptibility, Chiral Susceptibiltiy

- All results relative to the location of the strong coupling tricritical point: $T^{TCP}_{N_\tau=4} = 0.85$, $\mu_B{}^{TCP}_{N_\tau=4} = 1.99$



Results: Average Plaquette

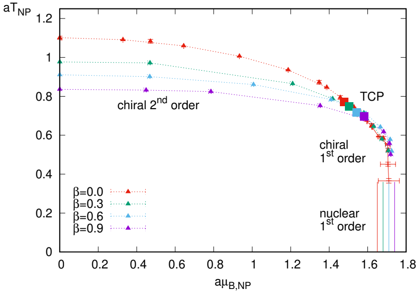
- All results relative to the location of the strong coupling tricritical point: $T^{TCP}_{N_\tau=4} = 0.85$, $\mu_B{}^{TCP}_{N_\tau=4} = 1.99$



- Average plaquette and its susceptibiltiy: no imprint of the chiral/nuclear transition

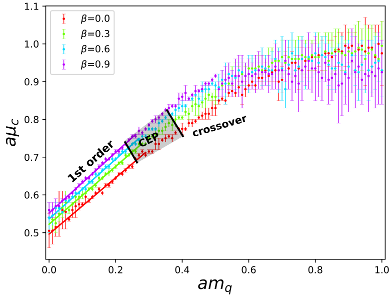
Phase Diagram in the Strong Coupling Regime $\beta > 0$

Phase Diagram in Chiral Limit:



- taking into account the β-dependent renormalization of aT and $a\mu_B$
- **tri-critical point** depends only slightly on β

m_q-Dependence of Nuclear Transition:



- **weak β-dependence of transition $\mu_c(m_q)$** at fixed $T \simeq 0$
- consistent with findings of other results (Meanfield, 3d-effective theory)
- location of nuclear CEP challenging

Prospects for the strong coupling expansion

- Higher orders $\mathcal{O}(\beta^n)$ with $n > 2$ may not be required for $\beta \lesssim 1$
- Continuum physics expected to emerge above $\beta \sim 2N_\mathrm{c}$
  $\rightarrow$ Large plaquette occupation numbers $(n_p, \bar{n}_p)$ cannot be neglegted.
- For $\beta > 1$: instead of Taylor expansion, a character expansion would have to be established (for staggered fermions: so far only a posteri identified):

$$\prod_p \exp \frac{\beta}{2N_c}(\mathrm{Tr}\, U_P + \mathrm{Tr}\, U_P^\dagger) = \prod_p \sum_\lambda u_\lambda(\beta/Nc)\chi_\lambda(U_p)$$

but with staggered fermions on top

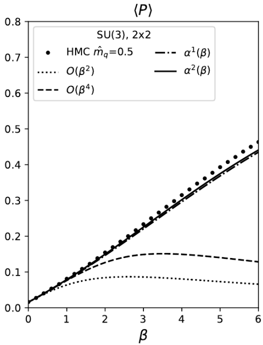
- Recursive relations for $\mathrm{SU}(3)$ via 3-step Lucas Polynomials, resulting in

$$\chi_\lambda(U_p) = \sum_{n_p, \bar{n}_p} c_\lambda(n_p, \bar{n}_p) Tr[U_p]^{n_p} Tr[U_p^\dagger]^{\bar{n}_p}$$

$$\alpha^r_{n_p, \bar{n}_p}(\beta) \equiv \sum_{\{\lambda\}_r} c_\lambda(n_p, \bar{n}_p) u_\lambda(\beta/3),$$

neglecting all $\lambda = (\lambda_1, \ldots)$ with $\lambda_1 > r$

- Sign problem would kick in at larger β



Taylor vs. character expansion

Quantum Simulations on a Quantum Annealer

- Quantum Annealer (D-Wave): array of qubits modeled by quantum spin glass H_{Ising}, with annealing process given by transverse external field:

$$H(s) = -A(t)\sum_i \sigma_x^i + B(t)H_{\mathrm{Ising}}$$

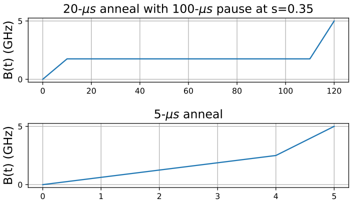
- Minimization: find optimal **binary solution vector** x

$$\chi^2 = x^T W x + p||Ax+b||^2 = x^T Q x + C,$$

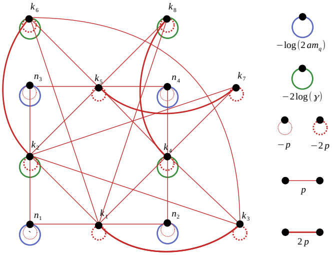
with QUBO (Quadradtic Unconstrained Binary Optimization) matrix:

$$Q = W + p\left(A^T A + \mathrm{diag}\left(2\mathrm{b}^{\mathrm{T}}\mathrm{A}\right)\right), \quad C = p\, b^T$$

with weight matrix W, constraint $(A$,$b)$ and the penality factor p



Annealing profile given by $B(t)$

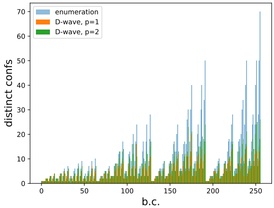


Problem Graph implementing QUBO matrix

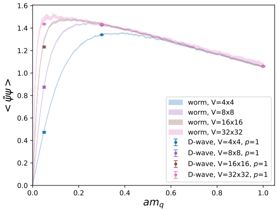
D-Wave Simulations for Strong Coupling LQCD

- For SC-LQCD: Demonstrated that it implements **importance sampling**

  [J. Kim, T. Luu, U. PRD 108 (2023)]
- Particularly useful at **low T**, where classical algorithms are expensive
- Large volumes addressed via histograms h obtained from D-Wave for 2×2 sub-lattices with fixed boundaries, iterating in parallel through sub-lattices
- Histograms approximate well true distribution for $p = 1$,
- Use hybrid stategy with Metropolis-Hastings $P_{\text{accept}} = e^{-S_{\text{new}} + S_{\text{old}}} \frac{h_{\text{old}}}{h_{\text{new}}}$ to improve on validity rate and acceptance rate



Number of histogram entries per boundary condition



Comparing results from D-Wave and Worm algorithm

[J. Kim, T. Luu, U. arXiv:2412.11677]

- Results at $\mathcal{O}(\beta)$ for $\mathrm{SU}(3)$ are underway, Metropolis-Hastings still feasible.

Gate-based Quantum Simulations for SCE: Preliminaries

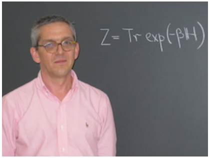
- Quantum annealers not flexible enough to address important aspects of a Hamiltonian
- Usual Hamiltonian approach to quantum simulations of LGTs: KS-Hamiltonian

$$\mathcal{H}_{\mathrm{KS}} = \frac{1}{2a}\left(g^2 \sum_{\ell\in\mathrm{Links}} E_\ell E_\ell + \frac{1}{g^2}\sum_{p\in\mathrm{Plaquettes}} \mathrm{Tr}[2\mathbb{1} - P_p + P_p^\dagger]\right)$$

  ($A_0 = 0$, operators in electric basis, Hilbert space must be truncated, different charge sectors)

- Hamiltonian derived from SCE at fixed order: Occupation number basis, everything discrete from the start
  $\rightarrow$ Hilbert space remains **finite-dimensional**
- No further truncation of Hilbert space needed
- Grassmann constraint and Gauss's law respected implicitly
- So far: no actual quantum simulations yet, emulation via `qiskit`

Euclidean Continuous Time Limit



Continuous time (CT) methods:

- make time direction continuous: $t \in [0, \beta]$, sample $Z(\beta)$ in terms of **decay probabilities** [Beard & Wiese, PRL 77 (1996) 5132] (for AFHM)

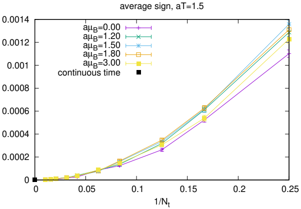
For Strong Coupling LQCD:

- Introduce **bare anisotropy** γ such that $\xi = \frac{a_s}{a_t} \neq 1$:
- Non-perturbative result: $\xi(\gamma) \approx \kappa\gamma^2 + \frac{\gamma^2}{1+\lambda\gamma^4}$, $\kappa = 0.781(1)$ [de Forcrand, Vairinhos, U., PRD 97 (2018)]
- Define the **continuous Euclidean time limit** (CT-limit):

$$N_\tau \to \infty, \quad \xi, \gamma \to \infty, \quad aT = \frac{\xi(\gamma)}{N_\tau} \simeq \kappa \mathcal{T}(\gamma, Nt), \quad \mathcal{T} = \frac{\gamma^2}{N_\tau} \quad \text{fixed}$$

- **only one parameter $\mathcal{T}$**
- non-perturbative factor κ directly in CT-limit!
- baryons are heavy: $\Delta f \simeq 10^{-5}$
- in continuous time limit $N_\tau \to \infty$: **baryons become static**

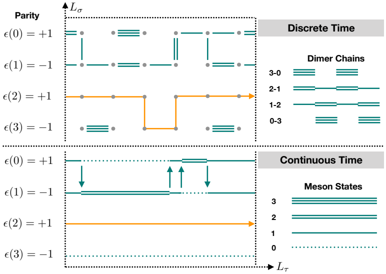
  $\Rightarrow$ finite density sign problem absent!



Δf vanishes for $N_\tau \to \infty$ ($a_t \to 0$)

From Meson Occupation Numbers to Hamiltonian ($N_f = 1$)

Correspondence between discrete and continuous time:



- alternating dimer chains (top) and **meson occupation numbers $\mathfrak{m}$** (bottom):
- multiple spatial dimers become **resolved in single spatial dimers**, oriented consistently due to even-odd ordering

- **conservation law:** for mesons connecting $\langle x, y \rangle$

$$\mathfrak{m}_x \mapsto \mathfrak{m}_x \pm 1 \quad \Leftrightarrow \quad \mathfrak{m}_y \mapsto \mathfrak{m}_y \mp 1$$

Derive Hamiltonian via **diagrammatic expansion** of $Z_{CT} = \lim_{\gamma, N_\tau \to \infty} Z_{N_\tau}(\gamma)$

- express the partition function as series in inverse temperature $\frac{1}{\mathcal{T}} = \frac{N_\tau}{\gamma^2}$:

$$Z_{\mathrm{CT}}(\mathcal{T}, \mu_{\mathcal{B}}) = \mathrm{Tr}_{\mathfrak{h}} \left[e^{(\hat{\mathcal{H}} + \hat{\mathcal{N}} \mu_{\mathcal{B}})/\mathcal{T}} \right], \; \hat{\mathcal{H}} = \frac{1}{2} \sum_{\langle \vec{x}, \vec{y} \rangle} \left(\hat{J}^+_{\vec{x}} \hat{J}^-_{\vec{x}} + \hat{J}^-_{\vec{x}} \hat{J}^+_{\vec{x}} \right), \; \hat{\mathcal{N}} = \sum_{\vec{x}} \hat{\omega}_x$$

Quantum Hamiltonian: Creation and Annihilation Operators

Creation $\hat{J}^+$ and **annihilation operators** $\hat{J}^- = (\hat{J}^+)^T$:

- contain the matrix elements $\langle \mathfrak{m}_1 | 1 | \mathfrak{m}_2 \rangle$ with $\hat{v}_{\mathbf{L}} = \langle 0|1|2\rangle = 1$, $\hat{v}_{\mathbf{T}} = \langle 1|1|1\rangle = \frac{\sqrt{3}}{4}$:

$$\hat{J}^+ = \left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & & \\ \hat{v}_{\mathbf{L}} & 0 & 0 & 0 & & \\ 0 & \hat{v}_{\mathbf{T}} & 0 & 0 & & \\ 0 & 0 & \hat{v}_{\mathbf{L}} & 0 & & \\ \hline & & & & 0 & 0 \\ & & & & 0 & 0 \end{array}\right), \quad \hat{\omega} = \left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ \hline & & & & 1 & 0 \\ & & & & 0 & -1 \end{array}\right)$$

- **local Hilbert space** per site: $|\mathfrak{h}\rangle = |\mathfrak{m}, \mathfrak{b}\rangle \in \mathbb{H}_{\mathfrak{h}} = [0, \pi, 2\pi, 3\pi; B^+, B^-]$
- block-diagonal structure due to vanishing commutator $[\hat{\mathcal{H}}, \hat{\mathcal{N}}] = 0$

Interpretation (generalized to $N_c \geq 3$):

- **pion current is conserved**
- Pauli saturation on the level of quarks as hadrons have fermionic substructure
  - $\rightarrow$ $|\mathfrak{m}\rangle$ bounded from above, $\mathfrak{m} \mapsto \mathfrak{s} = \mathfrak{m} - \frac{N_c}{2}$
  - $\rightarrow$ **particle-hole symmetry**, $N_c + 1$-dim. irrep of $\mathrm{SU}(2)$:

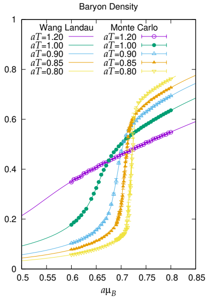
$$\hat{J}_1 = \frac{\sqrt{N_c}}{2}\left(\hat{J}^+ + \hat{J}^-\right), \quad \hat{J}_2 = \frac{\sqrt{N_c}}{2i}\left(\hat{J}^+ - \hat{J}^-\right), \quad \hat{J}_3 = \frac{N_c}{2}[\hat{J}^+, \hat{J}^-], \quad \hat{J}^2 = \frac{N_c(N_c+2)}{4}$$

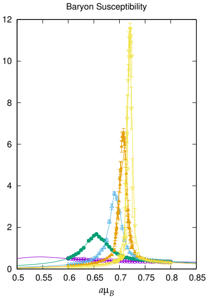
$$\hat{J}_3 \left|\frac{N_c}{2}, \mathfrak{s}\right\rangle = \mathfrak{s}\left|\frac{N_c}{2}, \mathfrak{s}\right\rangle, \qquad \hat{J}^2 \left|\frac{N_c}{2}\mathfrak{s}\right\rangle = \frac{N_c\,(N_c+2)}{4}\left|\frac{N_c}{2}, \mathfrak{s}\right\rangle, \qquad [\hat{J}^2, \hat{J}_3] = 0.$$

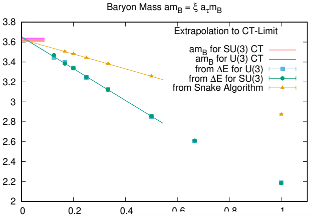
Quantum Monte Carlo for $N_f = 1$ Hamiltonian, Results

QMC is a continuous time **Worm algorithm**:

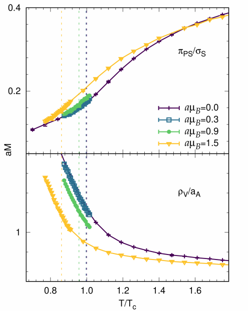
- Temporal locations uniformly distributed according to a **Poisson process:** $P(\Delta t) = e^{-\lambda \Delta t}$, $\Delta t \in [0, 1]$, λ is the "decay constant" for pion exchange







- Baryon mass could be unambiguously determined
- **meson pole masses** can be extracted from the Euclidean time correlators for various spin kernels
- pion mass M_π **sensitive to chiral transition**

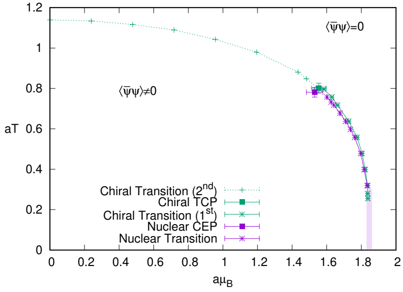


[Klegrewe, U. PRD 102 (2020)]

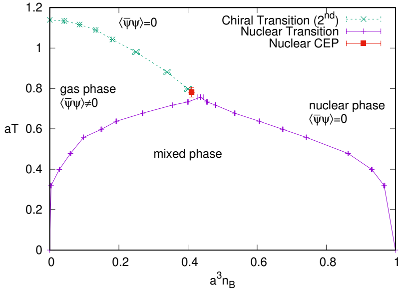
Phase Diagrams from $N_f = 1$ Hamiltonian LQCD (Chiral Limit)

From **Quantum Monte Carlo** / Density of States Method:

- obtain baryonic observables and phase diagrams to high precision



Grand-canonical phase diagram



Canonical phase diagram

[Klegrewe, U. PRD 102 (2020)]

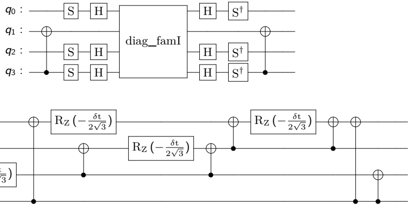
Gate-based Quantum Computing: Circuits

To identify the gate set required for quantum simulations ($N_\mathrm{f} = 1$):

- Qubitize the Hamiltonian $\mathcal{H}$ in mesonic and baryonic sector independently: $|\mathfrak{h}\rangle_x = |\mathfrak{m}\rangle_x \oplus |\mathfrak{b}\rangle_x$, 2 qubits for mesons, 1 control qubit for baryons
- Partition $\mathcal{H}_{meson}$ into mutually commuting even/odd parts and decompose each into **Pauli strings** (with c_i depending on $\hat{v}_\mathbf{L}$ and $\hat{v}_\mathbf{T}$):

$$\mathcal{H} = \sum_{x_e} \mathcal{H}_{x_e} + \sum_{x_o} \mathcal{H}_{x_o}, \qquad \mathcal{H}_{x_{e/o}} = \sum_i c_i P_i, \qquad P_i \in \{I, X, Y, Z\}^{\otimes(2+2)}$$

- Four families of Pauli strings, e.g. $\mathrm{Fam}_1 = \{(IX)_x(YY)_y, (IY)_x(YX)_y + x \leftrightarrow y\}$

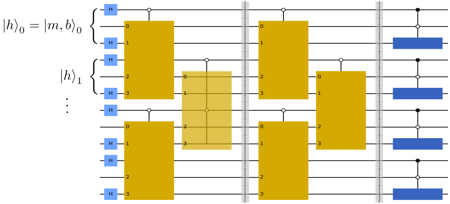


Top: four qubit quantum gate corresponding to Fam_1*.*
Bottom: diagonal gate of the above unitary, decomposed into elementary operations.

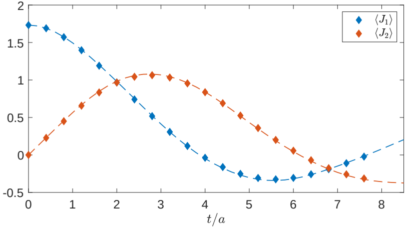
[M. Fromm, O. Philipsen, U., Ch. Winterowd, EPJ Quantum Technol. 11, 24 (2024)]

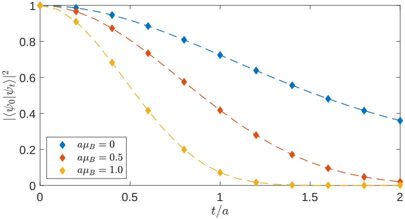
Gate-based Quantum Computing: Emulation

- Trotterized time evolution in 1+1 dimension, $L = 4$



Mesonic Trotter steps (yellow) and baryonic evolution (blue)





Mesonic $\hat{J}_1$, $\hat{J}_2$, exact results versus simulated with Trotter step size $\delta_t = 0.4$

Wave function overlap for various μ_B with Trotter step size $\delta_t = 0.2$

[M. Fromm, O. Philipsen, U., Ch. Winterowd, EPJ Quantum Technol. 11, 24 (2024)]

Partition Function: $N_{\mathrm{f}} > 1$

Local Hilbert space $\mathbb{H}_{\mathfrak{h}}$ via canonical sectors $B \in [-N_{\mathrm{f}}, -N_{\mathrm{f}}+1, \ldots, N_{\mathrm{f}}]$:

- $\mathbb{H}_{\mathfrak{h}}$ quickly grows with N_{f}, general formula: [U., Lattice 2014]:

$$Z_{\mathrm{stat}}(N_{\mathrm{c}}, N_{\mathrm{f}}) \sum_{B=-Nf}^{N_{\mathrm{f}}} \prod_{a=0}^{N_{\mathrm{c}}} \frac{a!(2N_{\mathrm{f}}+a)!}{(N_{\mathrm{f}}+a+B)!(Nf+a-B)!} e^{B\mu_B/T}$$

- coefficient encodes number of **hadronic states** $\mathfrak{h}$, for gauge group $\mathrm{SU}(3)$:
  - $N_{\mathrm{f}} = 2$: $d = [1, 20, 50, 20, 1] = 92$
  - $N_{\mathrm{f}} = 3$: $d = [1, 56, 490, 980, 490, 56, 1] = 2074$

Every state of the local Hilbert space can be described by a **set of charges**:

- $B \in \{-N_{\mathrm{f}}, \ldots N_{\mathrm{f}}\}$, $I \in \{-N_{\mathrm{c}}, \ldots N_{\mathrm{c}}\}$, $U, D \in \{0, \ldots N_{\mathrm{c}}\}$ for $N_{\mathrm{f}} = 2$
- $K_1, K_2 \in \{-N_{\mathrm{c}}, \ldots N_{\mathrm{c}}\}$, $S \in \{0, \ldots N_{\mathrm{c}}\}$ additionally for $N_{\mathrm{f}} = 3$

Quantum Hamiltonian for $N_{\mathrm{f}} = 2$

The Hamiltonian has ${N_{\mathrm{f}}}^2$ contributions, one for each pseudoscalar meson:

- Partition function for $N_{\mathrm{f}} = 2$:

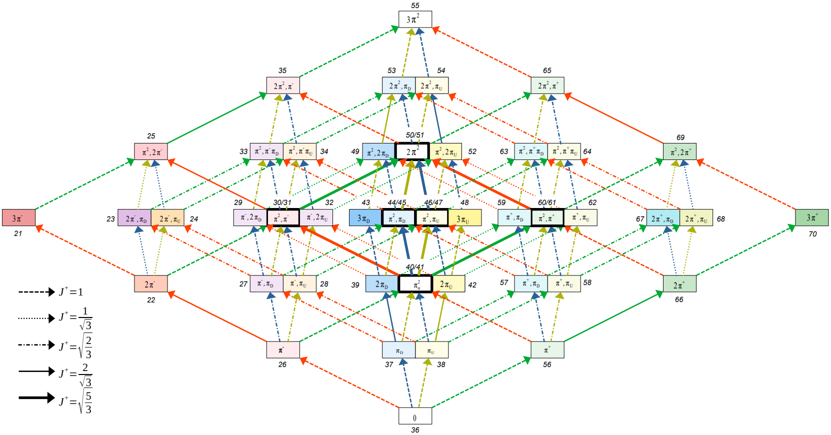
$$Z_{\mathrm{CT}}(\mathcal{T}, \mu_{\mathcal{B}}, \mu_{\mathcal{I}}) = \mathrm{Tr}_{\mathfrak{h}} \left[e^{(\hat{\mathcal{H}} + \hat{\mathcal{N}}_B \mu_{\mathcal{B}} + \hat{\mathcal{N}}_I \mu_{\mathcal{I}})/\mathcal{T}} \right] \qquad \mathfrak{h} \in \mathbb{H}_{\mathfrak{h}}$$

$$\hat{\mathcal{H}} = \frac{1}{2} \sum_{\langle \vec{x}, \vec{y} \rangle} \sum_{Q_i \in \{\pi^+, \pi^-, \pi_U, \pi_D\}} \left(\hat{J}^+_{Q_i, \vec{x}} \hat{J}^-_{Q_i, \vec{y}} + \hat{J}^-_{Q_i, \vec{x}} \hat{J}^+_{Q_i, \vec{y}} \right)$$

- For the transition $\mathfrak{h}_1 \mapsto \mathfrak{h}_2$, the **matrix elements** $\langle \mathfrak{h}_1 | Q_i | \mathfrak{h}_2 \rangle$ of $\hat{J}^\pm_{Q_i}$ are determined from Grassmann integration and diagonalization
- Only non-zero which are consistent with current conservation of all Q_i,
  and turn out to be positive!
  (Note: that requires the Euclidean continuous time limit, while for finite N_τ there remain negative weights for $N_{\mathrm{f}} > 1$)
- Again, $[\mathcal{H}, \hat{\mathcal{N}}_B] = 0$, however $[\mathcal{H}, \hat{\mathcal{N}}_I] \neq 0$

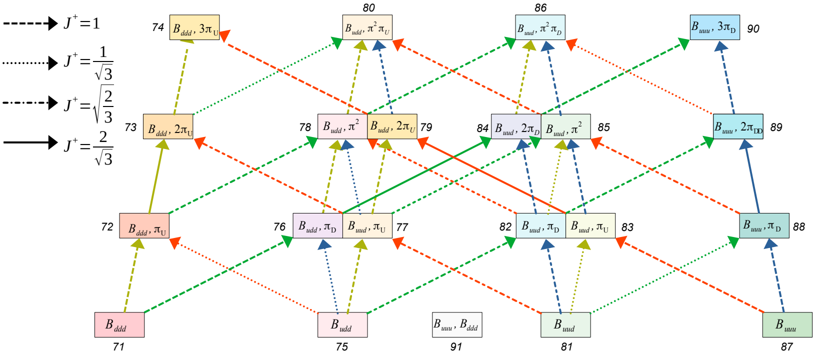
Hadronic States for $N_\mathrm{f} = 2$

Transitions according to J^+, quantum numbers $B = 0$, $I = -3, \ldots 3$, $m = 0 \ldots 6$:



Hadronic States for $N_\mathrm{f} = 2$

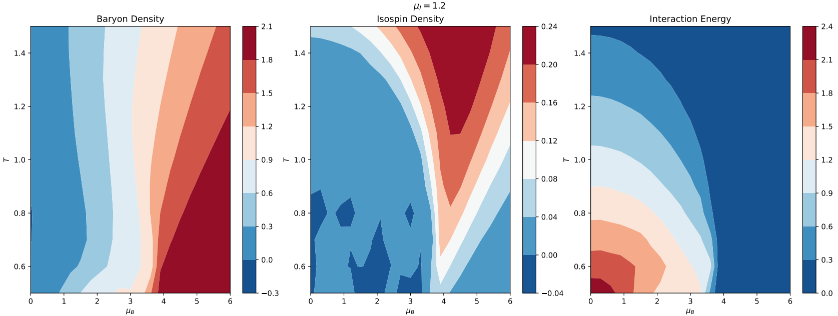
Transitions according to J^+, quantum numbers $B = 1$, $I = -3/2, \ldots 3/2$, $m = 0 \ldots 3$
and $B = 2$, $I = 0$, $m = 0$



Preliminary Results for $N_\mathrm{f} = 2$ CT-Worm simulations

$N_\mathrm{f} = 2$ allows to study SC-LQCD at both **finite baryon and isospin density:**

- enlarged $\mu_B - \mu_I - T$ phase diagram accessible
- Mean-field predicts: both the nuclear and chiral transition split up into two transitions as $\mu_I > 0$, first indications from MC simulations: with increasing μ_I, the plateau between the two transitions increases (lower T required)



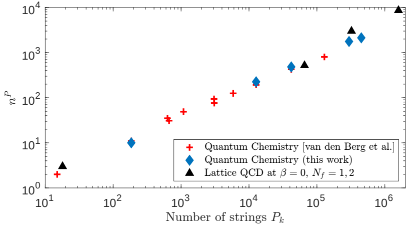
Contour plots of nuclear, isospin and chiral observables in the μ_B-T plane for $\mu_I = 1.2$

New physics expected:

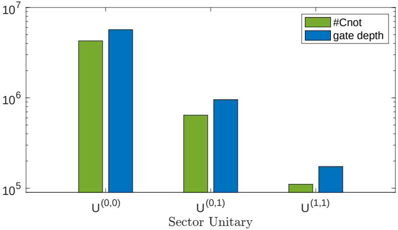
- single baryons can coexist with pions $\rightarrow$ **pion exchange** between nucleons
- **pion condensation** competes with nuclear phase

Prospects for Quantum Simulations for $N_\mathrm{f} = 2$

- Hilbert space 92-dimensional, 9 qubits required
- Too complex to determine minimal set of commuting families of Pauli strings
- With qubits: **high gate depth** for single Trotter step
- Time evolution of baryonic sectors mix (meson exchange for $B = 0, \pm 1$)

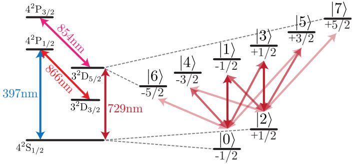


Number of partitions as a function of the number of Pauli strings



Number of entangled CNOT gates, gate depth in baryonic sectors (B_x, B_y)

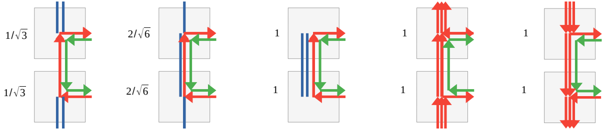
- Less complexity: qudits (d-level systems): each U an elementary operation
- With trapped ions, e.g. ^{40}Ca: 8 levels
- Entangled gate count: nn, $N_\mathrm{f} = 2$, $d > 2$: $\mathcal{O}(10^6)$ qubits $\mapsto \mathcal{O}(10^2)$ qudits



[M. Ringbauer et al., Nature Physics 18, 1053 (2022)]

Prospects for Quantum Simulations for $\beta > 0$

- At finite temperature: spatial plaquettes are suppressed over temporal plaquettes
- In the continuous time limit: only **temporal plaquettes** survive



- Temporal plaquettes are of same order as meson exchange, but also allows to **couple to baryons!** ($\hat{J}^{\pm}$ still block-diagonal for $N_{\mathrm{f}} = 1$):

$$J^{+} = \left(\begin{array}{cccc|cc} v_{M,0} & 0 & 0 & 0 & 0 & 0 \\ v_L & v_{M,1} & 0 & 0 & 0 & 0 \\ 0 & v_T & v_{M,2} & 0 & 0 & 0 \\ 1 & 0 & v_L & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & v_{\bar{B}} & 0 \\ 0 & 0 & 0 & 0 & 0 & v_B \end{array}\right), \quad v_{M,0} = \frac{1}{3g}, \quad v_{M,1} = \frac{2}{3g}, \quad v_{M,2} = \frac{1}{g}, \quad v_{\bar{B}} = \frac{1}{g}, \quad v_B = \frac{1}{g}.$$

- General SCE: Enlarged but **still finite Hilbert space**:
  (anti-) quarks $d_l, \bar{d}_l = 0, \ldots N_{\mathrm{c}}$ vary independently: 16-dimensional for $N_{\mathrm{c}} = 3$
- Still NN interaction $J^{+}J^{-}$, requiring 4×4 qubit coupling
- Caveat: Spatial plaquettes would enlarge Hilbert space further, no longer NN

Conclusions

Summary:

- Dual representation that is in principle not truncated in β established, caveat: it re-introduces the sign problem gradually with β
- TCP remains invariant when comparing $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta^2)$
- Nuclear transition at $\mathcal{O}(\beta)$ has small β-dependence
- Hamiltonian formulation based on CT-limit **sign problem-free** for $N_\mathrm{f} = 1, 2, 3$
- Matrix elements for the **creation and annihilation operators** $\hat{\mathcal{J}}^{\pm}$ have now been determined for $N_\mathrm{f} = 2$ and for $\mathcal{O}(\beta)$
- First exploratory $N_\mathrm{f} = 2$ QMC simulations at $\mu_B > 0$, $\mu_I > 0$ (lower T required)
- Strong coupling LQCD on a quantum annealer allows very low T
- Gate-based quantum computing feasible for $N_\mathrm{f} = 1$, not yet feasible for $N_\mathrm{f} = 2$ (huge QC resources required)

If TCP remains invariant for higher orders in β: CEP also exists in the continuum!