Is the Ni's solution of the Tolman-Oppenheimer-Volkoff problem without the maximum-mass limit applicable to the real neutron stars? A discussion.

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ABSTRACT. In 2011, Jun Ni published solution of the equations in the classical Tolman-Oppenheimer-Volkoff (TOV) modeling of spherically symmetric neutron star. The Ni's solution implies no upper-mass limit and the outer surface of modeled object always appears to be above the event horizon. In fact, Ni found an infinite variety of sets of the TOV-problem solutions. The original Oppenheimer-Volkoff project provides only a single set from this variety offered by general relativity. As originally Openheimer and Volkoff as Ni assumed the positive energy density and pressure (or zero in the vacuum outside the object). And, the gravity of every mass element of the object had the attractive character. Ni noted that this type of solution cannot be obtained in Newtonian physics. However, general relativity may not obey the limitations sourcing from the Newtonian gravity and, thus, it seems that the neutron-star models based on the Ni's solution are still applicable on real compact objets. We discuss the relevance of main objections against this applicability.

Two versions of general relativity (GR)

* **pure geometrical GR** (G-GR); it contains quantities: metric tensor, energy (energy density), pressure
* **GR with the Newtonian-type potential** (GR-NTP) and mass as its integral part; quantities additional to G-GR: Newtonian-type potential, mass
* since **GR-NTP** must obey the properties of Newtonian-type potential/mass, the size of g_{rr} in the case of spherically symmetric (S.S.) object is constrained as $|g_{rr}| \geq 1$; on contrary, $|g_{rr}| > 0$ **in G-GR**

SMOKING GUN of the astrophysics of relativistic compact objects: The Ni's solution of the Einstein field equations

* the solution is found for the S.S. TOV problem, but while the TOV model of relativistic compact object (RCO) was found within the GR-NTV ($|g_{rr}| \geq 1$), the Ni's solution assumes the G-GR, i.e. $0 < |g_{rr}| < 1$ is also permitted
* in the Ni's modeling of RCO, the integration of Einstein field equations (EFEs) starts in a radial distance $r > 0$
* inward processed integration practically always ends with $E = 0$ and $P = 0$ in a distance $r = R_{in} > 0$; hence, an **inner physical surface** of RCO is implied (only a single combination of initial integrated quantities at the RCO of given rest energy leads to the maximum E and P in $r = 0$; this combination is forced to be the case when the integration starts in $r = 0$)
* behaviours of $\rho = E/c^2$, g_{rr}, g_{tt}, $u = (r/2)(1 + 1/g_{rr})$, and U in an example of RCO are shown in Fig.1

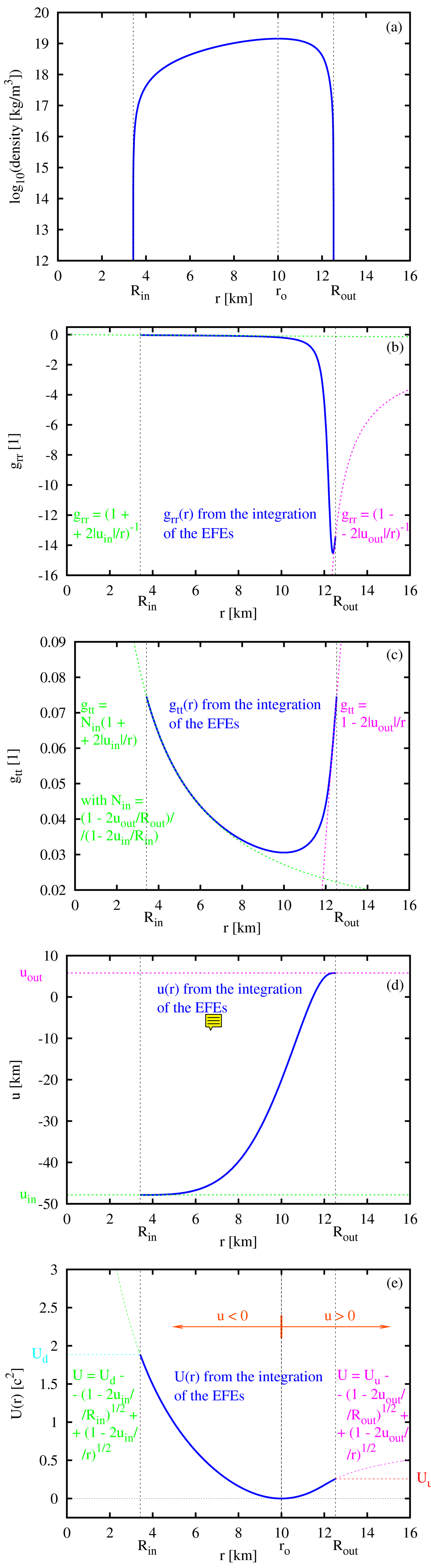

FIG. 1. The behaviors of density ($\rho = E/c^2$) (plot a), components of metric tensor g_{rr} (plot b) and g_{tt} (plot c), auxiliary metric quantity u (plot d), and re-calibrated effective gravitational potential U (plot e) in the interior as well as in the internal cavity and outside of an example RCO, which was constructed by using of the Ni's solution of EFEs. The gravitational mass of this object is 3.923 solar masses.

* Ni's solution provides the model of **stable RCO of what-ever large mass** (formally calculated); this fact evidences that the Oppenheimer-Volkoff upper-mass limit occurs due to the constraint $|g_{rr}| \geq 1$ and, therefore, only within the GR-NTP
* R_{in}, r_o, and R_{out} in their dependence on mass (quantity $M = c^2u/G$) are shown in Fig.2 for several sequences of RCO model
* $\mathbf{R_{out} > R_g}$, always (see Fig.2)
* **indication** (not proof, yet; see Fig.2) that $\mathbf{R_{out} \rightarrow R_g}$ (from outside) **in the limit** $\mathbf{M \rightarrow \infty}$, i.e. we would have to deliver an infinite amount of energy to a RCO to force its collapse below the event horizon if the indication is proved

Reason for inner surface and internal cavity: net gravitational attraction of upper material layers of RCO

Let us consider a S.S. material shell and its net gravity on a particle situated inside the shell.
* in Newtonian physics, the net gravity can be analytically calculated and is equal to zero
* in GR-NTP, the net gravity is postulated (in fact, in the context of Ni's solution) to be zero (Minkowski metrics)
* in G-GR, the net gravity can be ad hoc postulated to be zero ("ad hoc" because there is no actual reason for the postulate) or, as in this work, it can be derived integrating the EFEs in the way by Ni [let us the GR alone to work]; the **Ni's solution implies that the net gravity inside the shell is non-zero and oriented outward** from its center

Note on spherical symmetry

* in the Euclidean space of Newtonian physics, a distribution of matter, which is observed as spherically symmetric by the observer in its center, is also observed as spherically symmetric by an observer being in whatever position aside the center
* **in a curved space (in static case) of GR, the distribution of matter which is observed as spherically symmetric by the observer in its center, is not observed, in general, as spherically symmetric by an observer aside the center**; this cirmustance enables the existence of non-zero net gravity inside the material shell, which is spherically symmetric according to the observer located in its center

RCO with the outward oriented net gravitational attraction of upper layers

* near the outer surface inside RCO, the outward-oriented net gravity of upper layers is smaller than the inward-oriented gravity of inner layers, therefore the total net gravity is

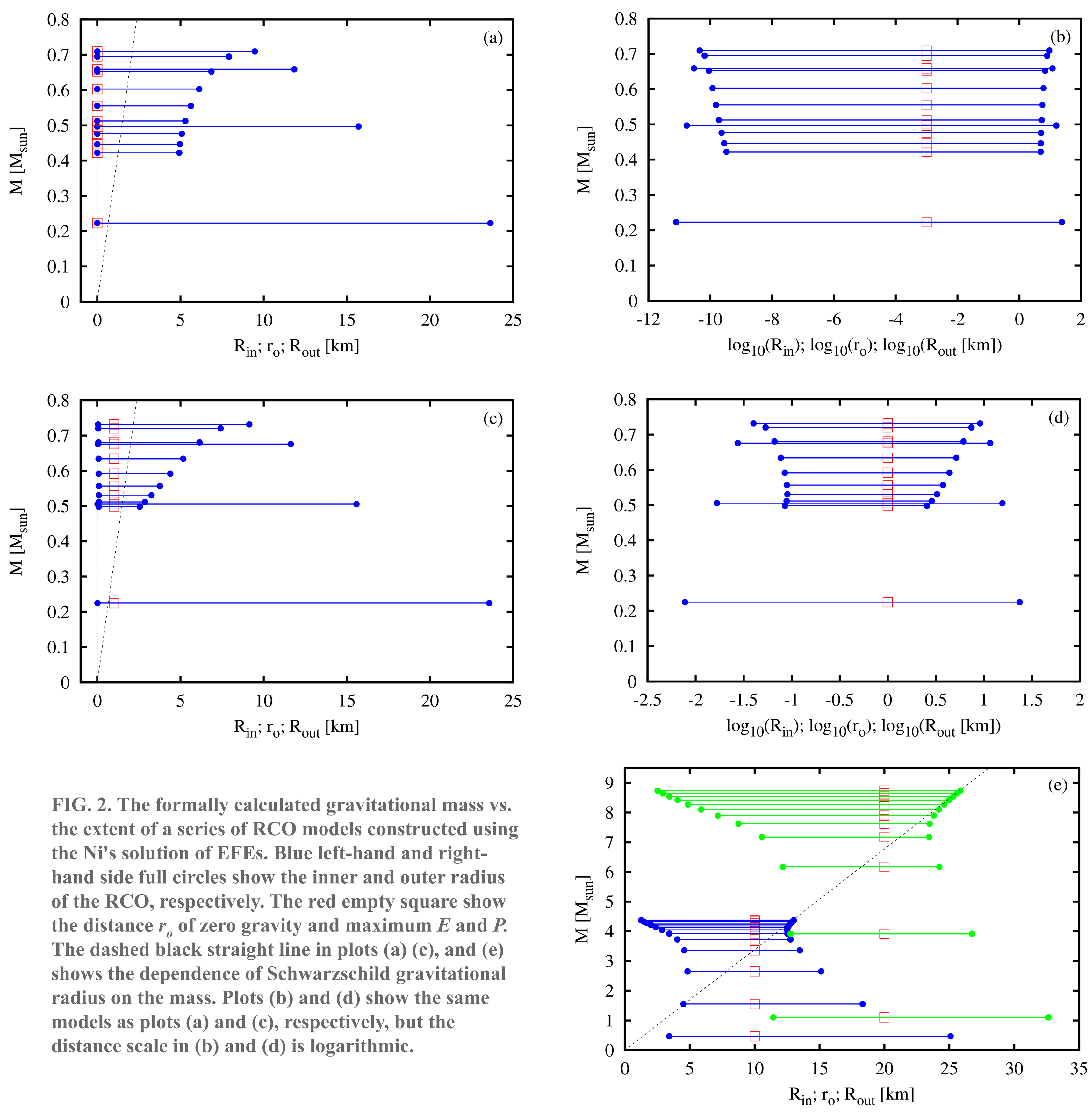

FIG. 2. The formally calculated gravitational mass vs. the extent of a series of RCO models constructed using the Ni's solution of EFEs. Blue left-hand and right-hand side full circles show the inner and outer radius of the RCO, respectively. The red empty square show the distance r_o of zero gravity and maximum E and P. The dashed black straight line in plots (a), (c), and (e) shows the dependence of Schwarzschild gravitational radius on the gravitational mass. Plots (b) and (d) show the same models as plots (a) and (c), respectively, but the distance scale in (b) and (d) is logarithmic.

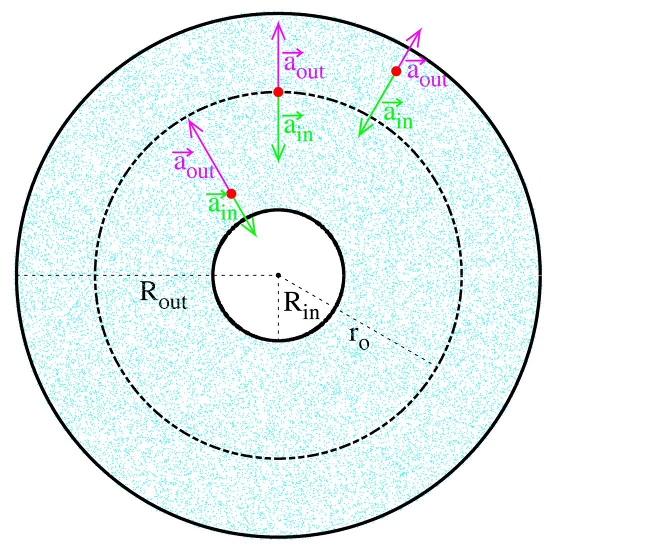

FIG. 3. The scheme of RCO described by the Ni's solution of the EFEs. The body of the RCO extends from the inner physical surface with radius R_{in} up to the outer physical surface with radius R_{out}. Three red full circles represents the test particles, which are attracted by the matter of the RCO. While the acceleration of the particle from the inner layers (with the radius smaller than the RCO-centric distance of given particle) is denoted by $\vec{a}_{in}$ and illustrated with a green arrow, the acceleration due to the gravity of upper layers (with radius larger than the RCO-centric distance of the particle) is denoted by $\vec{a}_{out}$ and illustrated with a violet arrow. Near the upper surface, $|\vec{a}_{in}| > |\vec{a}_{out}|$ and total acceleration is oriented inward. In the distance of zero net gravity, r_o, the gravitational attraction of inner layers is exactly balanced by that of outer layers, i.e. $|\vec{a}_{in}| = |\vec{a}_{out}|$. Near the inner surface, $|\vec{a}_{in}| < |\vec{a}_{out}|$, therefore the total net gravity is oriented outward, and is again balanced with the gradient of pressure, which is everywhere oriented against the gravitational attraction, according to the EFEs.

* near the inner surface inside RCO, the net gravity of upper layers dominates over the net gravity of inner layers and the total net gravity is oriented outward; the inner physical surface of RCO is formed by the same mechanism as its outer surface
* inside the RCO, there is a specific distance, r_o, in which the net gravity of inner layers is exactly balanced by the net gravity of upper layers; the total net gravity is zero; E and P reach their maxima in r_o
* the EFEs imply that the gradient of pressure must always be oriented against the gravitational acceleration; consequently, the orientation of dP/dr in $r < r_o$ is opposite to that of dP/dr in $r > r_o$

Almost identical Oppenheimer-Volkoff and Ni RCOs

* the distance of zero net gravity and maximum E and P inside the RCO can be arbitrarily small in the Ni's RCO
* considering the same initial Fermi impulse in both center of Oppenheimer-Volkoff object and in distance r_o of Ni's object, whereby $r_o << R_{out}$, we show the comparison of the behaviors of $\rho = E/c^2$ of these two objects in Fig.4

* as seen, the objects are practically identical down to the object-centric distance equal to $\sim 10^{-8}$ km
* the total energy (hence mass) of both objects is practically the same
* constructing a series of the Ni's objects all consisting of the same number of neutrons (i.e. having the same rest mass), the Ni's object with a significantly larger r_o than that in the example has the smallest total energy of all and is, therefore, preferred

Central singularity

* in central cavity of RCO: $g_{tt} = 1 + 2|u_{in}|/r$, therefore $g_{tt} \rightarrow \infty$ for $r \rightarrow 0$ (central singularity)
* since $dg_{tt}/dr < 0$ (see Fig.1c) in the cavity, every particle in this region is attracted away from the center and, consequently, **the central singularity is the Big-Bang type singularity**
* there is neither any local Lorentz frame in $r = 0$
* problem of the central singularity and non-existence of local Lorentz frame is, however, only problem in the theory, because no observer can ever visit or no material particle can ever enter the central singularity
* point $r = 0$ is the point of empty space (vacuum); the active agent shaping the metrics in the cavity, point $r = 0$ including, is the matter of RCO surrounding the cavity

Strict cosmic censorship

Since it seems that there is no other than the Big-Bang type central singularity and because of the indication $R_{out} \rightarrow R_g$ only in the limit $M \rightarrow \infty$, we can most probably establish the "strict cosmic censorship" within the G-GR:

No singularity, true or apparent, exists in the real universe and, therefore, in any theoretical description, except of the Big-Bang type singularity in the center of RCOs, if all conditions are realistic.

A more detailed description concerning the Ni's solution and its application to real objects can be found in paper [3].

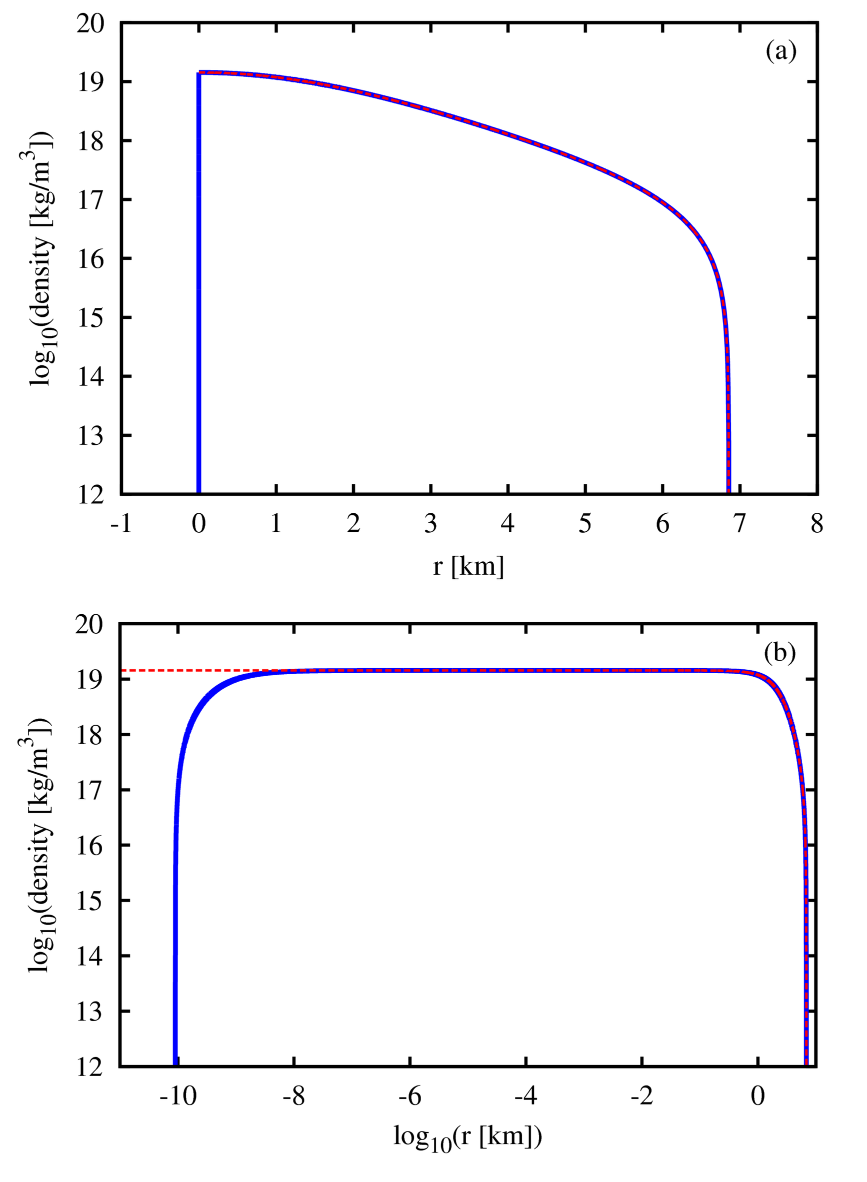

FIG. 4. The behavior of the density, $\rho = E/c^2$, inside the very similar RCOs constructed by using (i) the TOV solution (dashed red curve) and (ii) Ni's solution (solid blue curve) of the EFEs. The behavior is shown in two distance scales: linear in plot (a) and logarithmic in plot (b).

Acknowledgements.
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KEY REFERENCES (for all, see [3])
[1] Oppenheimer J.R. & Volkoff G.R.: 1939, Physical Review **55**, 374-381.
[2] Ni J.: 2011, Science China: Physics, Mechanics, and Astronomy **54**, 1304-1308.
[3] Neslušan L.: 2015, Journal of Modern Physics **6**, 2164-2183; www.scirp.org/journal/Paperinformation.aspx?PaperID=61794

Used symbols:
$\vec{a}_{in}$, $\vec{a}_{out}$ - acceleration of test particle inside the RCO due to its inner and outer material layers, respectively
c - speed of light
E - energy density
G - gravitational constant
g_{rr}, g_{tt} - radial and time components of metric tensor, respectively (in the case of S.S.)
M - gravitational mass of RCO
$\rho = E/c^2$ - mass density of RCO
R_g - Schwarzschild gravitational radius
R_{in}, R_{out} - inner and outer radius of RCO, respectively
r - radial distance from the center of RCO
r_o - radial distance of zero gravity and maximum E and P inside the RCO
U - effective, GR gravitational potential
$u = (r/2)(1 + 1/g_{rr})$ - auxiliary metric quantity (alternative form of g_{rr} component of metric tensor)
u_{in}, u_{out} - value of u in distance $r = R_{in}$ and $r = R_{out}$ (as well as in $r < R_{in}$ and $r > R_{out}$), respectively; both u_{in} and u_{out} are constants

Used abbreviations:
EFEs - Einstein field equations (here those for the spherical symmetry)
G-GR - geometrical general relativity
GR - general relativity
GR-NTP - general relativity with the Newtonian-type gravitational potential as its integral part
RCO - relativistic compact object (e.g., neutron star)
S.S. - spherical symmetry