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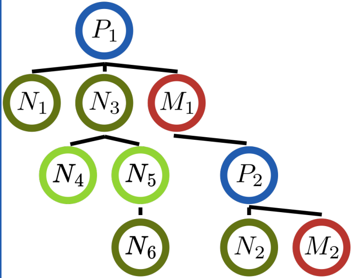
ETH, Institute for Theoretical Physics
High Performance Computational Physics group

Abelian Gauge Theories in 1+1D with Correlated Cluster Algorithms

Joao C. Pinto Barros

International SIGN25 workshop on the sign problem in QCD and beyond



Abelian Gauge Theories in 1+1D with Correlated Cluster Algorithms

Develop new Monte Carlo methods

Validate other classical and quantum approaches

Explore the physics of gauge theories available on near-term experiments

Outline

1. Cluster Algorithms
2. Applying the Correlated Cluster Algorithms to Gauge Theories
3. Multi-Flavor Correlated Cluster Algorithm
4. Conclusions

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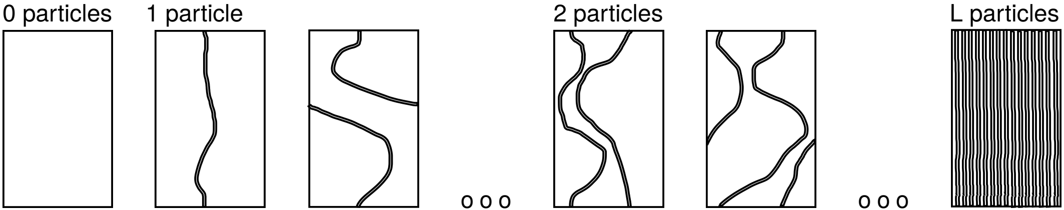
Worldline Formulation

$$\langle \mathcal{O} \rangle_\beta = \frac{\mathrm{Tr}\left(\mathcal{O} e^{-\beta H}\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)} = \frac{\sum_{\psi_0 \dots \psi_{N-1}} \langle \psi_0 \left| \mathcal{O} e^{-\varepsilon H_2} \right| \psi_{2N-1} \rangle \langle \psi_{2N-1} | \dots | \psi_1 \rangle \langle \psi_1 | \, e^{-\varepsilon H_1} \left| \, \psi_0 \right\rangle}{\sum_{\psi_0 \dots \psi_{N-1}} \langle \psi_0 | e^{-\varepsilon H_2} | \psi_{2N-1} \rangle \langle \psi_{2N-1} | \dots | \psi_1 \rangle \langle \psi_1 | \, e^{-\varepsilon H_1} | \, \psi_0 \rangle}$$

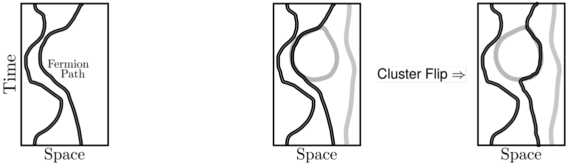
$$= \sum_{C \text{ worldline configuration}} O(C)\, p(C)$$



Mechanics of the Meron Cluster Algorithm

Path integral and cluster updates

$$\langle \mathcal{O} \rangle_\beta = \frac{\mathrm{Tr}\left(\mathcal{O} e^{-\beta H}\right)}{\mathrm{Tr}\left(e^{-\beta H}\right)}$$



U. Wolff, Phys. Rev. Lett. 62, 361 (1989)
H. G. Evertz, G. Lana, and M. Marcu, PRL 70, 875 (1993)
S. Chandrasekharan, U.J. Wiese, PRL 83 (1999)
N. Prokof'ev and B. Svistunov, Phys. Rev. Lett. 87, 160601 (2001)
S Chandrasekharan, J Cox, J C Osborn and U-J Wiese NPB 673 405 (2003)

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1. Cluster Algorithms

2. Applying the Correlated Cluster Algorithms to Gauge Theories

3. Multi-Flavor Correlated Cluster Algorithm

4. Conclusions

Wilson Formulation of $U(1)$ Schwinger Model

$$H = -t\sum_n c_n^\dagger U_n c_{n+1} + \text{h.c.} + m\sum_n (-1)^n c_n^\dagger c_n + g\sum_n \left(E_n + \frac{\theta}{2\pi}\right)^2 + U\sum_n c_n^\dagger c_n c_{n+1}^\dagger c_{n+1}$$

$$[E_n, U_m] = \delta_{mn} U_n \quad U_n U_n^\dagger = 1$$

$$U_n \left|E_n\right\rangle = \left|E_n + 1\right\rangle \quad U_n^\dagger \left|E_n\right\rangle = \left|E_n - 1\right\rangle$$

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Gauss' Law

$$G_n |\psi\rangle = 0 \qquad G_n = E_n - E_{n-1} - \rho_n$$

Charge

$$\rho_n = c_n^\dagger c_n - \frac{1-(-1)^n}{2}$$

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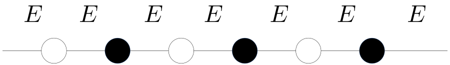
Bare Vacuum



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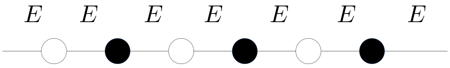
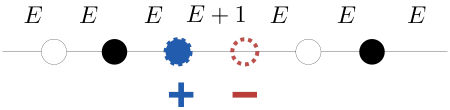


Hopping means pair creation

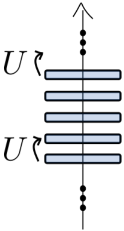


An Infinite Hilbert Space at Every Link

Gauge Invariance: $[E, U] = U$

Unitarity of the raising operators: $UU^\dagger = 1 \Rightarrow \left[U, U^\dagger\right] = 0$

Infinite Dimensional Hilbert Space at every link



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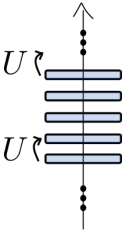
Infinite Dimensional Hilbert Space at every link

Can we make it finite?

- New opportunities for classical and quantum methods
- We can recover the same physics (at least at low energy)
- Explore new physical phenomena



Quantum Link Models

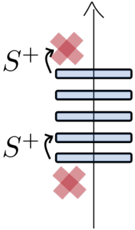
Replace the link operators with **spins**

$$U \rightarrow S^{+} = S^{x} + iS^{y}, \quad U^{\dagger} \rightarrow S^{-} = S^{x} - iS^{y}, \quad E \rightarrow S^{z} = S^{x} + iS^{y}$$

Gauge Symmetry preserved: $[S^{z}, S^{+}] = S^{+} \leftarrow [E, U] = U$

Unitarity broken: $S^{+}S^{-} \neq 1$

D. Horn, Phys. Lett. B100 (1981) 149.
P. Orland, D. Rohrlich, Nucl. Phys. B338 (1990) 647.
S. Chandrasekharan, U.-J. Wiese, Nucl. Phys. B492 (1997) 455.



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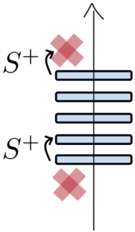
D. Horn, Phys. Lett. B100 (1981) 149.
P. Orland, D. Rohrlich, Nucl. Phys. B338 (1990) 647.
S. Chandrasekharan, U.-J. Wiese, Nucl. Phys. B492 (1997) 455.

QLM

- Continuum limit through D-theory

  R. Brower, S. Chandrasekharan, U.-J. Wiese, PRD 60, 094502 (1999)

- Wilson limit for $S \to \infty$ with $\frac{1}{\sqrt{S(S+1)}} S^+ \to U$



Truncated Link Models

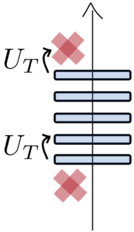
Simply truncate the electric field with a **cuttoff** S

$$U \to U_T, \quad U_T\,|E\rangle = |E+1\rangle,\, U_T\,|S\rangle = 0$$

Gauge Symmetry preserved: $[E, U_T] = U_T \leftarrow [E, U] = U$

Unitarity broken: $U_T U_T^\dagger \neq 1$

J.-Y. Desaules 1, A. Hudomal, D. Banerjee, A. Sen, Z. Papić, J. C. Halimeh, PRB 107, 205112 (2023)



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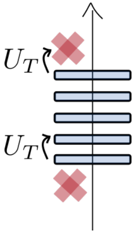
J.-Y. Desaules 1, A. Hudomal, D. Banerjee, A. Sen, Z. Papić, J. C. Halimeh, PRB 107, 205112 (2023)

TLM

- Unitarity preserved except near the cuttoff
- Wilson limit not directly recovered



$\mathbb{Z}_N$ Gauge Theories

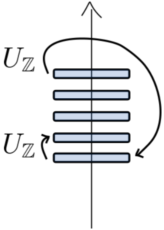
Replace $U\left(1\right)$ by a finite subgroup $\mathbb{Z}_N$

$$U \to U_{\mathbb{Z}}, \quad U_{\mathbb{Z}}\left|E\right\rangle = \left|E+1\right\rangle, U_{\mathbb{Z}}\left|N-1\right\rangle = \left|0\right\rangle$$

Gauge Symmetry broken: $[E, U_{\mathbb{Z}}] \neq U_{\mathbb{Z}}$

Unitarity preserved: $U_{\mathbb{Z}} U_{\mathbb{Z}}^{\dagger} = 11$

G. Magnifico, D. Vodola, E. Ercolessi, S. P. Kumar, M. Müller, A. Bermudez, PRB 100, 115152 (2019)



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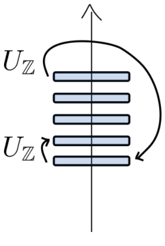
$\mathbb{Z}_N$

- Gauge symmetry broken to a subgroup
- $U(1)$ progressively better approximated by larger and larger subgroups



Satisfying Gauss' Law

For all these formulations we have a gauge symmetry $[G_n, H] = 0$

$$G_n = \underbrace{c_n^\dagger c_n + (1 - (-1)^n)/2}_{\text{charge } \rho_n} - \underbrace{E_n - E_{n-1}}_{\nabla E_n} = \rho_n - \nabla E_n$$

Physical sector: $G_n |\psi\rangle = 0$

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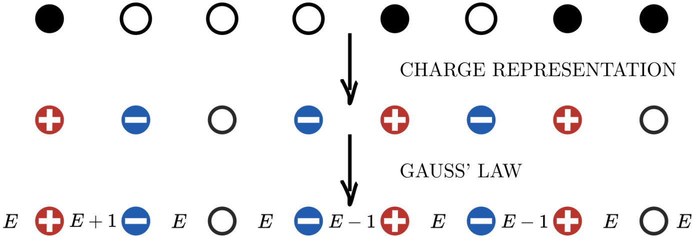
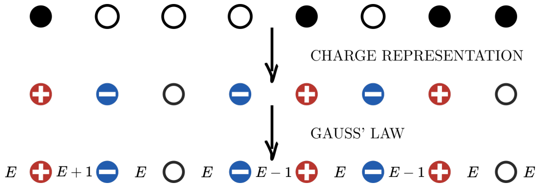


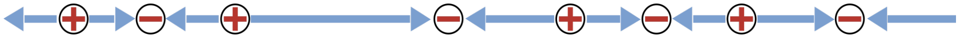
What makes it hard to sample configurations?



Gauss' Law - Constraints on States for Spin-$1/2$ Links

States that satisfy Gauss' law have an alternate positive/negative charge pattern.

We **can** have



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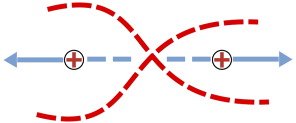

The difference of spins of any bounded region is equal to the total charge inside.

$$E_x - E_y = \text{charge between } x \text{ and } y$$

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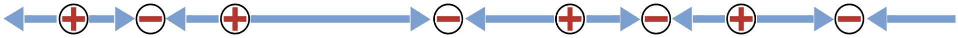

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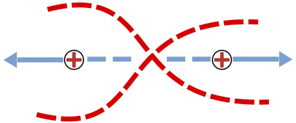

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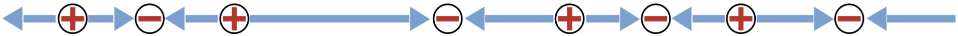

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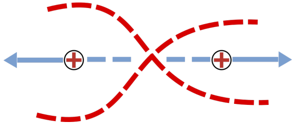

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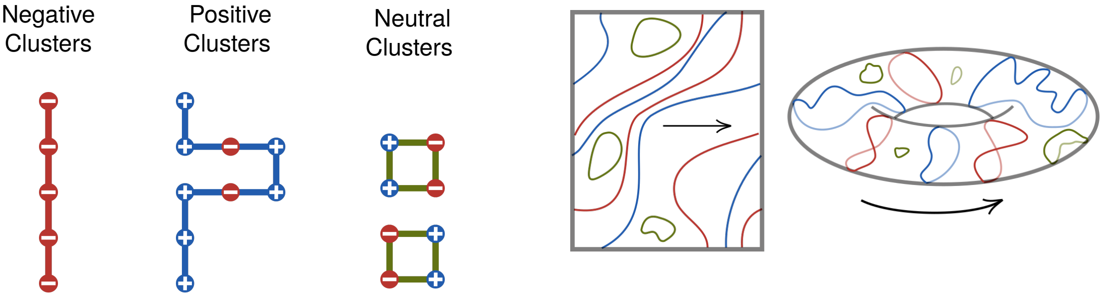
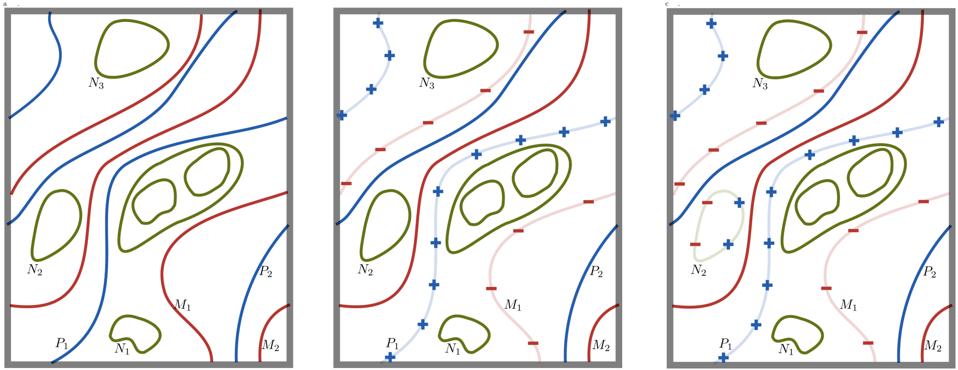
Higher spins follow a similar pattern.

The Different Types of Clusters

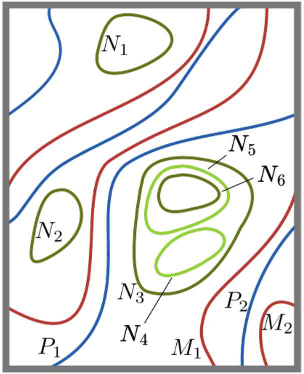


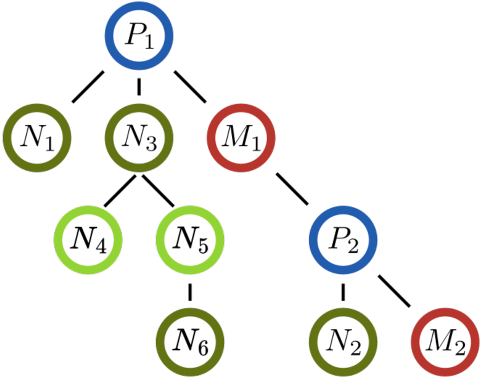
- Winding clusters, when flipped, create charges;
- Neutral clusters represent virtual processes of creation and annihilation of charges.

Satisfying Gauss' in the Spin 1/2 Quantum Link Model



Conditional Flipping





All allowed cluster flips can be sampled by inspecting a cluster tree.

We can now use the algorithm to explore different physical regimes.

Correlated Cluster Algorithm

1. Count number of ways to get each field value L from bottom up

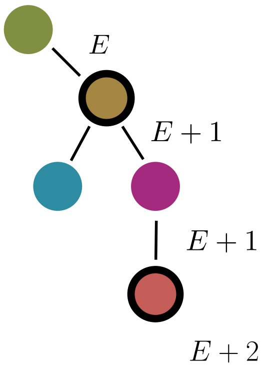
$$A_i(L) = \sum_{j \in \mathcal{C}_i} (A_j(L) + A_j(L - l_i))$$

2. Calculate flip probabilities from the top down

$$p_i = \frac{\sum_{j \in \mathcal{C}_i} A_j(L - l_i)}{A_i(L_i)}$$

Local terms in the Hamiltonian can also be added

$$A_i(L_i) = a_i\left(L_i\right) \sum_{j \in G(i)} A_j\left(L_i\right) + a_i'\left(L_i\right) \sum_{j \in G(i)} A_j\left(L_i - l_i\right)$$

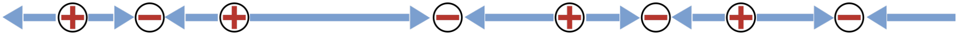


Phase Transition at finite mass with Spin $1/2$ Quantum Link Models

CP Symmetry Breaking with Spin-1/2

$$H = -t \sum_n c_n^\dagger \sigma_n^+ c_{n+1} + \text{h.c.} + m \sum_n (-1)^n c_n^\dagger c_n - 2t \sum_n c_n^\dagger c_n c_{n+1}^\dagger c_{n+1}$$

Small mass limit: Pair creation is common, **Symmetric phase**



Large mass limit: Pair creation is rare, CP**-broken phase**

E. Rico, T. Pichler, M. Dalmonte, P. Zoller, and S. Montangero PRL. 112, 201601 (2014)

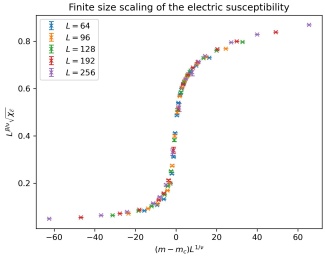
CP Symmetry Breaking with Spin-1/2: Numerical Data

$$H = -t \sum_n c_n^\dagger \sigma_n^+ c_{n+1} + \text{h.c.} + m \sum_n (-1)^n \, c_n^\dagger c_n - 2t \sum_n c_n^\dagger c_n c_{n+1}^\dagger c_{n+1}$$

Symmetric phase

CP-broken phase

$$\chi_\epsilon = \left\langle \left(\frac{1}{L} \sum_n \sigma_n^z \right)^2 \right\rangle$$



Non-universal critical mass:
$m_c \sim 0.24$

2d Ising critical exponents give curve collapse:
$\nu = 1$, $\beta = 1/8$

JPB, T. Budde, M. K. Marinkovic arXiv:2402.01039

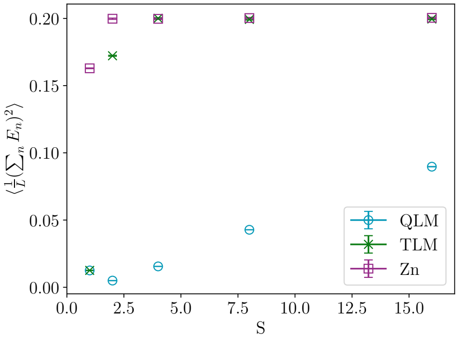
How large must the Hilbert Space per link be so that we converge?

Effects of Finite Hilbert Space per Link - Magnetization Squared

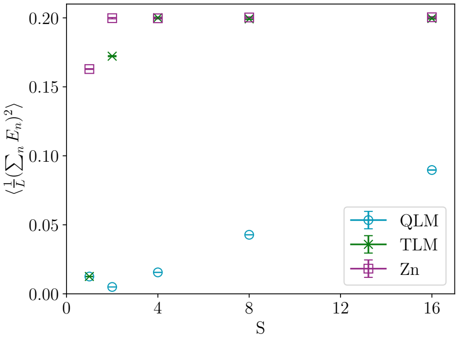
Square of the average electric field: $\left\langle \left(\frac{1}{L} \sum_n E_n \right)^2 \right\rangle$

For QLM this is the magnetization squared: $\left\langle \left(\frac{1}{L} \sum_n S^z_n \right)^2 \right\rangle$

$L = 128,\ \beta = 20,\ g^2a^2 = 10^{-4},\ m = 0$



$L = 128,\ \beta = 20,\ g^2a^2 = 10^{-3},\ m = 0$

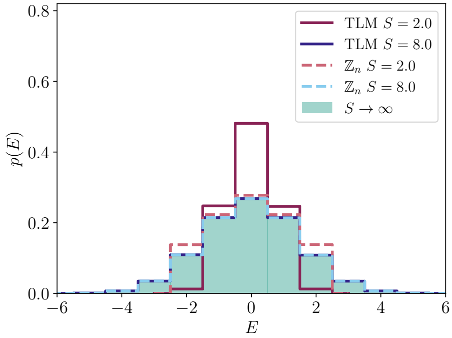


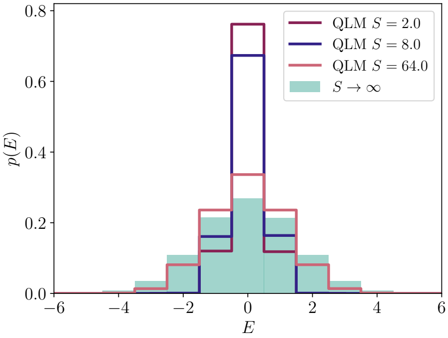
Effects of Finite Hilbert Space per Link - Electric Field Dstributions

Electric Field Distributions

$p(E)$ - Probability of finding a link with electric field value E

$$L = 128,\ \beta = 20,\ g^2a^2 = 10^{-4},\ m = 0$$

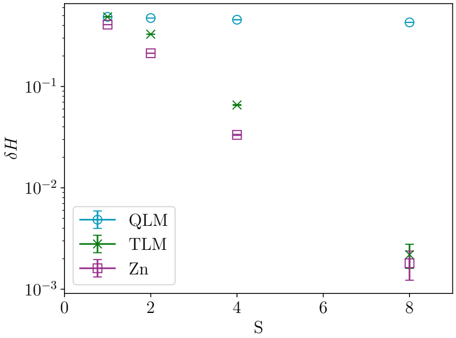




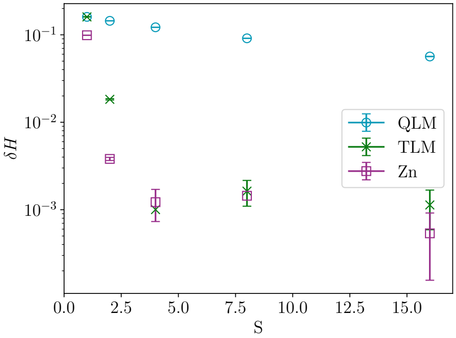
Effects of Finite Hilbert Space per Link - Helinger Distance

Helinger Distance $\delta H(p, p') = \frac{1}{\sqrt{2}} \sqrt{\sum_E \left(\sqrt{p(E)} - \sqrt{p'(E)} \right)^2}$

$L = 128,\ \beta = 20,\ (g/t)^2 = 10^{-4},\ m = 0$



$L = 128,\ \beta = 20,\ (g/t)^2 = 10^{-3},\ m = 0$

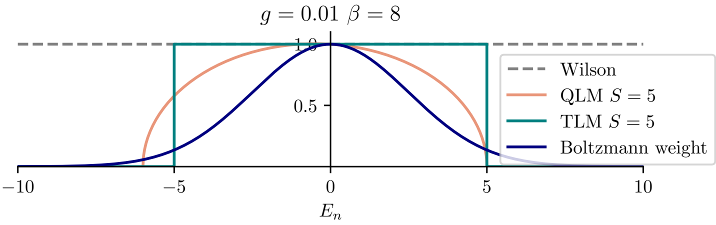


Why QLM should not be used as truncations schemes

What is the *relevant* Hilbert space?

$$w(E_n) \sim e^{-\beta g E_n^2}$$

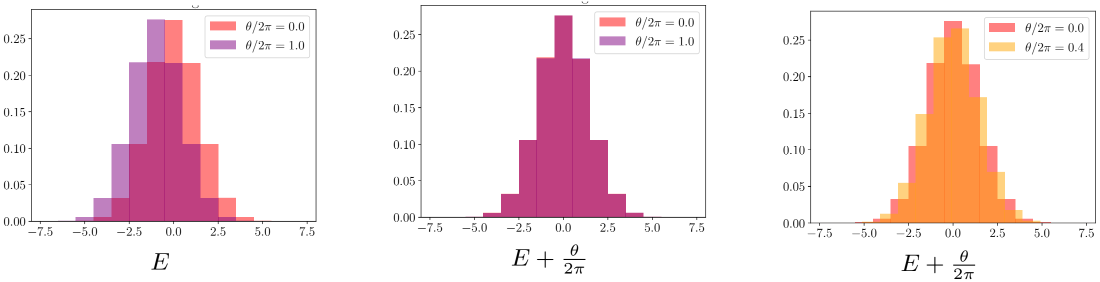
$$U_{\text{trunc}} |E\rangle = f(E) |E+1\rangle$$



There is no complex action problem when including non-zero θ angle

Inclusion of Topological θ Term: Histograms

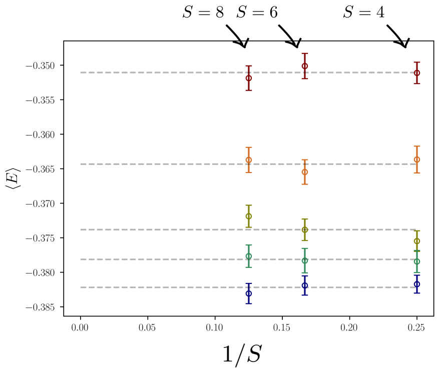
$$L = 128,\ \beta = 20,\ g/t = 10^{-2},\ m/t = 0.1$$

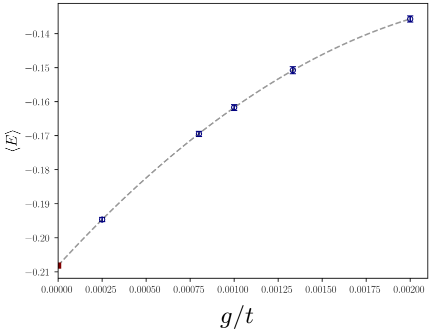


We do not need large spins to take the continuum limit

Continuum Limit $m/g = 0.01$

Re-checking spin convergence and continuum limit extrapolation at $\theta = \frac{4\pi}{5}$





Outline

1. Cluster Algorithms

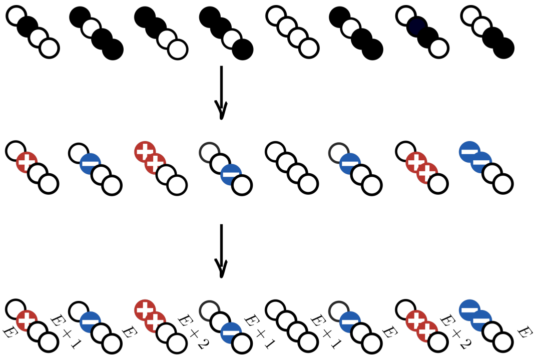
2. Applying the Correlated Cluster Algorithms to Gauge Theories

3. Multi-Flavor Correlated Cluster Algorithm

4. Conclusions

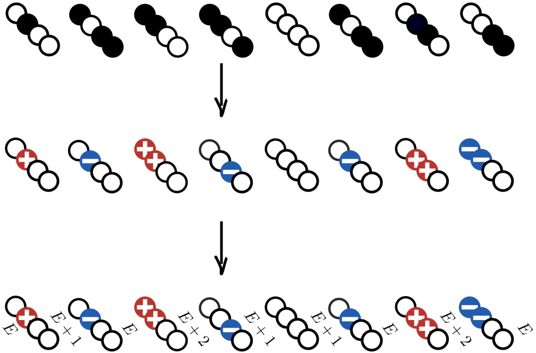
Inclusion of Multiple Flavors

- Gauge field still fixed up to a global constant
- Electric field can increase by more than one in successive links



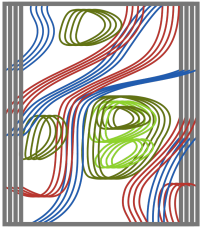
Inclusion of Multiple Flavors

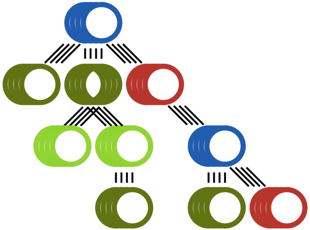
- Gauge field still fixed up to a global constant
- Electric field can increase by more than one in successive links
- This is a non-cyclic dependence and correlated cluster algorithms still apply (for properly reverse engineered Hamiltonians)



Cluster Tree with Multi-Flavor

Reverse-engineered Hamiltonian: same break-ups for all flavors





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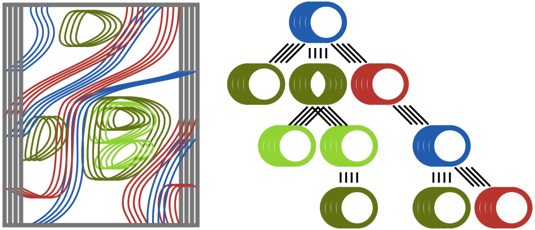

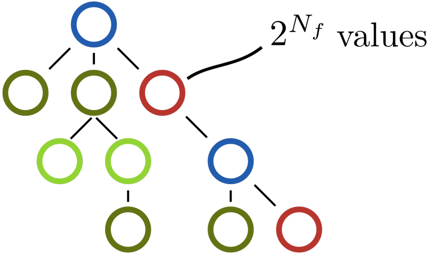

Equivalently: same tree with 2^{N_f} cluster states
(rather than flipped and non-flipped)

The Multi-Flavor Reverse Engineered Model

$$H = \left[-t\sum_i \left(c^\dagger_{i,1} U_i c_{i+1,1} + \text{h.c.}\right) - 2t\sum_i n_{i,1} n_{i+1,1}\right] \left[-t\sum_i \left(c^\dagger_{i,2} U_i c_{i+1,2} + \text{h.c.}\right) - 2t\sum_i n_{i,2} n_{i+1,2}\right]$$

$$\dots \left[-t\sum_i \left(c^\dagger_{i,N_f} U_i c_{i+1,N_f} + \text{h.c.}\right) - 2t\sum_i n_{i,N_f} n_{i+1,N_f}\right] + \sum_{i,f} m_f (-1)^i n_{i,f} + \frac{g}{2}\sum E_i^2$$

- Preserves relevant symmetries
- Has flavor symmetry if masses are equal
- Efficiently simulated by the correlated cluster algorithm

Phase Diagram?

Continuum Limit Spectrum?

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Conclusions

- Correlated cluster algorithm can be used to study $1+1$-d Abelian gauge theories
- **QLM should not be regarded as truncations** - TLM and $\mathbb{Z}_N$ are better tailored as truncations
- **Complex action problem** associated with θ-angle entirely **circumvented** with Monte Carlo
- Pathway to explore multi-flavor physics

JPB, T. Budde, M. K. Marinkovic - arXiv:2402.01039

T. Budde, M. Kristc Marinkovic, JPB - In preparation

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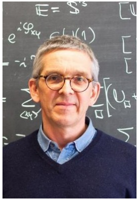


Meron Cluster Algorithm

We will be extending the algorithm that simulates the Hamiltonian

$$H = -t \sum_i c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} + 2 \left(c_i^\dagger c_i - \frac{1}{2} \right) \left(c_{i+1}^\dagger c_{i+1} - \frac{1}{2} \right)$$

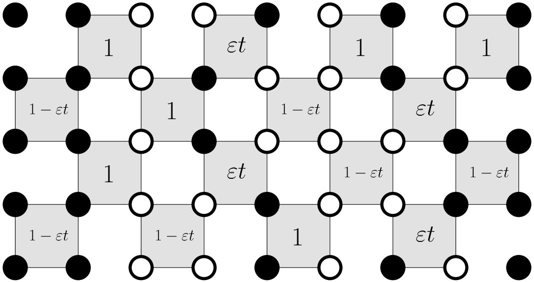
for a chain of spinless fermions

S. Chandrasekharan, U.J. Wiese, Phys.Rev.Lett. 83 (1999) 3116-3119

Meron Cluster Algorithm

$$Z = \mathrm{Tr}(e^{-\beta H}) = \sum_i \langle n_i| e^{-\beta H} |n_i\rangle \overset{\text{Trotter}}{\approx} \sum_i \langle n_i| \left(e^{-\frac{\beta}{N} H_{\text{even}}} e^{-\frac{\beta}{N} H_{\text{odd}}}\right)^N |n_i\rangle$$

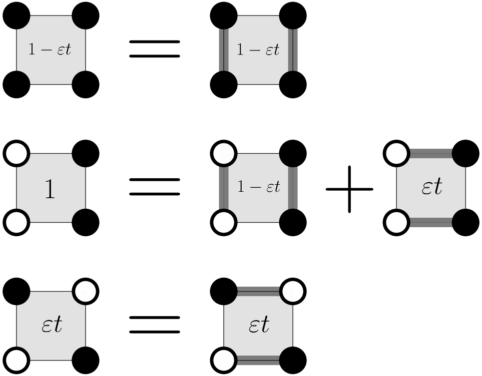
$$= \sum_{n_0, n_1, n_2, n_3, \ldots} \langle n_0| e^{-\epsilon H_{\text{even}}} |n_1\rangle \langle n_1| e^{-\epsilon H_{\text{odd}}} |n_2\rangle \langle n_2| \ldots |n_0\rangle$$



S. Chandrasekharan, U.J. Wiese, Phys.Rev.Lett. 83 (1999) 3116-3119

Cluster Algorithm for spinless fermions

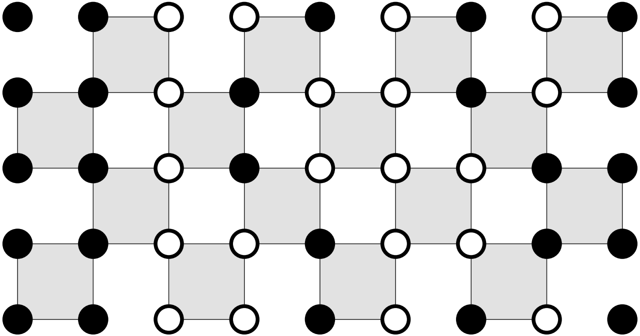
Add additional degrees of freedom: break-ups



S. Chandrasekharan, U.J. Wiese, Phys.Rev.Lett. 83 (1999) 3116-3119

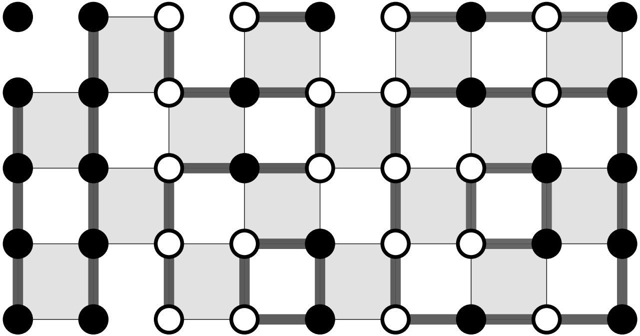
Mapping Fermion Paths to QLMs

Take Fermion Configuration



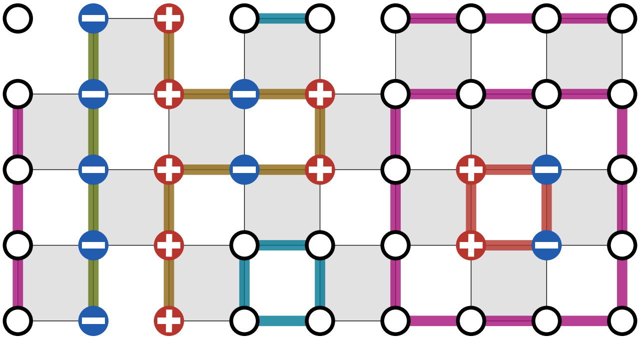
Mapping Fermion Paths to QLMs

Place break-ups as usual



Mapping Fermion Paths to QLMs

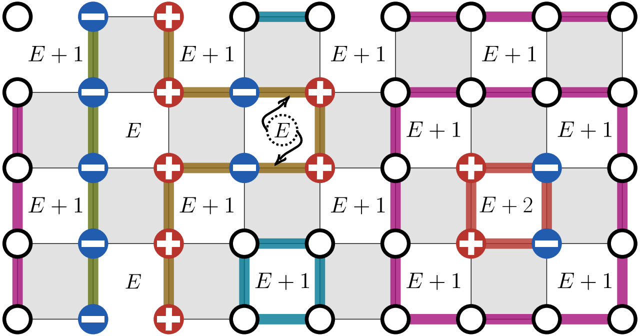
Translate to charge representation



⇒ Clusters flip between all charged and neutral

Mapping Fermion Paths to QLMs

Calculate links with Gauss's law



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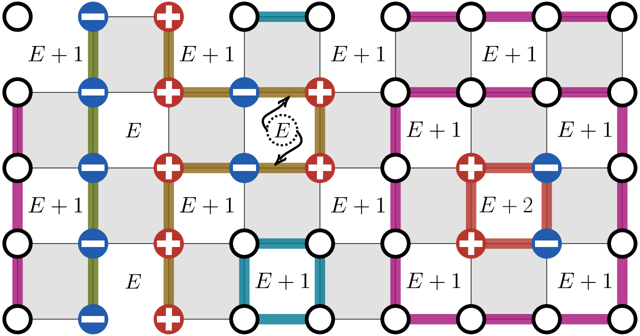
Calculate links with Gauss's law



$\Rightarrow$ Fields only change when crossing a charged cluster

Mapping Fermion Paths to QLMs

Order neighboring clusters into a tree