A relativistic metric extension of gravity based on the dynamics and lensing of individual, groups and clusters of galaxies

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A metric extension of gravity based on the Tully-Fisher law is presented. It is shown that the Tully-Fisher law extends from the dynamics of globular clusters up to the dynamics of groups of galaxies and how it can be consider as a modified version of Kepler's third law. With it, it follows that, at second perturbation order, lensing can be fully understood and that the corresponding γ PPN parameter is required to be one. It is briefly discussed how to construct a relativistic metric extension of gravity using these observational facts and its potential for understanding the dynamics of clusters of galaxies and of the expanding universe without the need to introduce any dark matter/energy entities for its description.

Introduction

The first solid step towards a full development of a non-relativistic theory of gravity was made by Newton in his *Philo-sophiæNaturalis Principia Mathematica* book (Newton, 1729). The starting point of this non-relativistic theory of gravity began with the third law of planetary motion published by Kepler in his *Harmonices Mundi* book (Kepler et al., 1619). For the known 7 planets back then, this law represents a relation between the mass of the sun M, a planet's particular distance to the sun r and the velocity v of a planet about the sun: $v \propto (M/r)^{1/2}$, for circular orbits. The requirement of centripetal balance during the motion of planets yields: $a = -v^2/r = -GM/r^2$ where the proportionality factor G is Newton's gravitational constant and the minus sign appears because of the attractive nature of gravity. The acceleration a produced by the sun on a test planet is thus given by a force inversely proportional to its separation from it and linearly depends on the sun's mass. The right hand side of the previous equation is the simplest form of the mathematical force of gravity introduced by Newton.
This elegant result was the culmination of Newton's great battle against the ideas of René Descartes where occult fluids permeated the cosmos producing vortices which made the planets to follow their observed trajectories (cf. Descartes & Gaukroger, 1998). The great success of gravity as a force of nature was well described by Mr. Coote, then Plumian Professor of Astronomy at the Cambridge Observatories, who wrote in the preface of the second edition of the Principia Mathematica book the following (Newton, 1729): (1) A critical paragraph to occult fluids "*...and moreover of fuppofing occult fluids, freely pervading the pores of bodies, endued with an all-performing fubrility, and agitated with occult motions; they now run out into dreams and chimera's, and neglect the true conftitution of things; which certainly is not to be expected from fallacious conjectures when we can fcarce reach it by the moft certain obfervations. Those who fetch from hypothefes the foundation on which they build their fpeculations, may form indeed an ingenius romance, but a romance it will ftill be*".
(2) A thought on the path to construct forces of nature when experimental (or observational) data is available: "*...but then they affume nothing as principle, that is not proved by phaenomena. They frame no hypothefes, nor receive them into philofophy otherwife than as queftions whofe truth may be difputed. They proceed therefore in a twofold method, fynthetical and analytical. From fome felect phaenomena they deduce by analyfis the forces of nature, and the more fimple laws of forces; and from thence by fynthefis fhew the conftitution of the reft.* ".
It seems that ideas of introducing occult entities to describe physical phenomena reoccurs from time to time. Consider for example the hypothetical aether (an extension of Descartes ideas denying the possibility of empty space), never detected and completely taken away by Einstein in order to build a more profound description of the physical world. Also, and most importantly for the modern perspectives of a modification of gravitational laws at astrophysical scales, is the introduction of non-baryonic dark matter and dark energy. A very interesting story also follows from the observed residuals of Neptune's orbit by Bouvard (1821), who concluded that: (i) the effect of the Sun's gravity, at such a great distance might differ from Newton's description; or (ii) the discrepancies might simply be observational error; or (iii) perhaps Uranus was being pulled, or perturbed, by an as-yet undiscovered planet. Future work by Le Verrier and Adams (Kollerstrom, 2014) lead to the discovery of Neptune (the baryonic dark matter perturber of the motion of Uranus). When anomalies on the advance of the perihelium of Mercury were later observed, Le Verrier postulated the existence of an hypothetical unknown planet called Vulcan, closer to the sun than Mercury (cf. Hsu & Fine, 2005). Vulcan was never observed and the relativistic extension made by Einstein when building up his general theory of relativity showed that almost all of the perihelium's advance of Mercury came from relativistic corrections of Newton's theory of gravity (cf. Clemence, 1947). The important lessons to learn from this story are that: (i) There could be unseen objects that perturbe gravitational orbits and (ii) there may be cases in which gravity needs to be extended. Note however that, in both cases, the anomalies were tiny to the observed dynamics. As opposed to this, the current dark matter/energy hypothesis require huge energy corrections in order to leave Einstein's or Newton's field equations untouched. As such, one may be more tempted to search for non-relativistic and relativistic extensions of gravity.

Non-relativistic MONDian construction

Our recent hypothesis on the existence of non-baryonic dark matter or modifications of gravity were seriously taken into account with studies of the baryonic Tully-fisher relation (e.g. Famaey & McGaugh, 2012, and references therein) through dynamical observations of galaxies. These astrophysical systems are not good enough in order to choose from dark matter hypothesis or from an extension of gravitational phenomena via a modification of Kepler's third law. To do so, recent work has been done in order to show that Kepler's third law of motion requires a modification, with dark matter components impossible to account for: (a) globular clusters dynamics (Hernandez & Jiménez, 2012; Hernandez et al., 2013) and (b) orbits of wide open binaries (Hernandez et al., 2012).
As such one can safely postulate that the Tully-Fisher law: $v \propto M^{1/4}$ signals the starting point in order to extend Newton's ideas of gravity, modifying Kepler's third law of motion. In the previous equation, v represents the velocity (or velocity dispersion for a dynamically pressure supported astrophysical system) and M is the mass (could be internal mass within a radius r) of the system. Similarly to Newton's approach, the requirement of centripetal balance means that the acceleration $a \propto v^2/r$ at a distance r from the configuration's centre and so (Mendoza, 2015): $a = -G_\mathrm{M} M^{1/2}/r$ where the constant of proportionality has been written as G_M and the minus sign has been introduced in order to manifest the attractive nature of the gravitational force. This last force equation can be seen as a motivation to suspect that a new theory of gravity needs to be developed in these astrophysical systems, since its right hand side represents a relation between the acceleration felt by a test body of mass M at a distance r. In this sense, the proportionality constant G_M can be seen as a new gravitational constant, with dimensions of squared length over squared time by the square root of mass.
This means that, in the same way as G is regarded as a fundamental constant of nature, G_M should aspire to the same privileged status. However, in order to gain merits in that direction, G_M should play an essential role in the description of relativistic phenomena on its corresponding scales. Nonetheless, it should be noted that the construction of the Newtonian and MONDian force equations are completely independent, since they both depend on different and unrelated data sets. As such, the constants G and G_M can safely be postulated as independent. Given this independence, one is allowed to think of both as equally fundamental.
Requiring gravity to be described by the Newtonian equation at some particular scales and behaving at some others according to the simplest MOND relation presented above, means that the scale invariance of gravity is necessarily broken. One can postulate that at some astrophysical scales gravity is Newtonian and requires modification at some others. The scale is not just a "fixed" distance scale. From the experimental astronomical evidence mentioned above it follows that the modified regime of gravity appears when the ratio of the mass of a given astrophysical system divided by its characteristic radius is sufficiently small as compared to the corresponding solar system value, which suggests that the transition scale is dynamical rather a simple fixed length. A given test particle sufficiently far away from a mass distribution is thus in this modified regime of gravity.
The approach introduced above for the description of gravitational phenomena departing from standard Newtonian gravity can be connected with the simplest version of the Modified Newtonian Dynamics (MOND) formula by replacing the constant G_M with a new constant a_0 introduced by Milgrom (1983) with dimensions of acceleration through the relation $\mathsf{a}_0 := G_\mathrm{M}^2/G$ and so the force equation can be written as:
$a = -\left(\mathsf{a}_0 G M\right)^{1/2}/r$.
Since Milgrom's acceleration constant $\mathsf{a}_0 \approx 1.2 \times 10^{-10}\,\mathrm{m\,s^{-2}}$ (Famaey & McGaugh, 2012) it follows that: $G_\mathrm{M} \approx 8.94 \times 10^{-11}\,\mathrm{m^2\,s^{-2}\,kg^{-1/2}}$.
In this regard it is quite important to notice that the formulation of Milgrom (1983) describes a modification on the dynamical sector of Newton's second law and not on the particular form of the gravitational force (cf. Milgrom, 2006). This is quite evident from the initial development of the theory, in which the requirement that the squared of the acceleration a^2 proportional to the Newtonian acceleration GM/r^2 implies flattening of rotation curves in spiral galaxies. In this relation, the proportionality constant a_0 with dimensions of acceleration, is required to be a fundamental quantity of nature. By doing so, the Tully-Fisher law is obtained as a consequence of the proposed modification of dynamics.
With the approach made here, it follows that the Tully-Fisher law forces the construction of a full gravitational theory in systems where Newtonian gravity does not work. In its simplest form, the developed theory must converge to the force equation $a = -\left(GMa_0\right)^{1/2}/r$. As such, no need for modification of Newton's second law needs to be introduced, since only a non-scale invariant character for the gravitational law is directly inferred from the observational data.
The introduction of G_M as a fundamental constant of gravity, rather than a_0 as a new fundamental acceleration scale, sheds light onto the strategy to follow to unveil the structure of the underlying theory. In fact, G_M points towards a modification on the gravitational sector, whereas a_0 could point towards a break down or possible extensions of special relativity (due to the existence of a universal acceleration scale, similarly as with the speed of light), with potentially dramatic implications even in non-gravitational systems.
If Newtonian gravity breaks at a certain scale, one can legitimately wonder whether the relativistic structure of gravitational interactions remains valid or may require a full reformulation. To explore these aspects one should study not only the dynamics of slow massive particles but also the motion of relativistic particles (such as photons) in astrophysical scenarios probing the gravitational field in this new regime. Being conservative, one may assume that Einstein's insights on the geometrical interpretation of gravity remain valid in this regime. As such, it is perfectly reasonable to assume that the Einstein Equivalence Principle remains valid, which implies that test particles satisfy the geodesic equation.
By knowing Tully-Fisher's modification of Kepler's third law and the geodesic equation, then at second order perturbation, the bending of light is completely determined up to a constant (Will, 1993). This is due to the fact that to this order of approximation the motion of photons only depends on the non-relativistic gravitational potential and a Parametrised Post Newtonian (PPN) parameter $\gamma = \text{const.}$, which measures the proportionality between the leading (second) order corrections of the time and spatial metric components in isotropic coordinates.
Additionally, it is natural to interpret the results by Hernandez et al. (2012) on the failure of Kepler's third law for wide binary systems, as a way to test a key aspect of the mathematical structure of the underlying theory of gravity, namely whether or not external boundary conditions influence the internal dynamics of local gravitational systems, which is sometimes referred to as an external field effect (Famaey & McGaugh, 2012). This effect means that for example, a gravitating system in the modified Keplerian regime embedded on an external standard Newtonian (or Keplerian) field, would behave in a Newtonian way. Hernandez et al. (2012) studied orbits of wide binary stars $\sim 1M_\odot$ separated by $\gtrsim 7000$ AU. These bound objects are embedded in our galaxy and are subject to its Newtonian gravity. As such, if an external field effect occurs, then these objects would orbit each other in a standard way, following Kepler's third law. However, their analysis shows that a violation of Kepler's third law occurs in these systems. The large statistics and precise astrometry to be obtained with the GAIA probe of the European Space Agency in the near future, should provide a strong test for the validity of Kepler's third law at scales yet to be explored. Furthermore, lensing observations strongly support the validity of the geodesic equation, implying that the effects of external gravitational fields can be removed by a suitable choice of local coordinates (a freely falling frame). To the light of these results, the idea of an external field effect appears as an artificial construction (possibly related to the specific mathematical realisations of the theory).
As explained by Mendoza et al. (2011) and Mendoza (2012), one can directly model many astrophysical observations if the modification is made in the force (gravitational) sector and not in the dynamical one. To do so, consider a test particle located at a distance r from a central mass M (it could be the mass within radius r, i.e. $M(r)$ producing a gravitational field. Since G and a_0 are fundamental physical constants related to gravitational phenomena, then the gravitational acceleration experienced by a test particle is given by

$$a = \mathsf{a}_0 g(x), \quad \text{where} \quad x := \frac{l_M}{r}, \qquad \text{and} \qquad l_M := \left(\frac{GM}{\mathsf{a}_0}\right)^{1/2}. \tag{1}$$

The length l_M plays an important role in the description of the theory and is such that when $l_M \gg r$, the strong Newtonian regime of gravity is recovered and when $l_M \ll r$ the weak MONDian regime of gravity appears. As such, the dimensionless acceleration (or ***transition function***) $g(x)$ is such that: $a/\mathsf{a}_0 = g(x) := x^2$, when $x \gg 1$ and equals x when $x \ll 1$. A general transition function

$$g(x) = x\,\frac{1 \pm x^{n+1}}{1 \pm x^n}. \tag{2}$$

was built by Mendoza et al. (2011). This non-singular function converges to the correct expected limits for any value of the parameter $n \geq 0$. For values $n \gtrsim 4$ it converges rapidly to the limit transition step function: $g(x)\big|_{n\to\infty} = x$ for $0 \leq x \leq 1$ and x^2, for $x \geq 1$. The parameter n needs to be found empirically by astronomical observations. The value found by Mendoza et al. (2011) for the rotation curve of our galaxy is $n \gtrsim 3$ and the one found by Hernandez (2012); Hernandez et al. (2012); Hernandez & Jiménez (2012) is $n \gtrsim 8$, with a minus sign selection on the numerator and denominator on the right hand side of equation (2). These authors have shown that a large value of n is coherent with solar system motion of planets, rotation curves of spiral galaxies, equilibrium relations of dwarf spheroidal galaxies and their correspondent relations in globular clusters, the Faber-Jackson relation and the fundamental plane of elliptical galaxies as well as with the orbits of wide binary stars. The $n = 3$ model in which a small, but measurable transition is obtained, has also been tested on earth and moon-like experiments by Meyer et al. (2011) and Exirifard (2013) respectively, showing that it is coherent with such precise measurements. In fact, these experiments also validate all $n \geq 3$ models.

Relativistic extension

As explained by Mendoza & Olmo (2015) it is possible to find, up to second perturbation order $\mathcal{O}(2)$, with pure theoretical arguments the metric components of a spherically symmetric space-time where the Tully-Fisher law holds. The $g_{00} = 1 + 2\phi/c^2 = 1 + {}^{(2)}g_{00}$ metric time component is obtained from the geodesic equation, which at this perturbation order for circular orbits is given by: $v^2/c^2 r = (1/2)\partial\, {}^{(2)}g_{00}/\partial r$ and so, by using the Tully-Fisher law the metric component ${}^{(2)}g_{00}$ can be obtained. The $g_{11} = 1 + {}^{(2)}g_{11} = -\frac{2\psi}{c^2}$ radial component can be obtained without requiring spherical symmetry, since the spatial part of the metric can be written as $g_{ik}\mathrm{d}x^i\mathrm{d}x^k$, with ${}^{(0)}g_{kl} = \delta_{kl}$ being the Minkowskian part. The second order perturbation corrections of g_{kl} could in principle involve other potentials (and not only ϕ or ψ). By a suitable choice of coordinates, one can get rid of the anisotropic contributions at the same perturbation order, which turns g_{kl} into a diagonal form. Given the isotropy of space, there is no preferred direction and so ${}^{(2)}g_{ik} \propto \delta_{ik}$. It is natural to expect that the leading order $\mathcal{O}(2)$ correction must be of the same order of magnitude as the gravitational potential ϕ. Accordingly $g_{kl} = \left(1 + 2\gamma\phi/c^2\right)\delta_{kl}$, where the PPN parameter γ is a proportionality constant. Returning to spherical coordinates the end result is that at second perturbation order:

$$\boxed{\begin{aligned} g_{00}(r) &= 1 + \frac{2\phi}{c^2} = 1 - \frac{2(GM_\mathrm{b}a_0)^{1/2}}{c^2}\ln\left(\frac{r}{r_\star}\right) = \frac{2r_\mathrm{g}}{l_M}\ln\left(\frac{r}{r_\star}\right) \\ g_{11}(r) &= -1 + \frac{2\psi}{c^2} = -1 - \gamma\frac{2(GM_\mathrm{b}a_0)^{1/2}}{c^2} = -1 - 2\gamma\frac{r_\mathrm{g}}{l_M}. \end{aligned}}$$

As noted by Mendoza et al. (2013), over the last few years it has become clear that the complete phenomenology of gravitational lensing, at the level of extensive massive elliptical galaxies (see e.g. Gavazzi et al., 2007; Koopmans et al., 2006; Barnabè et al., 2011), galaxy groups (see e.g. More et al., 2012), clusters of galaxies (see e.g. Newman et al., 2009; Limousin et al., 2007) and more recently spiral galaxies (see e.g. Dutton et al., 2011; Suyu et al., 2012) can be accurately modelled using total matter distributions having isothermal profiles, when treating the problem from the point of view of Einstein's general relativity. All these observations show that the dark matter halos needed to explain gravitational lensing under Einstein's general relativity obey the same Tully-Fisher scaling with total baryonic mass as the ones needed to explain the observed rotation curves of spiral galaxies. This means that for a given total baryonic mass, spiral and elliptical galaxies and groups of galaxies require dark matter halos having the same physical properties to explain the observations; from kinematics of rotation curves in the former case to gravitational lensing in the latter one (Dutton et al., 2011; Suyu et al., 2012). Under Einstein's general relativity the majority of these isothermal matter distribution, particularly at large radii, must be composed of a hypothetical dark matter. Using then the lens equation for a general metric theory of gravity is then possible to obtain empirically the metric component g_{11} with the end result that the PPN parameter $\boxed{\gamma = 1}$.
Bernal et al. (2011) and Carranza et al. (2013) have constructed an $f(\chi) = \chi^{3/2}$ extended metric theory of gravity which satisfies the metric and $\gamma = 1$ metric requirements mentioned above. In their description, a dimensionless Ricci scalar $\chi := A_\mathrm{M} R$, where $A_\mathrm{M} \propto r_\mathrm{g} l_\mathrm{M}$ is an area "coupling term" into the action, R is the standard Ricci scalar and $r_\mathrm{g} = GM/c^2$ is the gravitational radius . The weakest field limit of this theory describes converges in its simplest form to MOND and at second order perturbation it describes the dynamics of photons through lensing observations of individual, groups and clusters of galaxies (Mendoza et al., 2013).

Discussion

All together, both relativistic and non-relativistic proposals have shown that the dynamics of the solar system, elliptical, spherical, dSph galaxies can reproduce the observations (Mendoza et al., 2011), together with lensing observations (Mendoza et al., 2013), is capable of describing the dynamics of globular clusters (Hernandez & Jiménez, 2012). This approach has also been used in cosmology without the need to add dark matter and/or energy entities and is coherent with the current accelerated expansion of the universe Carranza et al. (2013). We have also shown that a 4th order perturbation expansion can correctly describe the dynamics of clusters of galaxies Bernal et al. (2015). The reason to do so can be thought in an analogous way as it occurred when studying the orbit of Mercury about a century ago. Its motions are mostly understood with Newton's theory of gravity. However it was necessary to add relativistic corrections to the underlying gravitational theory to account for the precession of its orbit. Mercury orbits at a velocity ~ 50km/s, implying a Lorentz factor of $\sim 10^{-4}$ and already relativistic corrections are required. Typical velocities of clusters of galaxies are $\sim 10^3$km/s with a Lorentz factor $\sim 10^{-3}$. This means that the dynamics of clusters of galaxies are about one order of magnitude more relativistic than the orbital velocity of Mercury and so, if the latter required relativistic corrections, then the necessity to describe the dynamics of clusters of galaxies with relativistic corrections are even more important.
The most important lesson to learn from the modified Kepler's third law (Tully Fisher law) is that, in the regions where it is applicable, the assumption that gravity is a geometrical phenomenon and that the Einstein Equivalence Principle holds, are sufficient to build a model independent approach of the relativistic regime at second perturbation order $\mathcal{O}(2)$, in complete analogy to the one used at solar system scales where the dynamics are compatible with Einstein's general relativity.

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