Lense-Thirring Precession in Strong Gravitational Fields

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Abstract

The exact frame-dragging (or Lense-Thirring (LT) precession) rates for Kerr, Kerr-Taub-NUT (KTN) and Taub-NUT spacetimes have been derived. Remarkably, in the case of the 'zero angular momentum' Taub-NUT spacetime, the frame-dragging effect is shown not to vanish, when considered for spinning test gyroscope. In the case of the interior of the pulsars, the exact frame-dragging rate monotonically decreases from the center to the surface along the pole and but it shows an 'anomaly' along the equator. Moving from the equator to the pole, it is observed that this 'anomaly' disappears after crossing a critical angle. The 'same' anomaly can also be found in the KTN spacetime. The resemblance of the anomalous LT precessions in the KTN spacetimes and the spacetime of the pulsars could be used to identify a role of Taub-NUT solutions in the astrophysical observations or equivalently, a signature of the existence of NUT charge in the pulsars.

1 Introduction

Stationary spacetimes with angular momentum (rotation) are known to exhibit an effect called Lense-Thirring (LT) precession whereby locally inertial frames are dragged along the rotating spacetime, making any test gyroscope in such spacetimes *precess* with a certain frequency called the LT precession frequency [1]. This frequency has been shown to decay as the inverse cube of the distance of the test gyroscope from the source for large enough distances where curvature effects are small, and known to be proportional to the angular momentum of the source. The largest precession frequencies are thus expected to be seen very close to the source (like the surface of a pulsar, or the horizon of a black hole), as well as for spacetimes rotating very fast with large angular momenta.

Earlier analyses of the LT effect [2] assume slowly rotating ($r >> a$, a is the Kerr parameter of the rotating spacetime due to a compact object like a black hole) spacetime for the test gyroscope [3]. Thus, the rotating spacetime solution is usually approximated as a Schwarzschild spacetime, and the effect of rotation is confined to a perturbative term added on to the Schwarzschild metric. This leads to the standard result for LT precession frequency in the weak field approximation, given by [2]

$$\vec{\Omega}_{LT} = \frac{1}{r^3}[3(\vec{J}.\hat{r})\hat{r} - \vec{J}] \tag{1}$$

where, $\hat{r}$ is the unit vector along r direction. In a recent work reported in ref. [4], an alternative approach based on solving the geodesic equations of the test gyroscope numerically, *once again* within the weak gravitational field approximation, is used to compute the frame-dragging effect for galactic-centre black holes.

In another very recent related work [5], Hackman and Lammerzahl have given an expression of LT precession (orbital plane precession) valid up to *first order* in the Kerr parameter a for a general axially symmetric Plebanski-Demianski spacetime. This is obviously a good approximation for slowly-rotating compact objects. The LT precession rate has also been derived [6] through solving the geodesic equations for both Kerr and Kerr-de-Sitter spacetimes at the *polar orbit* but these results are not applicable for orbits which lie in orbital planes other than the polar plane. We understand that observations of precession due to locally inertial frame-dragging have so far focused on spacetimes where the curvatures are small enough; e.g., the LT precession in the earth's gravitational field which was probed recently by Gravity Probe B [7]. There has been so far no attempt to measure LT precession effects due to frame-dragging in strong gravity regimes.

Two motivating factors may be cited in support of such a contention. First of all, the near-horizon physics of black holes and that of the outer layers of neutron stars emitting X-rays from their accretion discs also might need to be reanalyzed in view of the nontrivial LT precession of test geodesics in their vicinity. With upcoming X-ray observatories, as well as multi-wavelength strong gravity space probes currently under construction, which envisage to make observations of possible frame-dragging effects in strong gravity situations in the near future, the need to go beyond the weak field approximation is paramount. A recent work by Stone and Loeb [8] has estimated the effect of weak-field LT precession on accreting matter close to compact accreting objects. While there are claims that what has been estimated in this work pertains more to orbital plane precession, rather than precession of a test gyroscope (which remains the classic example of LT precession), it is obvious that in the vicinity of the spacetime near the surface of pulsars (respectively, the horizons of black holes), the large LT precession of test gyroscopes ought to manifest in changes in the predicted X-ray emission behaviour originating from modifications in the behaviour of infalling timelike geodesics of accreting matter particles due to the LT precession. Thus, there is sufficient theoretical motivation to compute LT precession rates in the strong gravity regime, in a bid towards a prediction that future probes of the inertial frame dragging effect, in such a regime, may correlate with.

2 Exact LT precession frequency in stationary spacetime & its applications

The exact LT precession frequency of a test gyroscope in strongly curved stationary spacetimes, analyzed within a 'Copernican' frame, is expressed as a co-vector given in terms of the timelike Killing vector fields K of the stationary spacetime, as (in the notation of ref. [9])

$$\tilde{\Omega} = \frac{1}{2K^2} * (\tilde{K} \wedge d\tilde{K}) \tag{2}$$

where, $\tilde{K}$ & $\tilde{\Omega}$ denote the one-form dual to K & Ω, respectively. Note that $\tilde{\Omega}$ vanishes if and only if $(\tilde{K} \wedge d\tilde{K}) = 0$. This happens only for a static spacetime.

Using the coordinate basis form of $K = \partial_0$, the co-vector components are easily seen to be $K_\mu = g_{\mu 0}$. Thus, the vector field corresponding to the LT precession co-vector can be expressed in coordinate basis as

$$\Omega = \frac{1}{2} \frac{\epsilon_{ijl}}{\sqrt{-g}} \left[g_{0i,j} \left(\partial_l - \frac{g_{0l}}{g_{00}} \partial_0 \right) - \frac{g_{0i}}{g_{00}} g_{00,j} \partial_l \right] \tag{3}$$

The remarkable feature of the above equation (3) is that it is applicable to any arbitrary stationary spacetime (irrespective of whether it is axisymmetric or not); it gives us the exact rate of LT precession in such a spacetime. For instance, a 'non-rotating' Newman-Unti-Tamburino [10] (NUT) spacetime is known to be spherically symmetric, but still has an angular momentum (dual or 'magnetic' mass [11]); we use Eq.(3) to compute the LT precession frequency in this case as well. This result is rather general, because, there is only one constraint on the spacetime : that it must be stationary, which is the only necessary condition for the LT precession. The utility of this equation is that; if any metric ($g_{\mu\nu}$) contains all 10 (4×4) elements non-vanishing, it can be used to calculate the LT precession in that spacetime. In this case, the precession rate depends only on non-zero $g_{0\mu}(\mu = 0, 1, 2, 3)$ components, not on any other non-zero off-diagonal components of the metric. Thus, this equation also reveals that the LT precession rate is completely determined by the metric components $g_{0\mu}$, and is quite independent of the other components (in co-ordinate basis).

Now, we should discuss that why a stationary spacetime shows this frame-dragging effect irrespective of whether it is axisymmetric or not. A spacetime is said to be stationary if it possesses a timelike Killing vector field ξ^a; further, a stationary spacetime is said to be static if there exists a spacelike hypersurface Σ which is orthogonal to the orbits of the timelike isometry. By Frobenius's theorem of hypersurface orthogonality, we can write for a static spacetime,

$$\xi_{[a} \nabla_b \xi_{c]} = 0 \tag{4}$$

If $\xi^a \neq 0$ everywhere on Σ, then in a neighbourhood of Σ, every point will lie on a unique orbit of ξ^a which passes through Σ. From the explicit form of a static metric, it can be seen that the diffeomorphism defined by $t \to -t$ (the map which takes each point on each Σ_t to the point with the same spatial coordinates on Σ_{-t}), is an isometry. The "time translation" symmetry, $t \to t + constant$ is possessed by all stationary spacetimes. Static spacetimes on the other hand, possess an additional symmetry, "time reflection" symmetry over and above the "time translation symmetry". Physically, the fields which are time translationally invariant can fail to be time reflection invariant if any type of "rotational motion" is involved, since the time reflection will change the direction of rotation. For example, a rotating fluid ball may have a time-independent matter and velocity distribution, but is unable to possess a time reflection symmetry [12]. In the case of stationary spacetimes, the failure of the hypersurface orthogonality condition (Eq.4) implies that neighbouring orbits of ξ^a "twist" around each other. The twisting of the orbits of ξ^a is the cause of that *extra precession* in stationary non-static spacetimes.

2.1 Exact LT precession rate in Kerr spacetime

One can now use Eq. (3) to calculate the precession rate of a test gyroscope in a Kerr spacetime to get the LT precession in a strong gravitational field. In Boyer-Lindquist coordinates, the Kerr metric is written as,

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \quad (5)$$

where, a is Kerr parameter, defined as $a = \frac{J}{M}$, the angular momentum per unit mass and

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2. \quad (6)$$

Now, using Eq.(3) we can obtain the following expression of LT precession rate in Kerr spacetime [13]

$$\vec{\Omega}_{LT}^K = 2aM\cos\theta\frac{r\sqrt{\Delta}}{\rho^3(\rho^2 - 2Mr)}\hat{r} - aM\sin\theta\frac{\rho^2 - 2r^2}{\rho^3(\rho^2 - 2Mr)}\hat{\theta}. \quad (7)$$

This is the LT precession rate where no weak gravity approximation has been made. In slow-rotation limit ($r >> a$), the Kerr metric is approximated as a Schwarzschild metric with the cross term ($g_{\phi t}d\phi dt$), that is

$$ds^2 = ds_{Sch}^2 - \frac{4Ma\sin^2\theta}{r}d\phi dt \quad (8)$$

and Eq. (7) reduces to Eq. (1) which is quite well-known to us.

2.2 Exact LT precession rate in Kerr-Taub-NUT spacetime

The Kerr-Taub-NUT spacetime is geometrically a stationary, axisymmetric vacuum solution of Einstein equation with Kerr parameter (a) and NUT charge (n). If the NUT charge vanishes, the solution reduces to the Kerr geometry. The metric of the Kerr-Taub-NUT spacetime is

$$ds^2 = -\frac{\Delta}{p^2}(dt - Ad\phi)^2 + \frac{p^2}{\Delta}dr^2 + p^2 d\theta^2 + \frac{1}{p^2}\sin^2\theta(adt - Bd\phi)^2 \quad (9)$$

with

$$\begin{aligned}\Delta &= r^2 - 2Mr + a^2 - n^2, p^2 = r^2 + (n - a\cos\theta)^2, \\ A &= a\sin^2\theta + 2n\cos\theta, B = r^2 + a^2 + n^2.\end{aligned} \quad (10)$$

As the spacetime has an intrinsic angular momentum (due to the Kerr parameter a), we can expect a non-zero frame-dragging effect which cam be written as

$$\vec{\Omega}_{LT}^{KTN} = \frac{\sqrt{\Delta}}{p}\left[\frac{a\cos\theta}{\rho^2 - 2Mr - n^2} - \frac{a\cos\theta - n}{p^2}\right]\hat{r} + \frac{a\sin\theta}{p}\left[\frac{r - M}{\rho^2 - 2Mr - n^2} - \frac{r}{p^2}\right]\hat{\theta} \quad (11)$$

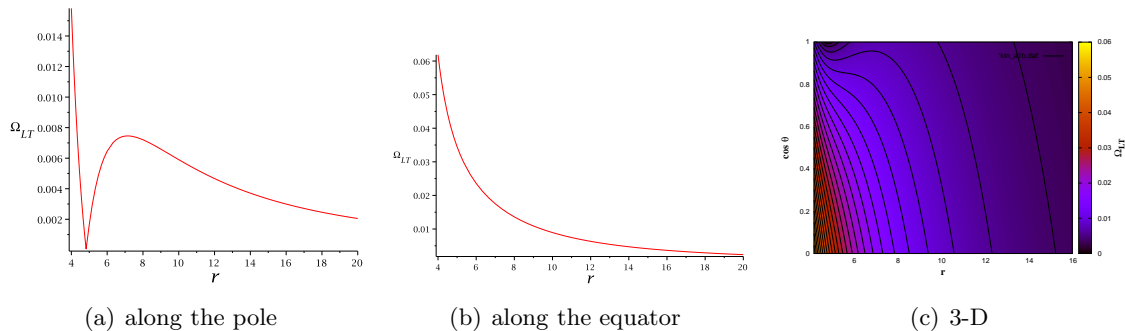(a) along the pole   (b) along the equator   (c) 3-D

Figure 1: *Plot of Ω_{LT} vs r and $\cos\theta$ in the KTN spacetime for $a = n = M = 1$ [14]*

where, $\rho^2 = r^2 + a^2 \cos^2\theta$. In contrast to the Kerr spacetime, where the source of the LT precession is the Kerr parameter a, the Kerr-Taub-NUT spacetime has an extra somewhat surprising feature : the LT precession does not vanish even for vanishing Kerr parameter $a = 0$, so long as the NUT charge $n \neq 0$. This means that though the orbital angular momentum (J) of this spacetime vanishes, the spacetime does indeed exhibit an intrinsic *spinlike* angular momentum (at the classical level itself). One can show that inertial frames are dragged along this orbitally *non-rotating* NUT spacetime with the precession rate [13]

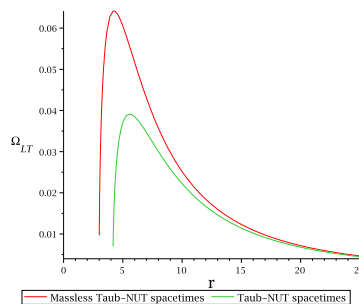$$\vec{\Omega}_{LT}^{MTN} = \frac{n(r^2 - 2Mr - n^2)^{\frac{1}{2}}}{(r^2 + n^2)^{\frac{3}{2}}}\hat{r}.\tag{12}$$

Figure 2: Plot of r vs Ω_{LT} for $n = 3$ & $M = 1$

2.3   Exact LT precession rate inside the rotating neutron stars

The rotating equilibrium models considered in this paper are stationary and axisymmetric. Thus we can write the metric inside the rotating neutron star as the following Komatsu-Eriguchi-Hachisu (KEH) [15] form:

$$ds^2 = -e^{\gamma+\sigma}dt^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2) + e^{\gamma-\sigma}r^2 \sin^2\theta(d\phi - \omega dt)^2\tag{13}$$

where γ, σ, α, ω are the functions of r and θ only. In the whole paper we have used the geometrized unit ($G = c = 1$). We assume that the matter source is a perfect fluid with a stress-energy tensor given by

$$T^{\mu\nu} = (\rho_0 + \rho_i + P)u^\mu u^\nu + Pg^{\mu\nu}\tag{14}$$

where ρ_0 is the rest energy density, ρ_i is the internal energy density, P is the pressure and u^μ is the matter four velocity. We are further assuming that there is no meridional circulation of the matter so that the four-velocity u^μ is simply a linear combination of time and angular Killing vectors.

In orthonormal coordinate basis, the modulus of the exact LT precession rate inside the rotating neutron star is:

$$\Omega_{LT} = |\vec{\Omega}_{LT}(r,\theta)| = \frac{e^{-(\alpha+\sigma)}}{2(\omega^2 r^2 \sin^2\theta - e^{2\sigma})} \cdot$$

$$\left[\sin^2\theta[r^3\omega^2\omega_{,r}\sin^2\theta + e^{2\sigma}(2\omega + r\omega_{,r} - 2\omega r\sigma_{,r})]^2 + [r^2\omega^2\omega_{,\theta}\sin^3\theta + e^{2\sigma}(2\omega\cos\theta + \omega_{,\theta}\sin\theta - 2\omega\sigma_{,\theta}\sin\theta)]^2\right]^{\frac{1}{2}}.$$

$$(15)$$

The expressions of $\sigma(r,\theta), \alpha(r,\theta), \omega(r,\theta)$ have been extracted from [15] and the frame-dragging rates have been calculated using the rotating neutron star `rns` code [16] and plotted from the origin to the surface for some pulsars [17].
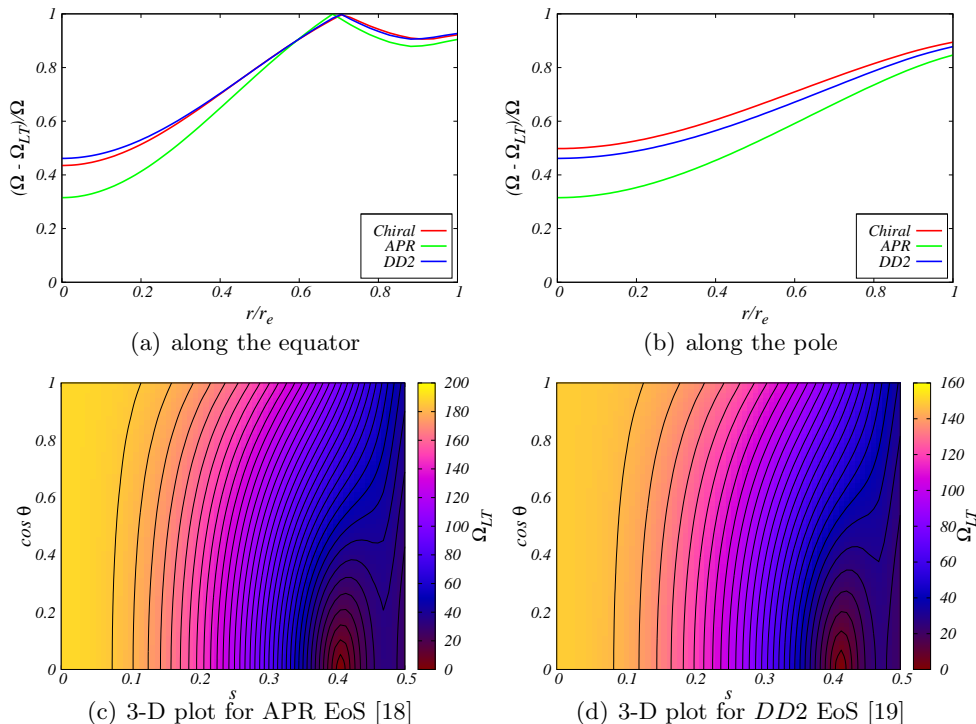

(a) along the equator

(b) along the pole

(c) 3-D plot for APR EoS [18]

(d) 3-D plot for DD2 EoS [19]

Figure 3: *LT effect is calculated from the origin to the surface in the interior of the pulsar J0737-3039A* $(M = 1.337 M_\odot, \Omega = 276.8\, s^{-1})$ *(taken from [17])*

3   Conclusions & Outlook

In contrast to most calculations of the LT precession rate in the literature, which focus on the weak-field approximation, this article has discussed in some detail the problem of the exact LT precession formula in any stationary spacetime and derived the exact LT precession formula for full Kerr, KTN and other spacetimes. The weak-field approximation of LT precession (Eq. 1) for the Kerr spacetime has been then shown to emerge straightforwardly from our general formulation. Interestingly, we have shown that in the *non-rotating* and *spherically symmetric* Taub-NUT spacetime the LT precession does not vanish. Applying the general LT formula we have also obtained the exact frame-dragging rate inside the rotating neutron stars. It is known to us that Lynden-Bell and Nouri-Zonoz [20] first highlighted about the observational possibilities for NUT charges or (gravito)magnetic monopoles and they claimed that the signatures of such spacetime might be found in the spectra of supernovae, quasars, or active galactic nuclei. In our case, the resemblance of the anomalous LT precessions in the KTN spacetimes and the spacetime of the pulsars could be the indirect proof of the existence of

the NUT charge ((gravito)magnetic monopoles) inside the pulsars and we also suggest that such a signature could be used to identify a role of Taub-NUT solutions in the astrophysical observations.

Though any strong gravity measurement has not been performed till now the analogue models of black holes can offer an alternative option of the indirect measurement of strong gravity LT effect in a comparatively accessible laboratory setup. We deduce precise estimate [21] of the angular velocity of the precession of a test spin outside the ergoregion of a fluid mechanical rotating dumb hole in acoustic spacetimes. It is our hope that with present technological expertise in manipulating analogue black holes, experimentalists will be able to successfully verify our estimate and hence, more importantly, the predicted strong gravity LT effect. Accretion disk theory and the related astrophysical phenomena have been investigated using Newtonian or post-Newtonian Gravity. The strong gravity LT effect has not been considered in those calculations. As the accretion disks extend to the vicinity of black hole horizons associated with the very strong gravity regime, the effect of the frame-dragging may have to be taken into account. It will be interesting to investigate the changes to the standard accretion disc theory that this inclusion might entail. The exact frame-dragging rates have already been derived inside and outside of the rotating neutron stars by us. Now, it would be worth to study the quasi-periodic-oscillations (QPOs) in the case of accreting pulsars.

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