Saturation of the f-mode Instability in Neutron Stars

Pantelis Pnigouras, Kostas D. Kokkotas, Daniela D. Doneva
Eberhard Karls University of Tübingen, 72076 Tübingen, Germany

Marco Surace
Eberhard Karls University of Tübingen, 72076 Tübingen, Germany
University of Portsmouth, PO1 3FX Portsmouth, UK

January 30, 2016

Abstract

Due to the Chandrasekhar-Friedman-Schutz (CFS) instability, the f-mode (fundamental oscillation) in a newborn neutron star is driven unstable by the emission of gravitational waves. This star is usually the result of a core-collapse supernova explosion, but may also be the aftermath of a binary neutron star merger, where a rapidly rotating, supramassive configuration is formed, before its collapse to a black hole. The instability is halted by nonlinear coupling to other modes of the star, which drain energy and saturate it. Depending on the saturation point, the generated gravitational wave signal could be detected by the next generation gravitational wave detectors and, thus, provide useful information about the neutron star equation of state.

1 Introduction

Inferring the equation of state of matter at supranuclear densities is proven a very hard task, both for QCD and terrestrial experiments. If studied with the right tools, neutron stars offer a unique probe of matter in extreme conditions. Asteroseismology, namely the study of stellar oscillations [1], could be used to deduce the internal structure of these dense objects, with an additional help from gravitational waves. *Gravitational wave asteroseismology* [2, 3] is concerned with stellar deformations that may produce a significant gravitational wave signal. The different waveforms produced by different models can then be used as templates, in order to look for this weak signal in the detector's noisy background.

During the 1970s, Chandrasekhar [4], Friedman, and Schutz [5, 6] discovered that nonradial oscillation modes in a rapidly rotating star are prone to a secular instability, due to the emission of gravitational waves (*CFS instability*). Since the instability is suppressed by viscous effects [7], only a few modes can indeed develop the instability in a relatively short time scale: low-multipole f- and r-modes, which describe the fundamental and inertial oscillations of the fluid respectively, are among the most promising sources of gravitational waves from individual stars.

The exponential growth of the unstable mode will, however, eventually be halted by nonlinear effects. Nonlinear mode coupling, during which the unstable mode's energy is drained by other modes of the star, was shown to saturate the r-mode instability very efficiently, at quite low amplitudes [8–11]. This would make the associated gravitational wave signal potentially detectable with second-generation interferometers, such as Advanced LIGO, only from sources within the local galactic group [12, 13].

Hitherto, the f-mode saturation amplitude is chosen ad hoc, mainly based on upper limits obtained by nonlinear hydrodynamic simulations (e.g. Ref. [14]). Nevertheless, the secular time scale of the instability prevents these simulations, which remain stable only on dynamical time scales, from making a robust estimate. A conclusive result about the saturation point of an unstable f-mode can only come via the study of the nonlinear coupling paradigm.

2 The Oscillation Modes—Linear Perturbation Scheme

Assuming a star which is uniformly rotating with an angular velocity Ω, the fluid equations, in the frame rotating with the star, are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = -\frac{\nabla p}{\rho} - \nabla \Phi, \tag{2}$$

and

$$\nabla^2 \Phi = 4\pi G \rho. \tag{3}$$

Equations (1)–(3) are the continuity, Euler, and Poisson equations, respectively, and have to be supplemented with an equation of state $p = p(\rho, \mu)$, where μ usually corresponds to entropy or composition.

Perturbing the system above to linear order and seeking harmonic solutions, we get an eigenvalue equation which, accompanied with the appropriate boundary conditions, gives the eigenfrequencies ω and the eigenfunctions $\boldsymbol{\xi}$ of the oscillation modes of the star.

3 The f-mode Instability

According to the standard multipole expansion of the power radiated in the form of gravitational waves (GWs), by an oscillation mode with frequency ω [15],

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{GW}} = -\sum_{l_{\min}}^{\infty} N_l \, \omega \, (\omega - m\Omega)^{2l+1} \left(|\delta D_l^m|^2 + |\delta J_l^m|^2\right), \tag{4}$$

where δD_l^m and δJ_l^m are the mass and current multipole moments respectively, l and m are the spherical harmonic indices of the mode, $l_{\min} = \max(2, |m|)$, and N_l is a constant.

That is, the emission of gravitational radiation damps the mode, unless $\omega(\omega - m\Omega) < 0$, in which case the energy grows. The onset of the instability occurs when $\omega/m = \Omega$, namely when the *pattern speed* of the mode matches the angular velocity of the star. Since $\omega - m\Omega$ is the inertial-frame frequency of the mode (ω is measured in the corotating frame), this means that a mode which is retrograde for slow rotation can be dragged forwards by fast rotation and appear as prograde.

The instability is, however, suppressed by viscous effects [7]. Shear viscosity (SV), due to particle scattering, and bulk viscosity (BV), due to the disturbance of β-equilibrium by the perturbation, dominate at low and high temperatures, respectively. The instability operates in the area where

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{GW}} + \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{SV}} + \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{BV}} > 0, \tag{5}$$

with E denoting the energy of the mode. The region in the "temperature–angular velocity" plane where this inequality is satisfied is called the mode's *instability window* (Fig. 1).

4 Mode Coupling—Quadratic Perturbation Scheme

By perturbing Eqs. (1)–(3) to quadratic, instead of linear, order, *mode coupling* is introduced and modes can no longer be described independently. It can be shown that modes couple in triplets, if a resonance condition is satisfied between their frequencies; their amplitudes are then described by [16, 17]

$$\dot{Q}_\alpha = \gamma_\alpha Q_\alpha + i\omega_\alpha \mathcal{H} Q_\beta Q_\gamma e^{-i\Delta\omega t}, \tag{6a}$$

$$\dot{Q}_\beta = \gamma_\beta Q_\beta + i\omega_\beta \mathcal{H} Q_\gamma^* Q_\alpha e^{i\Delta\omega t}, \tag{6b}$$

$$\dot{Q}_\gamma = \gamma_\gamma Q_\gamma + i\omega_\gamma \mathcal{H} Q_\alpha Q_\beta^* e^{i\Delta\omega t}, \tag{6c}$$

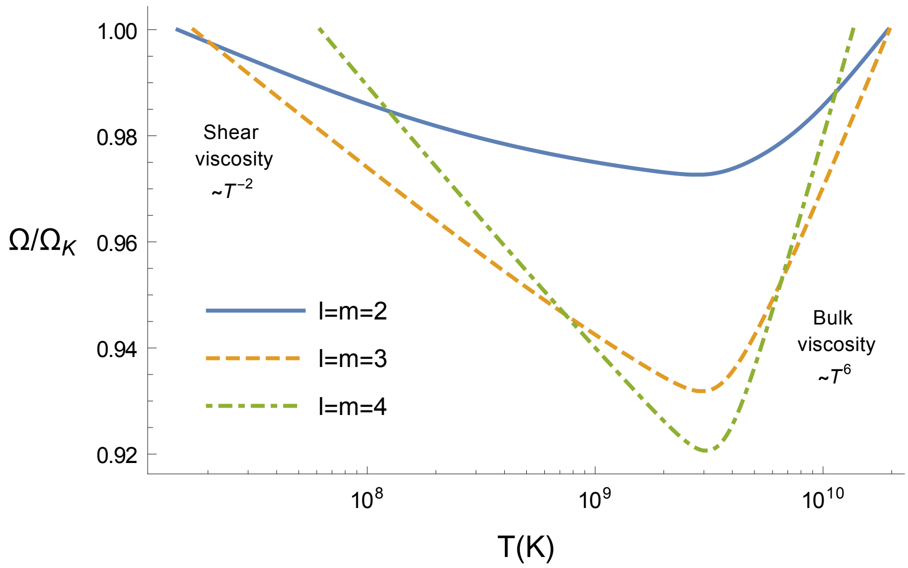

Figure 1: Quadrupole, octupole, and hexadecapole f-mode instability windows, for a Newtonian star with $p \propto \rho^3$, $M = 1.4\,M_\odot$, and $R = 10\,\mathrm{km}$. The angular velocity Ω, normalized to the Kepler (mass shedding) limit, is plotted against the temperature.

where $\gamma_i = (\mathrm{d}E_i/\mathrm{d}t)/(2E_i)$ are their growth/damping rates, $\mathcal{H}$ is the triplet's coupling coefficient, and $\Delta\omega = \omega_\alpha - \omega_\beta - \omega_\gamma$ their frequency mismatch. The efficiency of the coupling depends on how close to resonance the modes are ($\Delta\omega \approx 0$) and the value of the coupling coefficient $\mathcal{H}$. Also, for saturation to be achieved, the unstable (parent) mode ($\gamma_\alpha > 0$) has to couple to two stable (daughter) modes ($\gamma_{\beta,\gamma} < 0$). When this happens, a *parametric resonance* occurs: after the *parametric threshold* (PT) is crossed by the parent mode, the daughter modes start growing by draining energy from it. The system eventually saturates if two additional conditions are met, given by [16, 17]

$$|\gamma_\beta + \gamma_\gamma| \gtrsim \gamma_\alpha \quad \text{and} \quad |\Delta\omega| \gtrsim |\gamma_\alpha + \gamma_\beta + \gamma_\gamma|. \tag{7}$$

The saturation amplitude of the parent mode is

$$|Q_\alpha^{\mathrm{sat}}|^2 \approx |Q_{\mathrm{PT}}|^2 = \frac{\gamma_\beta\gamma_\gamma}{\omega_\beta\omega_\gamma\mathcal{H}^2}\left[1 + \left(\frac{\Delta\omega}{\gamma_\beta + \gamma_\gamma}\right)^2\right]. \tag{8}$$

The two cases of successful and unsuccessful saturation are presented in Fig. 2.

5 Results

The saturation amplitude of the quadrupole ($l = m = 2$) and octupole ($l = m = 3$) f-modes throughout their instability windows, for a Newtonian polytrope, is shown in Fig. 3. Its scaling with temperature, along lines of constant angular velocity, is [18]

$$|Q_{\mathrm{PT}}| \propto \begin{cases} T^{-1}, & T \lesssim 10^9\,\mathrm{K} \\ T^3, & T \gtrsim 10^9\,\mathrm{K} \end{cases} \quad \text{for} \quad \Omega = \textit{const.} \tag{9}$$

which is a result of the temperature dependence of the daughters' damping rates [cf. Eq. (8)]. This scaling justifies the higher values of the saturation amplitude near the edges of the window, where the damping rates due to viscosity are larger.

An additional feature, noticeable in the results for the octupole f-mode, are these horizontal bands which seem to behave differently from the "background". These are mainly attributed to the

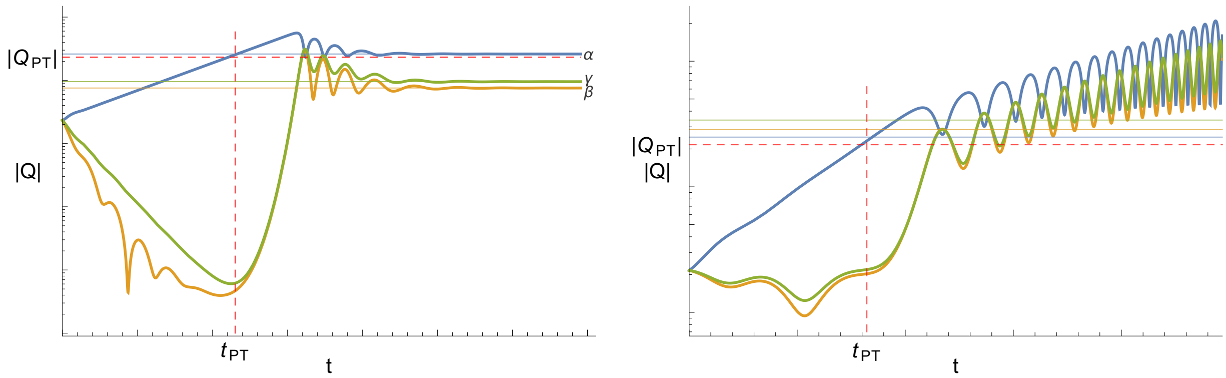

Figure 2: After the parametric threshold (horizontal dashed line) is crossed by the parent, the daughters start growing by draining energy from it. *Left:* The three modes saturate around their equilibrium solution (horizontal solid lines). *Right:* The saturation conditions (7) are not fulfilled and the triplet diverges from its equilibrium solution.

occurrence of a very fine resonance between the parent f-mode and some daughter pair, which only appears for a specific angular velocity of the star.

As a newborn neutron star cools down, it might enter the instability window from the right, at which point the unstable mode will start growing. After it saturates, the star will continue to cool, until thermal equilibrium is established. Then, it will spin down at $T \sim 10^9\,\mathrm{K}$ [19], due to the emission of gravitational waves, until it finally exits the window. The saturation amplitude is usually considered constant during this process. However, based on Fig. 3, it might change considerably during the neutron star evolution through the instability window.

Apart from supernova-derived neutron stars, another interesting case involves merger-derived, supramassive neutron stars. These are products of the coalescence of a neutron star binary, which are supported by rotation against collapse to a black hole. Should they survive for enough time, they could develop a very strong f-mode instability [20] and, thus, produce a significant gravitational wave

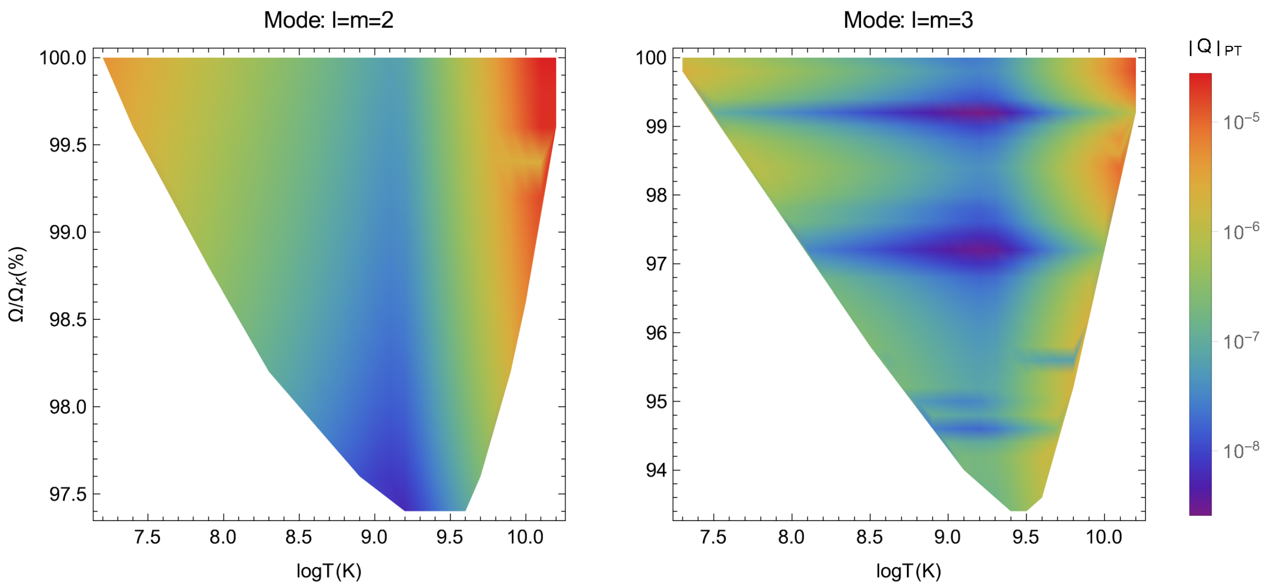

Figure 3: Contour plots of the saturation amplitude for the quadrupole and octupole f-modes, versus temperature and angular velocity (as a fraction of the Kepler limit), for a Newtonian star with $p \propto \rho^3$, $M = 1.4\,M_\odot$, and $R = 10\,\mathrm{km}$. The normalization used is $E_{\mathrm{mode}} = |Q|^2 M c^2$.

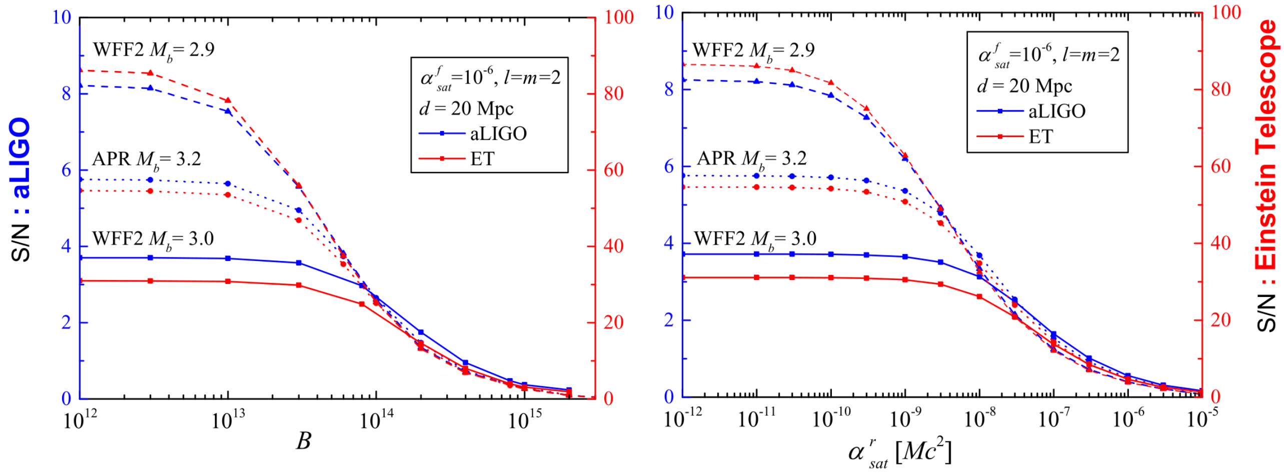

Figure 4: Signal-to-noise ratio of a quadrupole f-mode, from relativistic, supramassive neutron stars with different baryon masses and equations of state, versus the magnetic field strength (*left*) and the r-mode saturation amplitude (*right*). Blue lines and axes correspond to Advanced LIGO and red ones to the Einstein Telescope. The distance to the source is set to 20 Mpc and the f-mode saturation amplitude to 10^{-3} (here, $\alpha_{\rm sat} = |Q_\alpha^{\rm sat}|^2$).

signal. According to Ref. [20], the signal could be detectable from sources at the Virgo cluster with second-generation gravitational wave interferometers, such as Advanced LIGO, or even further with third-generation detectors, like the Einstein Telescope (cf. Fig. 4).

The strength of the gravitational wave signal obtained from such instabilities is determined by the saturation amplitude of the unstable mode. However, just a high absolute value of the saturation amplitude does not imply a strong signal. In addition to the f-modes, unstable r-modes and magnetic fields also contribute to the spin down of the star. This means that the f-mode instability also has to compete with these mechanisms. As shown in Fig. 4, if the intensity of the magnetic field or the r-mode saturation amplitude are too high, the f-mode signal is rendered unimportant and will not be detected.

Finally, the superposition of unresolved gravitational wave signals from unstable f-modes throughout the universe could be detected in the form of a stochastic background. As shown in Ref. [21], the stochastic background from supernova-derived neutron stars could be detectable even with second-

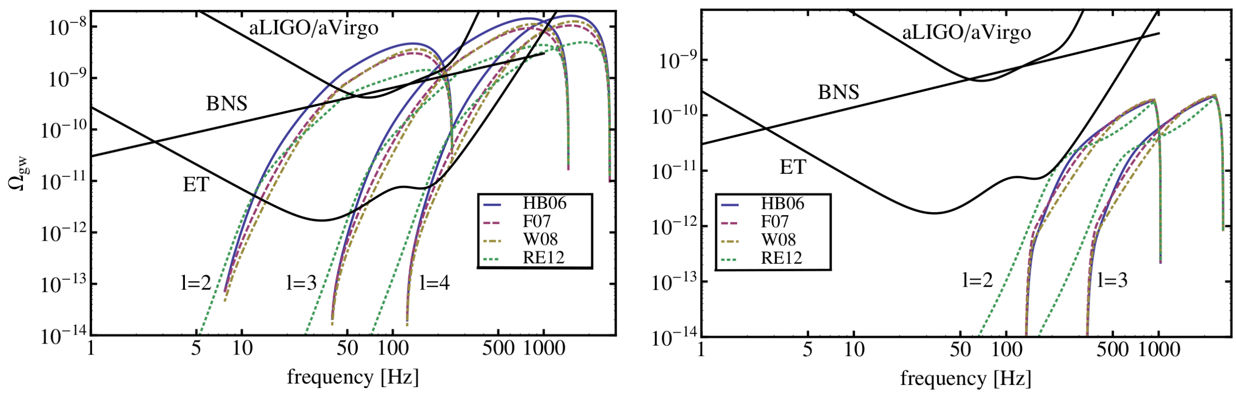

Figure 5: Dimensionless energy density $\Omega_{\rm gw}$ of the stochastic gravitational wave background from quadrupole, octupole, and hexadecapole f-modes, versus the observed frequency, for supernova-derived (*left*) and merger-derived (*right*) neutron stars. Results are shown for different cosmic star formation rate models. The detection limits of second- and third-generation interferometers, as well as the stochastic background due to coalescing binary neutron stars, are also plotted.

generation interferometers. This is presented in Fig. 5.

Acknowledgements

We acknowledge the support of the German Science Foundation (DFG) via SFB/TR7.

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