Phase Diagram of the Schwinger Model by Adiabatic Preparation of States on a Quantum Simulator

Oleg Kaikov

Université Paris-Saclay, CEA-List

SIGN25, 21$^{\text{st}}$ January 2025

Based on: arXiv:2407.09224

work in progress

OK, T. Saporiti, V. Sazonov, M. Tamaazousti

Motivation

Phase diagrams are often studied numerically by Monte Carlo simulations

Monte Carlo methods can suffer from the sign problem (e.g. in LQCD)

Quantum computing is unaffected by the sign problem

Classical optimization problem in variational quantum algorithms can be challenging

L. Bittel, M. Kliesch: Phys. Rev. Lett. **127**, 120502 (2021)

Part I: Phase structure via quantum adiabatic evolution without modifying the model?

Quantum computers are noisy

Part II: Mitigation of quantum errors from Part I by classically accessible regimes?

Part I: Phase structure by adiabatic evolution

General idea

Goal: Study phase diagram of a quantum physical system

Approach: Adiabatic preparation of states on a quantum device

Method: Direct application of the adiabatic theorem

Example: The Schwinger model

The Schwinger model (continuum)

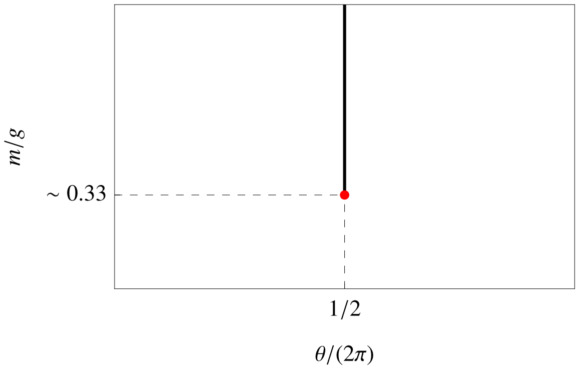
(1+1)-dim. U(1) gauge theory $\hbar = c = 1$

$$\mathcal{H} = -i\overline{\psi}\gamma^1\left(\partial_1 - igA_1\right)\psi + m\overline{\psi}\psi + \frac{1}{2}\left(\dot{A}_1 + \frac{g\theta}{2\pi}\right)^2$$

J. Schwinger: Phys. Rev. **128**, 2425 (1962)

Temporal gauge: $A_0 = 0$

Gauss's law: $-\partial_1 \dot{A}^1 = g\overline{\psi}\gamma^0\psi$



The Schwinger model (lattice)

Lattice discretization via Kogut-Susskind staggered fermions with OBC

J. Kogut and L. Susskind: Phys. Rev. D **11**, 395 (1975)

Re-scale to make dim.-less: $H \to \frac{2}{ag^2} H$

Enforce vanishing total charge: $H \to H + \lambda(\sum_{n=0}^{N-1} Q_n)^2$ with $Q_n = \frac{1}{2}(Z_n + (-1)^n)$

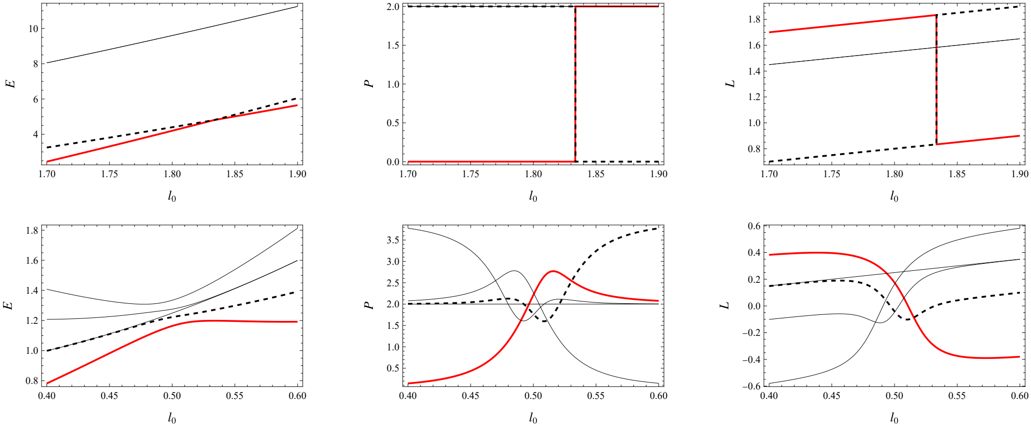
$$
\begin{aligned}
H = {} & \frac{x}{2} \sum_{n=0}^{N-2} (X_n X_{n+1} + Y_n Y_{n+1}) + \frac{1}{2} \sum_{n=0}^{N-2} \sum_{k=n+1}^{N-1} (N - k - 1 + \lambda) Z_n Z_k \\
& + \sum_{n=0}^{N-2} \left(\frac{N}{4} - \frac{1}{2} \left\lceil \frac{n}{2} \right\rceil + l_0 (N - n - 1) \right) Z_n + \frac{m_{\text{lat}}}{g} \sqrt{x} \sum_{n=0}^{N-1} (-1)^n Z_n \\
& + l_0^2 (N-1) + \frac{1}{2} l_0 N + \frac{1}{8} N^2 + \frac{\lambda}{4} N
\end{aligned}
$$

N: even number of sites, a: lattice spacing, $x = 1/(ag)^2$, $l_0 = \theta/(2\pi)$

T. Angelides, P. Naredi, A. Crippa, K. Jansen, S. Kühn, I. Tavernelli, and D. S. Wang: npj Quantum Inf. **11**, 6 (2025)

1^{st}-order-PT & no-PT regions

Observables: $E\,,\qquad P=\frac{N}{2}+\frac{1}{2}\sum_{n=0}^{N-1}(-1)^n Z_n\,,\qquad L=l_0+\frac{1}{4}+\frac{1}{2}\sum_{n=0}^{N/2-2}Z_n+\frac{1}{4}Z_{N/2-1}$



The adiabatic theorem with a gap condition

Consider a quantum system parameterized by a time-dependent Hamiltonian $H(t)$ and prepared in one of its instantaneous eigenstates

The system remains in that state if:

1. The Hamiltonian is varied sufficiently slowly, and
2. There is a non-zero gap Δ between the eigenvalue of that state and the rest of the spectrum for all t

P. Ehrenfest: Ann. Phys. **356**, 327 (1916)
M. Born and V. Fock: Z. Phys. **51**, 165 (1928)
T. Kato: J. Phys. Soc. Jpn. **5**, 435 (1950)
J. E. Avron and A. Elgart: Comm. Math. Phys. **203**, 445 (1999) & refs. therein
etc.

Adiabaticity: $T_{\text{dyn}} |\langle \dot{H} \rangle| \ll \Delta$

$$T_{\text{dyn}} \sim \frac{1}{\Delta}$$

Non-adiabaticity parameter:

$$\varepsilon \sim \frac{|\langle \dot{H} \rangle|}{\Delta^2} \ll 1$$

Phase structure by avoiding crossings

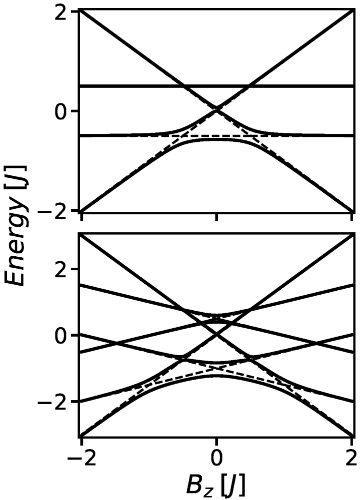
1-dim. isotropic XY model with PBC

$$H = -\sum_{n=1}^{N} \left[\frac{J}{4}(X_n X_{n+1} + Y_n Y_{n+1}) + \frac{B_z}{2} Z_n + \frac{B_x}{2} X_n \right]$$

$[S_z, H] = [\sum_n Z_n/2, H] \neq 0$

1. Evolve $|0\rangle^{\otimes N}$ with $H(B_z(t))$ for fixed B_x
2. Find B_z^{crit} by midpoint of step in magnetization
3. Repeat 1.-2. for different B_x
4. Extrapolate B_z^{crit} for $B_x \to 0$

+ Advantageous in case of many level-crossings
− Cannot determine order of PT
− Cannot be applied if phase structure is not known



A. Francis, E. Zelleke, Z. Zhang, A. F. Kemper, and J. K. Freericks: Symmetry **14**(4), 809 (2022)

The adiabatic theorem without a gap condition

What happens if there is a level-crossing?

Let E_α be an energy level, e a unit of energy, C a dimensionless constant, $\tau = t/T$

If each $(E_\alpha(\tau) - E_\beta(\tau))$, $\alpha \neq \beta$, has a finite number of zeros of max. the r-th order, and in the vicinity of each zero point τ_0 it holds that $\frac{e}{|E_\alpha(\tau) - E_\beta(\tau)|} < \frac{C}{|\tau - \tau_0|^r}$,
then the transition probability to a different energy level is $p_{\mathrm{tr}} = O\left[(eT)^{-2/(r+1)}\right]$

M. Born and V. Fock: Z. Phys. **51**, 165 (1928)

Adiabatic evolution of multiple states

In application to the Schwinger model: $H(l_0(t))$ $\qquad$ $l_0(t) = l_0^{\text{min}} + \left(l_0^{\text{max}} - l_0^{\text{min}}\right) t/T$

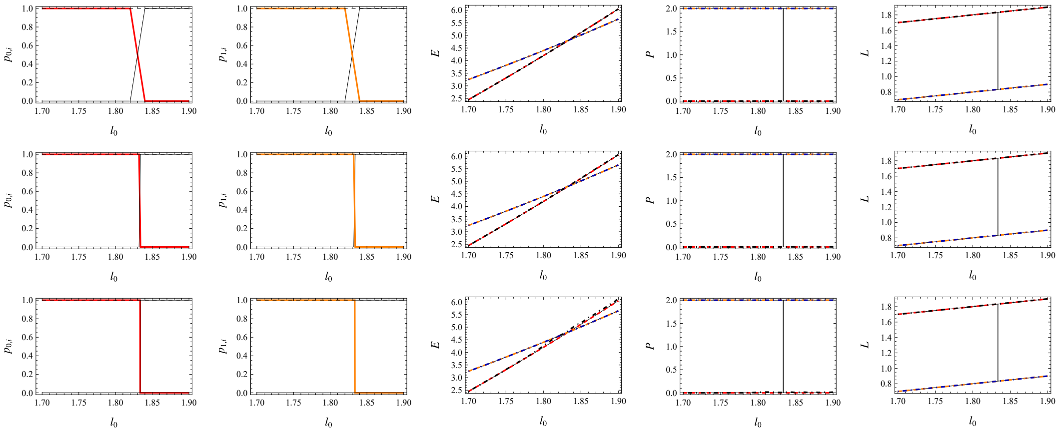
1. Choose evolution time T and error bound $\mathcal{B}$
2. Evolve lowest-energy eigenstates $\{|\alpha\rangle\}$ at l_0^{min} separately
3. Compute energy levels $E_\alpha(l_0)$
4. Compute deviation (error) in $E_\alpha(l_0)$ w.r.t. previous iteration

$$\mathcal{E} = \frac{1}{\epsilon} \sum_\alpha \sqrt{\frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \left[E_\alpha^{\text{old}}\left(l_0(t_i)\right) - E_\alpha\left(l_0(t_i)\right)\right]^2}$$

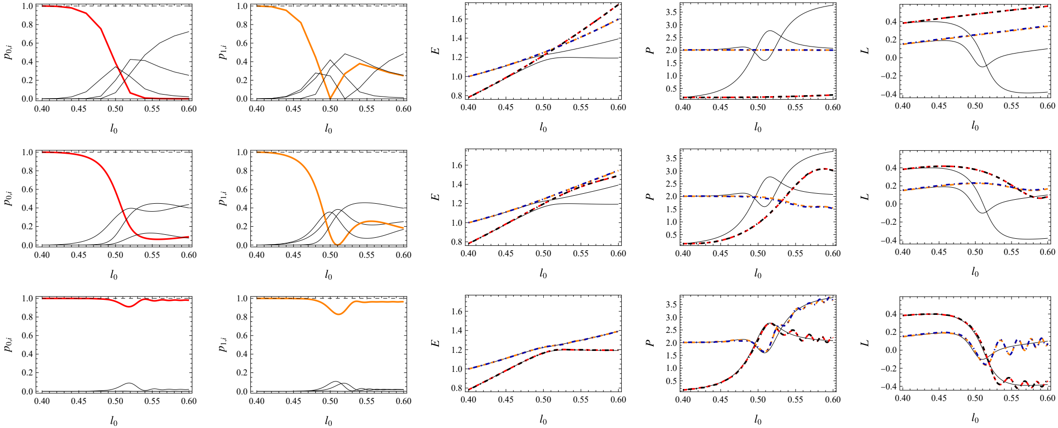
5. Increase T
5. Repeat 2.-5. until $\mathcal{E} < \mathcal{B}$

Assumption of asymptotic convergence: $\mathcal{E} \to 0 \quad \Rightarrow \quad T$ sufficiently large

Numerical results: PT



Numerical results: No PT



Part II: Quantum error mitigation via classically accessible regimes

General idea

Insight: Certain phase diagram regions are accessible by classical computations (e.g. sign problem-free regimes, particular phases)

Method: Global Randomized Error Cancellation (GREC)

V. Sazonov and M. Tamaazousti: Phys. Rev. A **105**, 042408 (2022)

Example: 1^{st}-order-PT & no-PT regions of the Schwinger model

Questions:

1. Can GREC be realized for EM in AE?
2. Does GREC EM of energy levels learned in one phase transfer to another?

GREC method

Inspired by Zero Noise Extrapolation (ZNE) and Probabilistic Error Cancellation (PEC)

Y. Li and S. C. Benjamin: Phys. Rev. X **7**, 021050 (2017)

K. Temme, S. Bravyi, and J. M. Gambetta: Phys. Rev. Lett. **119**, 180509 (2017)

1. Generate quantum circuits with R realizations of random additional noise (AN)
2. Measure observable $A(\lambda)$ on entire parameter domain
3. $\langle A(\lambda)\rangle^{\mathrm{ED}} = \sum_{r=1}^{R} \eta_r \langle A(\lambda)\rangle_r^{\mathrm{AN}} + \eta_0$

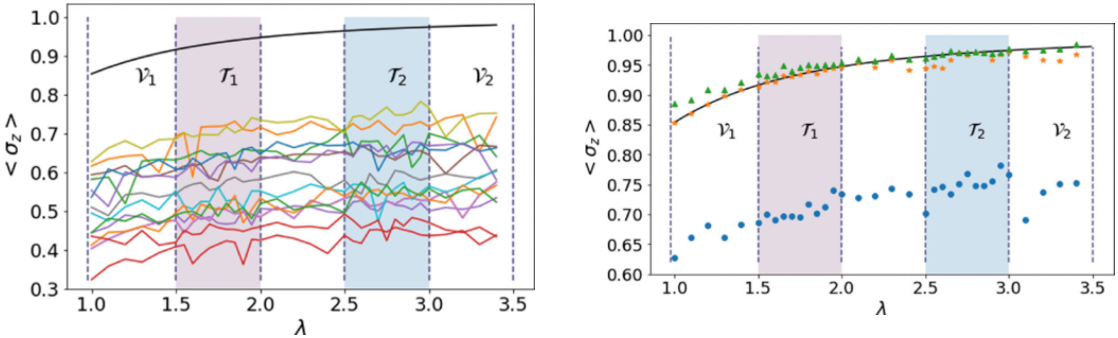
   $\rightarrow$ Learn $\eta_0, \eta_1, \ldots, \eta_R$ in "easy" (CL & QU) regime
4. Predict $\langle A(\lambda)\rangle^{\mathrm{EM}} = \sum_{r=1}^{R} \eta_r \langle A(\lambda)\rangle_r^{\mathrm{AN}} + \eta_0$ in "hard" (only QU) regime

V. Sazonov and M. Tamaazousti: Phys. Rev. A **105**, 042408 (2022)

GREC example

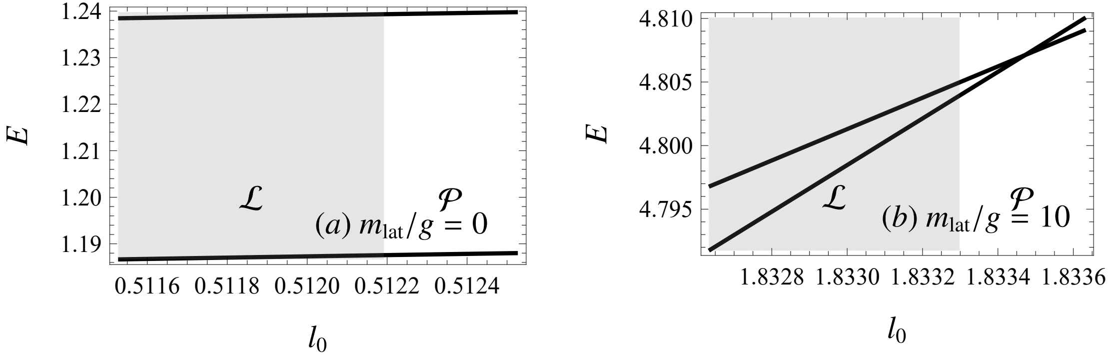
Transverse field antiferromagnetic Ising model

$$H = \sum_{n=1}^{N-1} X_n X_{n+1} + \lambda \sum_{n=1}^{N} Z_n + Y_1 \prod_{n=2}^{N-1} Z_n Y_N$$



V. Sazonov and M. Tamaazousti: Phys. Rev. A **105**, 042408 (2022)

GREC in application to the Schwinger model



Added noise

Weak error model: $p = 10^{-6}$ of Z flip in RZ gate

Time-evolution mitigation: choose to add noise only to H (not to $|\text{in}\rangle$)

Add noise at the level of H (alternatively: directly at circuit level)

To preserve Hermiticity of H, each Pauli P in H is mapped as:
$P \to U_P^\dagger P U_P$ with U_P a complex 2×2 matrix

For simplicity, take U_P to be a linear combination of I, X, Y, Z

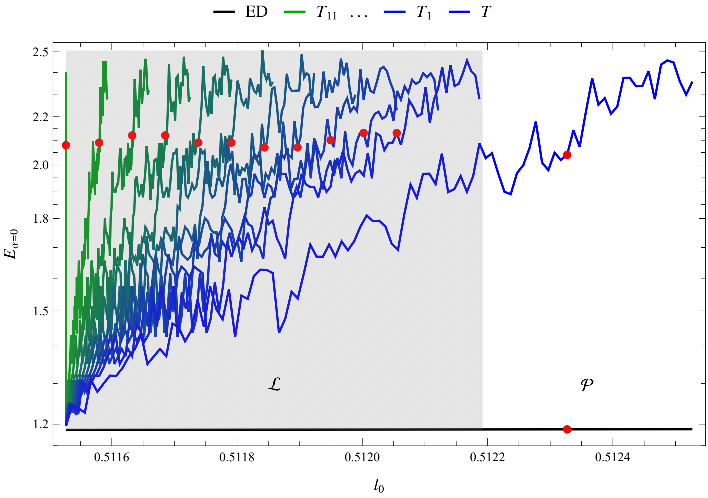
For simplicity, restrict U_P to a rotation: $U_P(\theta, \phi, \lambda)$

Set added noise comparable in magnitude to error

Add noise only to Z: $U_P(\theta, \phi, \lambda) \to U_P(0, 0, \lambda)$

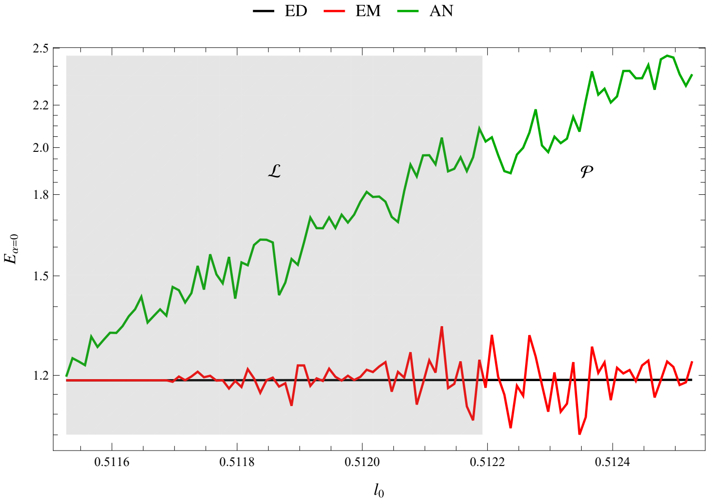
GREC for adiabatic evolution

$m_{\text{lat}}/g = 0,\ \alpha = 0$



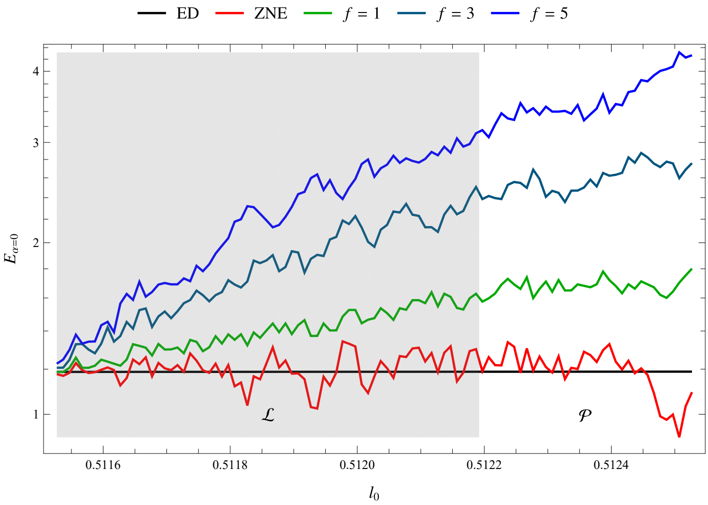
GREC mitigation

$m_{\text{lat}}/g = 0,\ \alpha = 0$



ZNE

$m_{\text{lat}}/g = 0,\ \alpha = 0$



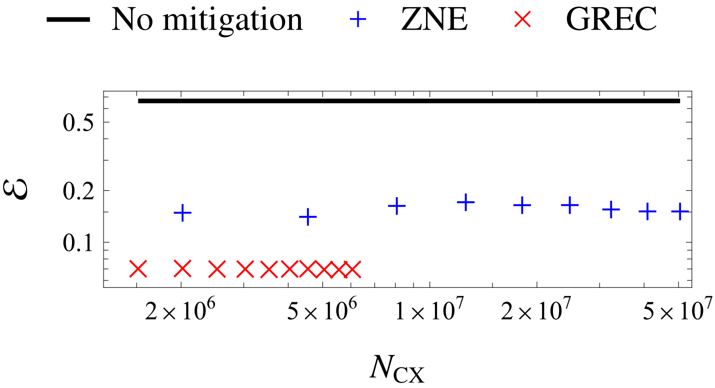
GREC vs ZNE

$$N_{\mathrm{G}}^{\mathrm{ZNE}} = n_{\mathrm{levels}} \frac{n_{\delta t}}{2}(n_{\delta t}+1) P_{\mathrm{ZNE}}^2 n_{\mathrm{G}/\delta t}$$

$$N_{\tilde{\mathrm{G}}}^{\mathrm{GREC}} = n_{\mathrm{levels}} \frac{n_{\delta t}}{2}(n_{\delta t}+1) P_{\mathrm{GREC}} n_{\tilde{\mathrm{G}}/\delta t}$$

$$\mathcal{E} = \sum_{\alpha} \sqrt{\frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \left(E_{\alpha}^{\mathrm{ED}}(l_{0,i}) - E_{\alpha}^{\mathrm{EM}}(l_{0,i})\right)^2}$$

$$m_{\mathrm{lat}}/g = 0, \ \alpha = 0, 1$$



Outlook

2 extensions/hybrids:

Generate lines with different magnitudes of added noise, and

1. Perform ZNE
2. Perform GREC

Related work: Next talk

T. Saporiti: BBGKY Hierarchy for Zero Noise Extrapolation in Quantum Error Mitigation

work in progress

OK, T. Saporiti, V. Sazonov, M. Tamaazousti

Summary

Part I:

Phase structure by direct application of adiabatic theorem without modifying the model

Evolve multiple lowest-energy eigenstates individually with $H(\theta(t))$

Applicable in case of no knowledge about the phase structure

Part II:

GREC can be realized for EM in AE

GREC EM of energy levels transfers across phases

Parameters

$$N = 6, \quad V = N/\sqrt{x} = 30, \quad \lambda = 100, \quad m_{\mathrm{lat}}/g \in \{0, 10\}, \quad n_{\mathrm{shots}} = 10^4$$

Part I:

$\delta t = 0.5$

For iteration i: $T_i = \begin{cases} 5i\,, & 1 \leq i \leq 9 \\ 10\,T_{i-9}\,, & i \geq 10 \end{cases}$

Suzuki-Trotter order 6

Part II:

$\delta t = 0.1$

$\mathrm{T} = 10$

Lie-Trotter (order 1)

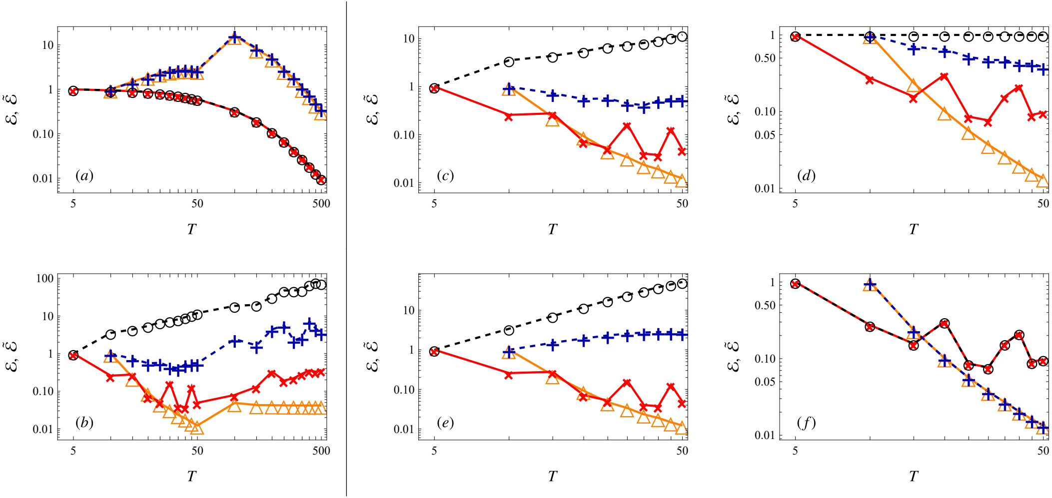
$\delta l_0 = 10^{-5}$

Simulator basis gates: CX, I, RZ, RX, X

Initial states

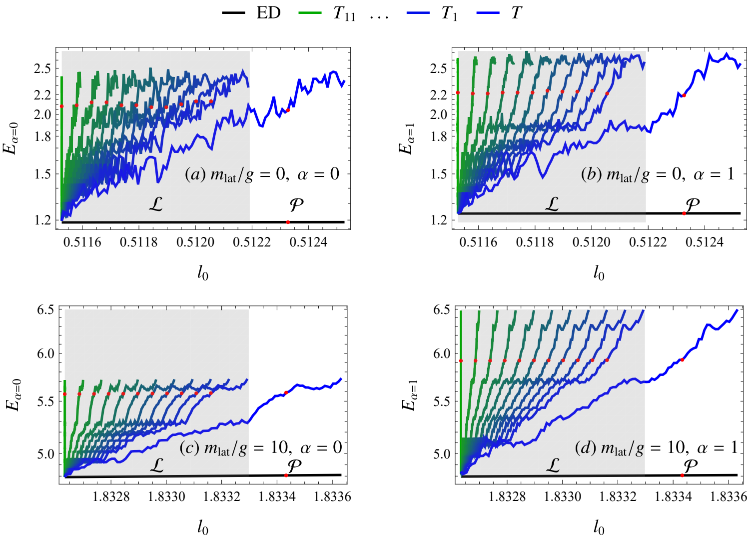
1. For fixed m_{lat}/g, $l_0 = l_0^{\text{min}}$, λ, consider only terms of H with Z_n and $Z_n Z_k$
   Eigenspectrum: 2^N states $|s\rangle^{\otimes N}$ with $|s\rangle \in \{|0\rangle, |1\rangle\}$
2. Select states by total charge $\sum_n^{N-1} Q_n$:
   Let $|Q_A^{\text{tot}}| < |Q_B^{\text{tot}}|$ with $Q_A^{\text{tot}}, Q_B^{\text{tot}} = -N/2, \ldots, N/2$ for subsectors $\{|A\rangle\}$, $\{|B\rangle\}$
   By $H \to H + \lambda(\sum_{n=0}^{N-1} Q_n)^2$, it holds $E_B - E_A \approx \lambda[(Q_B^{\text{tot}})^2 - (Q_A^{\text{tot}})^2]$
3. Evolve states adiabatically from $x(t=0) = 0$ to $x(t=T) = x_{\text{target}}$
4. For sufficiently large m_{lat}/g, 2 lowest eigenstates are well approximated by:
   $|10\rangle^{\otimes N/2}$ with $P = 0$, $L = l_0$, and
   $|11\rangle \otimes |10\rangle^{\otimes (N-4)/2} \otimes |00\rangle$ with $P = 2$, $L = l_0 - 1$

Convergence

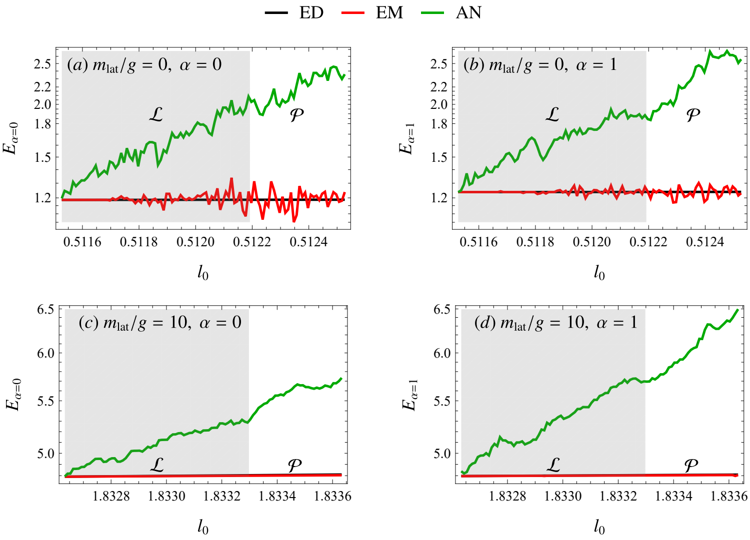


△ $\mathcal{E}_{\mathrm{CL}}$ + $\mathcal{E}_{\mathrm{QU}}$ × $\tilde{\mathcal{E}}_{\mathrm{CL}}$ ○ $\tilde{\mathcal{E}}_{\mathrm{QU}}$ RHS: $\rightarrow$: $\delta t \searrow$, $\downarrow$: shot noise $\rightarrow 0$

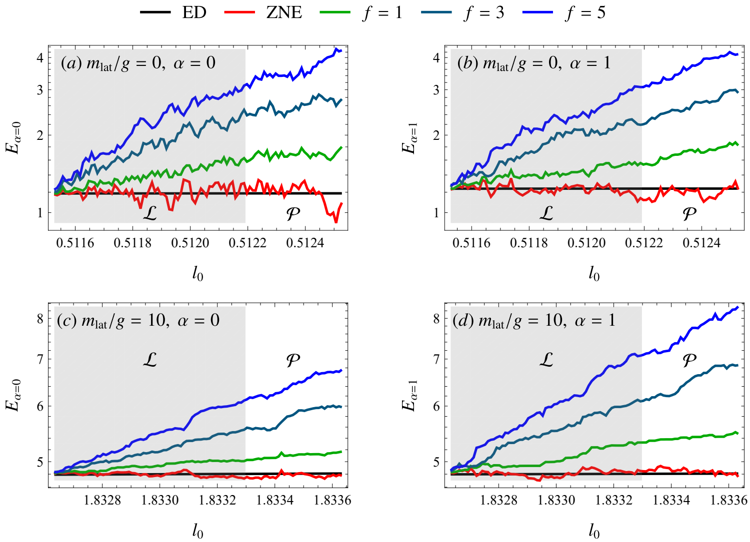
GREC for adiabatic evolution



GREC mitigation

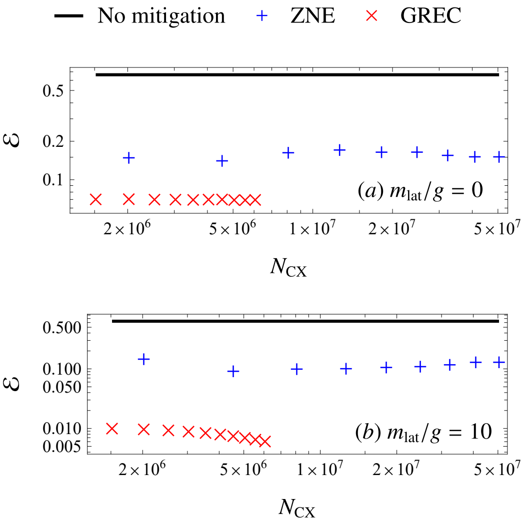


ZNE



ED ZNE $f = 1$ $f = 3$ $f = 5$
(a) $m_{\rm lat}/g = 0,\ \alpha = 0$
(b) $m_{\rm lat}/g = 0,\ \alpha = 1$
(c) $m_{\rm lat}/g = 10,\ \alpha = 0$
(d) $m_{\rm lat}/g = 10,\ \alpha = 1$
$\mathcal{L}$
$\mathcal{P}$
$E_{\alpha=0}$
$E_{\alpha=1}$
l_0
0.5116 0.5118 0.5120 0.5122 0.5124
1.8328 1.8330 1.8332 1.8334 1.8336

GREC vs ZNE



No mitigation
ZNE
GREC
0.5
0.2
0.1
ε
(a) $m_{\text{lat}}/g = 0$
2×10^6
5×10^6
1×10^7
2×10^7
5×10^7
N_{CX}
0.500
0.100
0.050
0.010
0.005
ε
(b) $m_{\text{lat}}/g = 10$
2×10^6
5×10^6
1×10^7
2×10^7
5×10^7
N_{CX}

GREC vs ZNE in $\mathcal{P}$